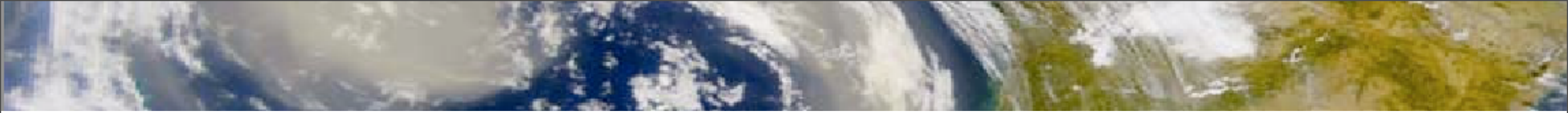
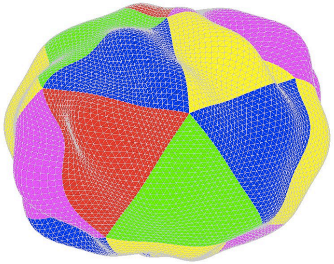
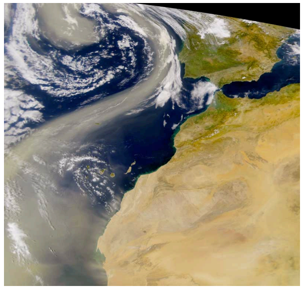
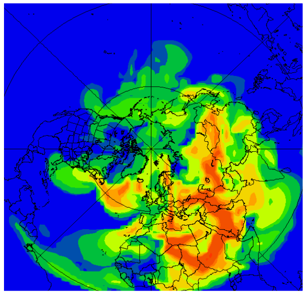


On the discretization and time integration of geophysical fluid dynamics equations on a rotating spheroid

J. Pudykiewicz
RPN, Environment Canada
May 15, 2015

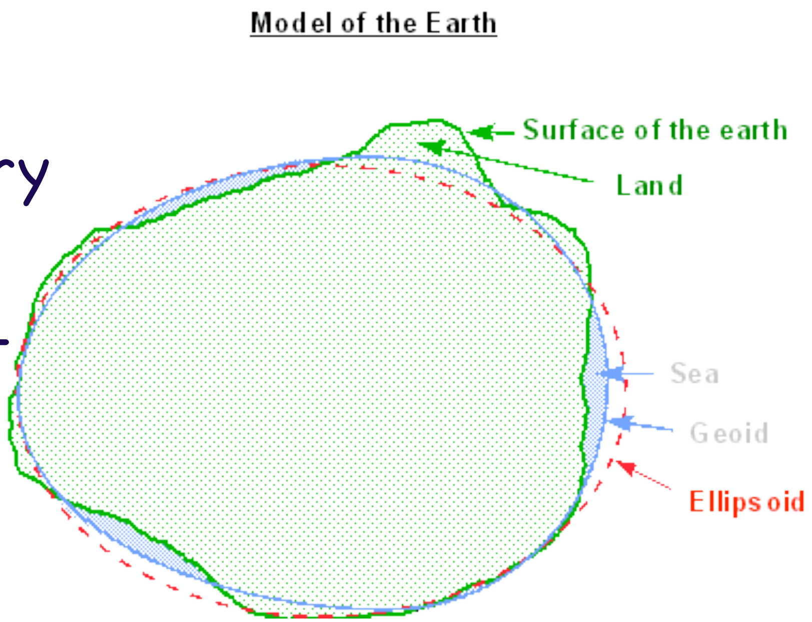


Motion of fluid in non-inertial reference frame
Rotation, curvature and gravity

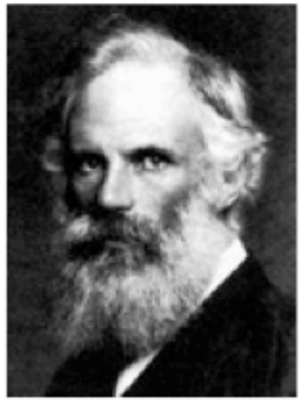
When trying to formulate equations of geophysical fluid on a rotating spheroid one quickly realizes that the problem is best described using mathematical methods of topology, tensor calculus and Riemannian geometry

We have curvature and the non-inertial system
(observer can not distinguish accelerations from gravity)

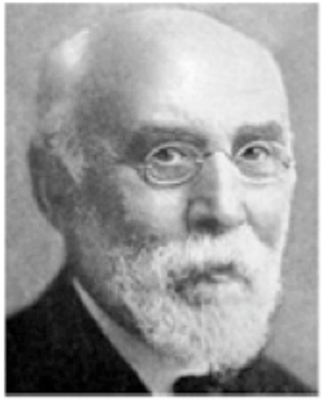


Special relativity

Michelson



FitzGerald



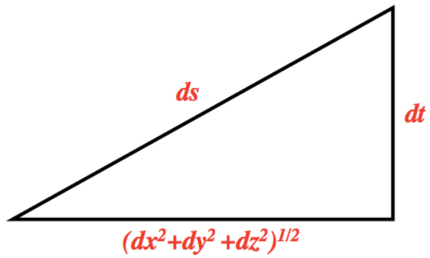
Lorentz

Poincaré

Invariance of time space interval

Einstein in 1905

☆ The notion of ***SPACE-TIME***



$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)$$

$$ds^2 = c^2dt^2 - \left[dr^2 + r^2d\theta^2 + r^2 Sin^2\theta d\phi^2\right]$$

Equations of mechanics written as 4-D divergence of mass-energy tensor

General Relativity


Gauss


Riemann


Grossmann


Hilbert


Einstein in 1916

❑ In General

The interval is given by:

$$ds^2 = \sum_{i,j=0}^{n} g_{ij}\, dx^i dx^j$$

- g_{ij} is the *metric tensor* (Riemannian Tensor) :
  - Tells us how to calculate the distance between 2 points in any given spacetime

Curvature of the space-time depends on distribution of mass, non-inertial systems are equivalent to gravitation

How relativity theory was used in meteorology?

Mc Vittie (1949) Systematic treatment of moving axes in hydrodynamics
Proc. Roy. Soc. series A, vol 196, Issue 1045

The time and space are connected even in the framework of classical physics, all tensors are therefore four dimensional

This is the first time when 4-D tensors in the sense of Einstein (The meaning of Relativity) were applied to write down meteorological equations

Defrise review of the tensorial formalism

First complete formalization of four dimensional tensor calculus in meteorology was presented by Defrise (1967), (this work is not very well known, Google scholar search on April 23rd, 2015 shows citation number = 5)

Defrise (1967) Tensor Calculus in Atmospheric Mechanics, Academic Press (series: Advances in Geophysics)

Beyond four dimensions

$$\frac{\partial}{\partial t}(g^{1/2}\rho) + \frac{\partial}{\partial x^i}(g^{1/2}\rho u^i) = 0$$

$$\frac{\partial}{\partial t}(g^{1/2}\rho u^i \vec{g_i}) + \frac{\partial}{\partial x^i}(g^{1/2}(\rho u^j u^i + p g^{ji})\vec{g_j}) = 0$$

$$\frac{\partial}{\partial t}(g^{1/2} e) + \frac{\partial}{\partial x^i}(g^{1/2}(e+p)u^i) = 0$$

$$\mathrm{div}_5 \mathbf{T} = 0 \qquad \frac{\partial}{\partial x^\alpha}(g^{1/2} T^{\alpha\beta}\vec{g_\beta}) = 0$$

Vinokur (1974) Conservation Equations of Gasdynamics in Curvilinear Coordinate Systems, J. Comp. Phys., vol 14, pp 105-125

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p \begin{bmatrix} g^{11} & g^{12} & g^{13} & 0 & u^1 \\ g^{21} & g^{22} & g^{23} & 0 & u^2 \\ g^{31} & g^{32} & g^{33} & 0 & u^3 \\ 0 & 0 & 0 & 0 & 0 \\ u^1 & u^2 & u^3 & 0 & 0 \end{bmatrix}$$

$$u^4 = 1$$

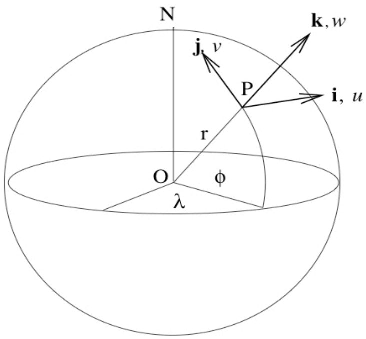
$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 \\ g_{31} & g_{32} & g_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$u^5 = e/\rho = \mathcal{E} + \frac{1}{2} g_{ij} u^i u^j$$

Vinokur (1974) Conservation Equations of Gasdynamics in Curvilinear Coordinate Systems, J. Comp. Phys., vol 14, pp 105-125

Charron, Zadra and Girard (2013) Four-dimensional tensor equations for a classical fluid in an external gravitational field, QJRSC, vol 140, pp 908-916

$$ds^2 = dt^2 + \left(r\cos\phi\left(d\lambda + \omega dt\right)\right)^2 + (r d\phi)^2 + \left(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial \lambda}d\lambda + \frac{\partial r}{\partial \phi}d\phi + \frac{\partial r}{\partial \eta}d\eta\right)^2 = g_{\mu\nu}dx^\mu dx^\nu$$



$$dx^\mu = (dt, d\lambda, d\phi, d\eta)$$

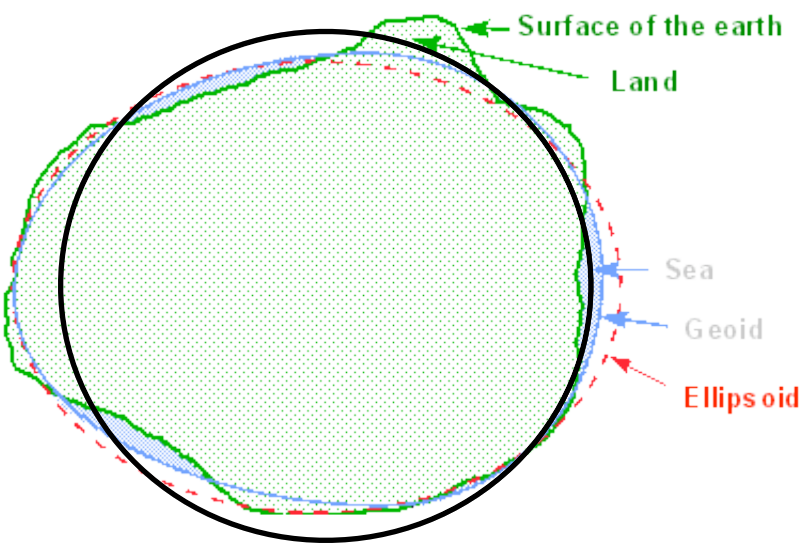
$$\begin{bmatrix}
\left[1 + (\Omega r\cos\varphi)^2 + (\frac{\partial r}{\partial t})^2\right] & \left[\Omega(r\cos\varphi)^2 + \frac{\partial r}{\partial t}\frac{\partial r}{\partial \lambda}\right] & \left[\frac{\partial r}{\partial t}\frac{\partial r}{\partial \varphi}\right] & \left[\frac{\partial r}{\partial t}\frac{\partial r}{\partial \eta}\right] \\
\left[\Omega(r\cos\varphi)^2 + \frac{\partial r}{\partial t}\frac{\partial r}{\partial \lambda}\right] & \left[(r\cos\varphi)^2 + (\frac{\partial r}{\partial \lambda})^2\right] & \left[\frac{\partial r}{\partial \lambda}\frac{\partial r}{\partial \varphi}\right] & \left[\frac{\partial r}{\partial \lambda}\frac{\partial r}{\partial \eta}\right] \\
\left[\frac{\partial r}{\partial t}\frac{\partial r}{\partial \varphi}\right] & \left[\frac{\partial r}{\partial \lambda}\frac{\partial r}{\partial \varphi}\right] & \left[r^2 + (\frac{\partial r}{\partial \varphi})^2\right] & \left[\frac{\partial r}{\partial \varphi}\frac{\partial r}{\partial \eta}\right] \\
\left[\frac{\partial r}{\partial t}\frac{\partial r}{\partial \eta}\right] & \left[\frac{\partial r}{\partial \lambda}\frac{\partial r}{\partial \eta}\right] & \left[\frac{\partial r}{\partial \varphi}\frac{\partial r}{\partial \eta}\right] & \left[(\frac{\partial r}{\partial \eta})^2\right]
\end{bmatrix}$$

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p(g^{\alpha\beta} - g^{\alpha 0} g^{0\beta})$$

$$F^\alpha = -\rho(g^{\alpha\beta} - g^{\alpha 0} g^{0\beta}) \frac{\partial}{\partial x^\alpha} \Phi$$

$$\begin{bmatrix} 1 & -\Omega & 0 & g^{03} \\ -\Omega & \left(\Omega^2 + \frac{\sec^2\phi}{r^2}\right) & 0 & g^{13} \\ 0 & 0 & \frac{1}{r^2} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{bmatrix}$$

Model of the Earth



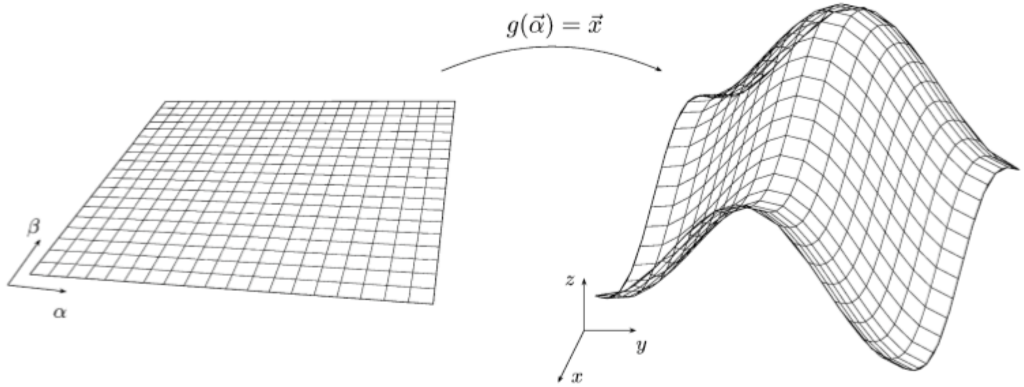
$$g^{03} = g^{30} = -\left(\frac{\partial r}{\partial \eta}\right)^{-1} \left(\frac{\partial r}{\partial t} - \Omega \frac{\partial r}{\partial \lambda}\right)$$

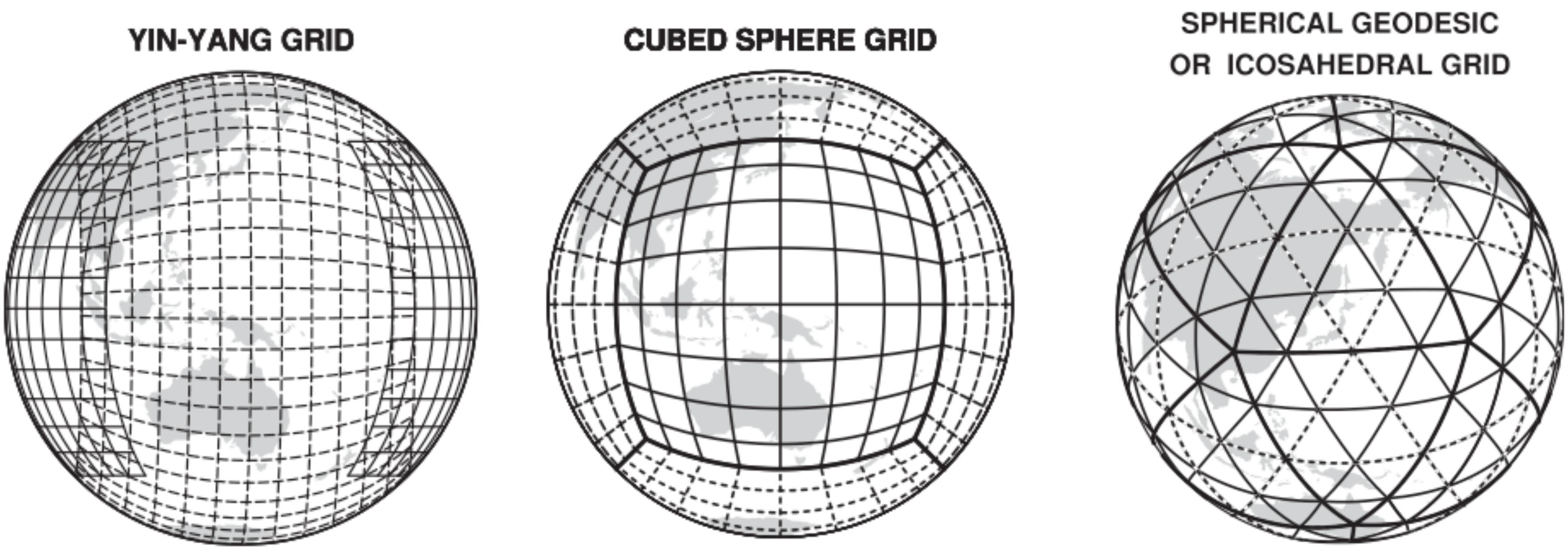
$$g^{13} = g^{31} = \Omega \left(\frac{\partial r}{\partial \eta}\right)^{-1} \left(\frac{\partial r}{\partial t} - \Omega \frac{\partial r}{\partial \lambda}\right) - \frac{\sec^2\phi}{r^2} \left(\frac{\partial r}{\partial \eta}\right)^{-1} \left(\frac{\partial r}{\partial \lambda}\right)$$

$$g^{23} = g^{32} = -\frac{1}{r^2} \left(\frac{\partial r}{\partial \eta}\right)^{-1} \left(\frac{\partial r}{\partial \phi}\right)$$

$$g^{33} = \left(\frac{\partial r}{\partial \eta}\right)^{-2} \left[1 + \left(\frac{\partial r}{\partial t} - \Omega \frac{\partial r}{\partial \lambda}\right)^2 + \frac{\sec^2\phi}{r^2} \left(\frac{\partial r}{\partial \lambda}\right)^2 + \frac{1}{r^2} \left(\frac{\partial r}{\partial \phi}\right)^2\right]$$

Discretization of the equations written in arbitrary curvilinear coordinates





We said before: “Time - space continuum is not homogeneous and isotropic, its properties are described by metric tensor”

What are alternative ways to represent non-Euclidean space?

The answer is provided by the theory of discrete topological spaces

This theory is used in many areas including General Relativity and differential geometry (Ricci flow equation is one of the best examples)

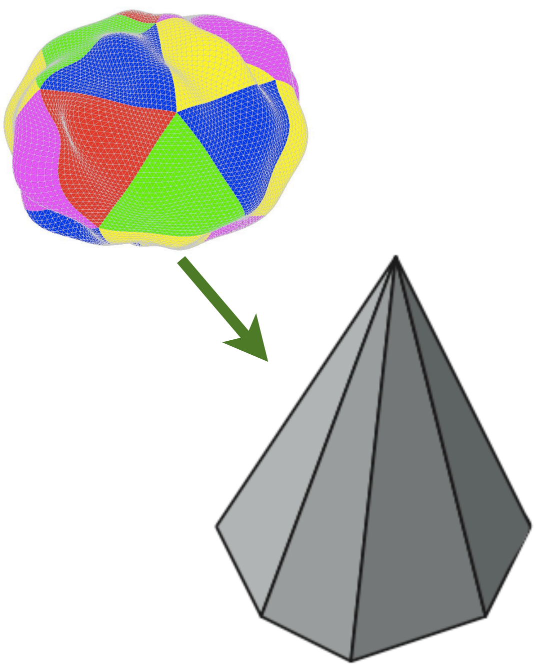
Tulio Regge (1961) General Relativity without coordinates

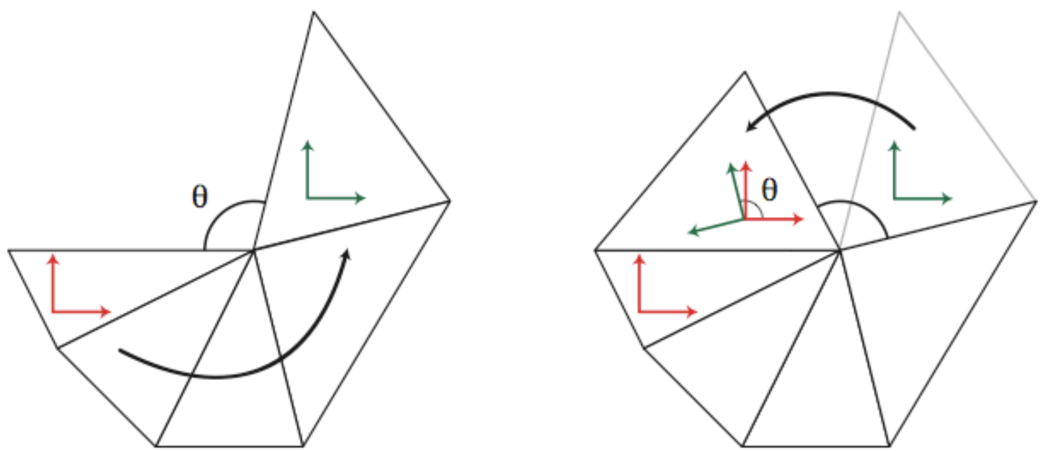
Il Nuovo Cimento, vol XIX no 3

1) use Discrete Exterior Calculus to write approximation of operators
2) Curvature in this case is defined by the lengths of edges of the space frame approximating the manifold
3) The mathematical basis is provided by the theory of discrete topological spaces

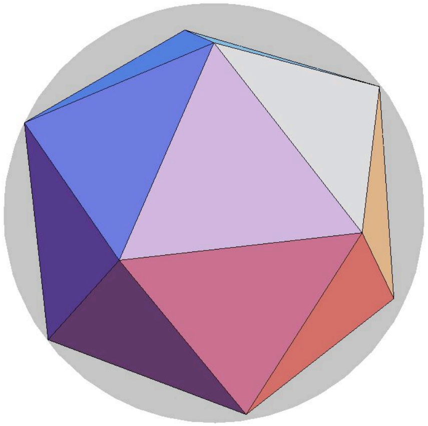
K_p is the **angle defect** at a point p, given by

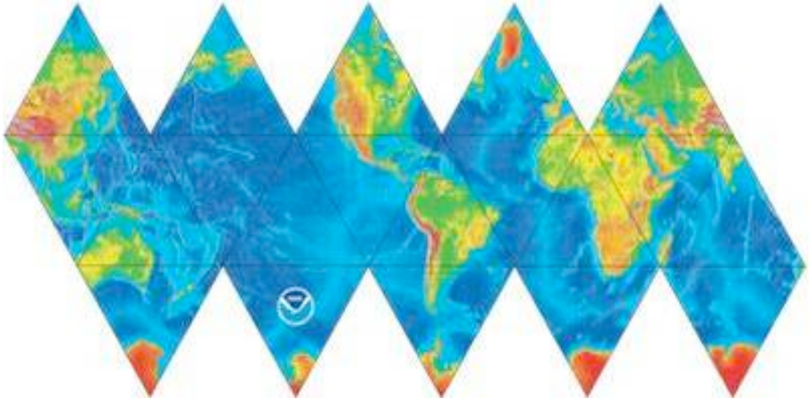
$$K_p = 2\pi - \sum_i \theta_i \, .$$

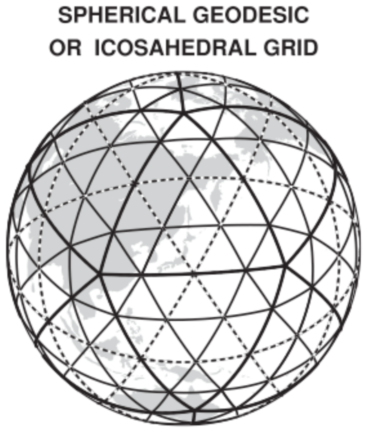


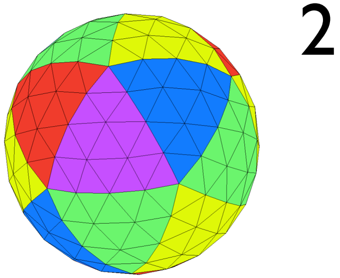


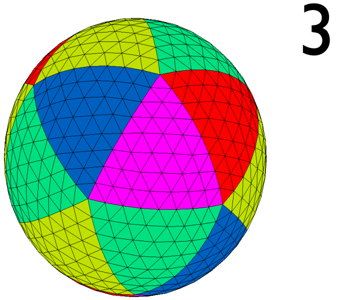
Simplex defined on the basis of icosahedron

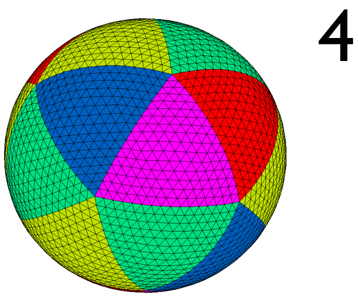


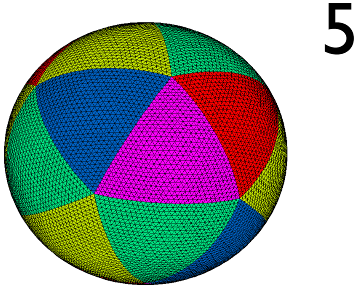




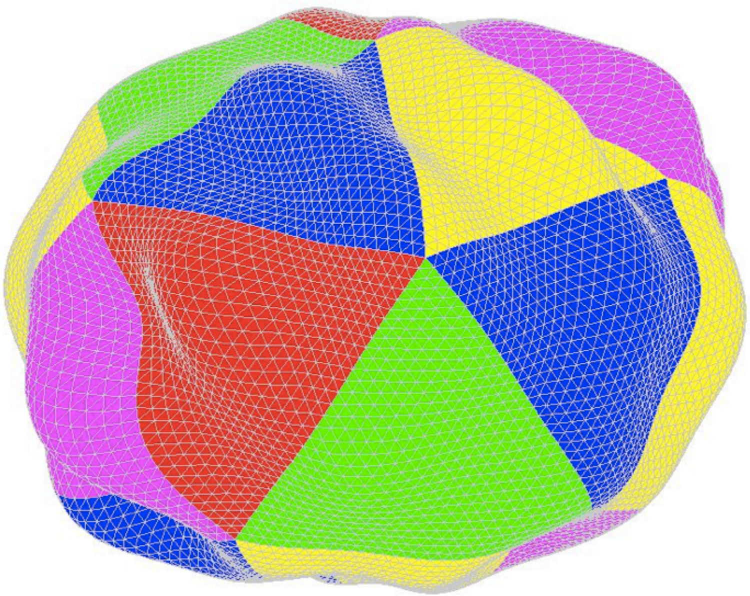


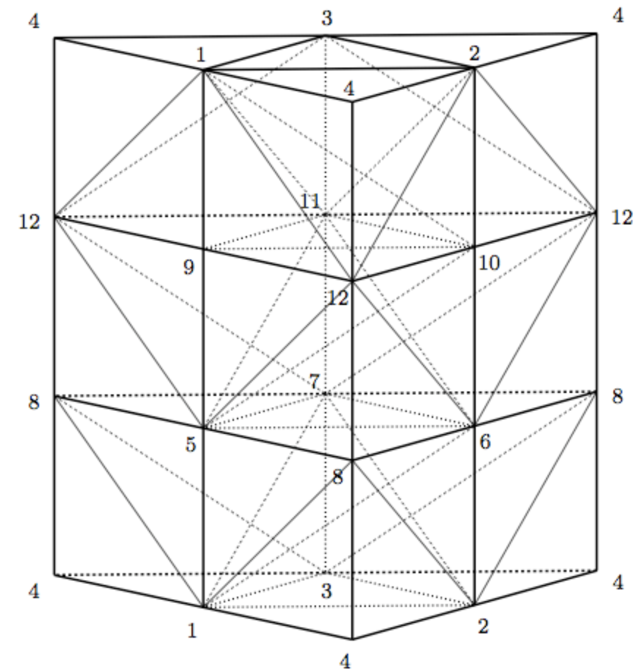






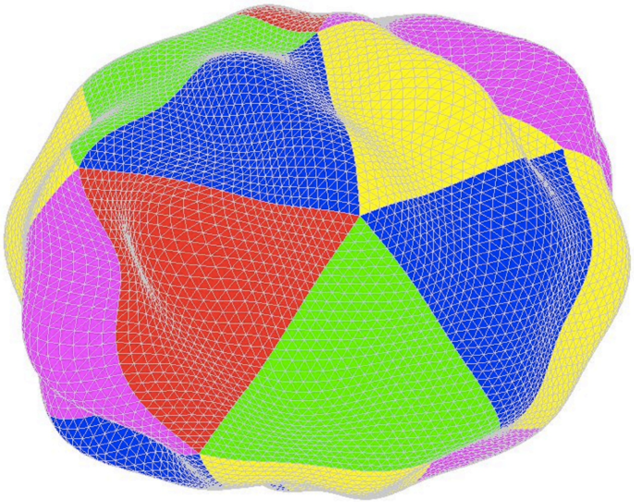
4	2562	450
5	10 242	225
6	40 962	112
7	163 842	56
8	655 362	28
9	2 621 442	14
10	10 485 762	7
11	41 943 042	3.5

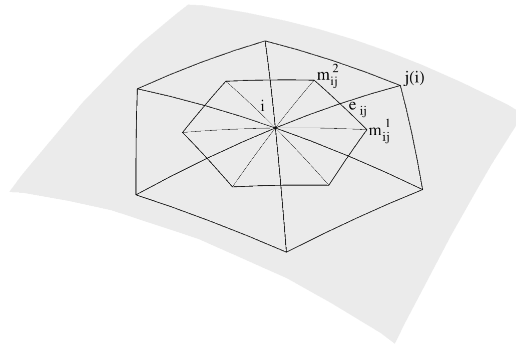




The supporting discrete manifold on the arbitrary spheroid is build upon icosahedron, the vertical structure is either tetrahedral or constructed on the basis of triangular prisms

approximation of the differential operators is based on the concepts of Discrete Exterior Calculus





Generalized Stokes theorem:

$$\int_{\Omega} \mathbf{d}\omega = \int_{\partial\Omega} \omega$$

$$\mathrm{div}\mathbf{A}|_i = \frac{1}{S_i}\sum_{j(i)}(\mathbf{A}_{ij}\cdot\mathbf{n}_{ij})\delta l_{ij}$$

$$\mathrm{curl}\mathbf{A}|_i = \frac{1}{S_i}\sum_{j(i)}(\mathbf{A}_{ij}\cdot\mathbf{t}_{ij})\delta l_{ij}$$

$$\nabla\phi|_i = \frac{1}{S_i}\sum_{j(i)}(\phi_{ij}-\phi_i)\mathbf{n}_{ij}\delta l_{ij}$$

Mimetic properties of the discretization, conservation of energy, vorticity and other invariants, link to the method of Arakawa Jacobian, theory of Nambu brackets, preservation of symmetries

The complete system for geophysical fluid could be written as a dynamical system

$$\frac{dU}{dt} = F(U)$$

$$U(t) = \{u_1(t), \ldots, u_m(t)\}^T$$

On of the first ideas was to separate the linear (stiff) and nonlinear terms

$$\frac{dU}{dt} = \mathcal{L}U + N(U(t))$$

Use the implicit scheme for the linear term and the explicit for the nonlinear part

It is much better to use the dynamic linearization

$$\frac{dU}{dt}(t) = F_n + A_n(U(t) - U_n) + R(U(t))$$

$$R(U(t)) = F(U(t)) - F(U_n) - \frac{dF}{dU}(U_n)(U(t) - U_n)$$

$$A_n = \frac{dF}{dU}(U_n)$$ Jacobian

$$U_n = U(t_n)$$ State vector

$$F_n = F(U_n)$$ Forcing term

After using of the integrating factor

$$e^{-A_n t}$$

and integrating over $[t_n, t_n + h_n]$

followed multiplication by $e^{A_n(t_n+h_n)}$

we obtain

$$U(t_n + h_n) = U_n + (e^{A_n h_n} - I)A_n^{-1} F_n +$$

$$\int_{t_n}^{t_n+h_n} e^{A_n(t_n+h_n-t)} R(U(t))dt$$

Depending on the quadrature used we will obtain different versions of the exponential integration schemes. Their common property is that the desired solution could be expressed as weighted sum of "phi functions"

$$\phi_0 = e^A \qquad \phi_1 = \frac{e^A - I}{A} \qquad \phi_2 = \frac{e^A - (I + A)}{A^2}$$

$$\phi_l(A) = A\phi_{l+1} + \frac{1}{l!}$$

"phi functions" are evaluated using **phipm** algorithm with Arnoldi iterative procedure to evaluate the vectors spanning Krylov space

Semi-discrete form of fluid equations

$$F_x = -(\mathbf{W}_{xx}\{u_x\} + \mathbf{W}_{yx}\{u_y\} + \mathbf{W}_{zx}\{u_z\}) - \mathbf{G}_x(\{u^2/2\}) -$$

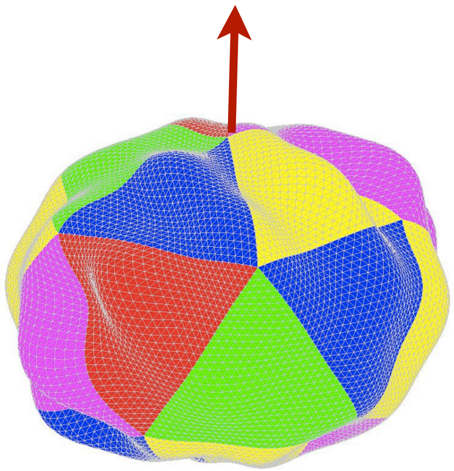
$$R * (\mathbf{IT} * \mathbf{G}_x)\{qp\} + 2(\mathbf{\Omega} \times \{u\})_x + n_x * gt_1 * \{T_p\}$$

$$F_y = -(\mathbf{W}_{xy}\{u_x\} + \mathbf{W}_{yy}\{u_y\} + \mathbf{W}_{zy}\{u_z\}) - \mathbf{G}_y(\{u^2/2\}) -$$

$$R * (\mathbf{IT} * \mathbf{G}_y)\{qp\} + 2(\mathbf{\Omega} \times \{u\})_y + n_y * gt_1 * \{T_p\}$$

$$F_z = -(\mathbf{W}_{xz}\{u_x\} + \mathbf{W}_{yz}\{u_y\} + \mathbf{W}_{zz}\{u_z\}) - \mathbf{G}_z(\{u^2/2\}) -$$

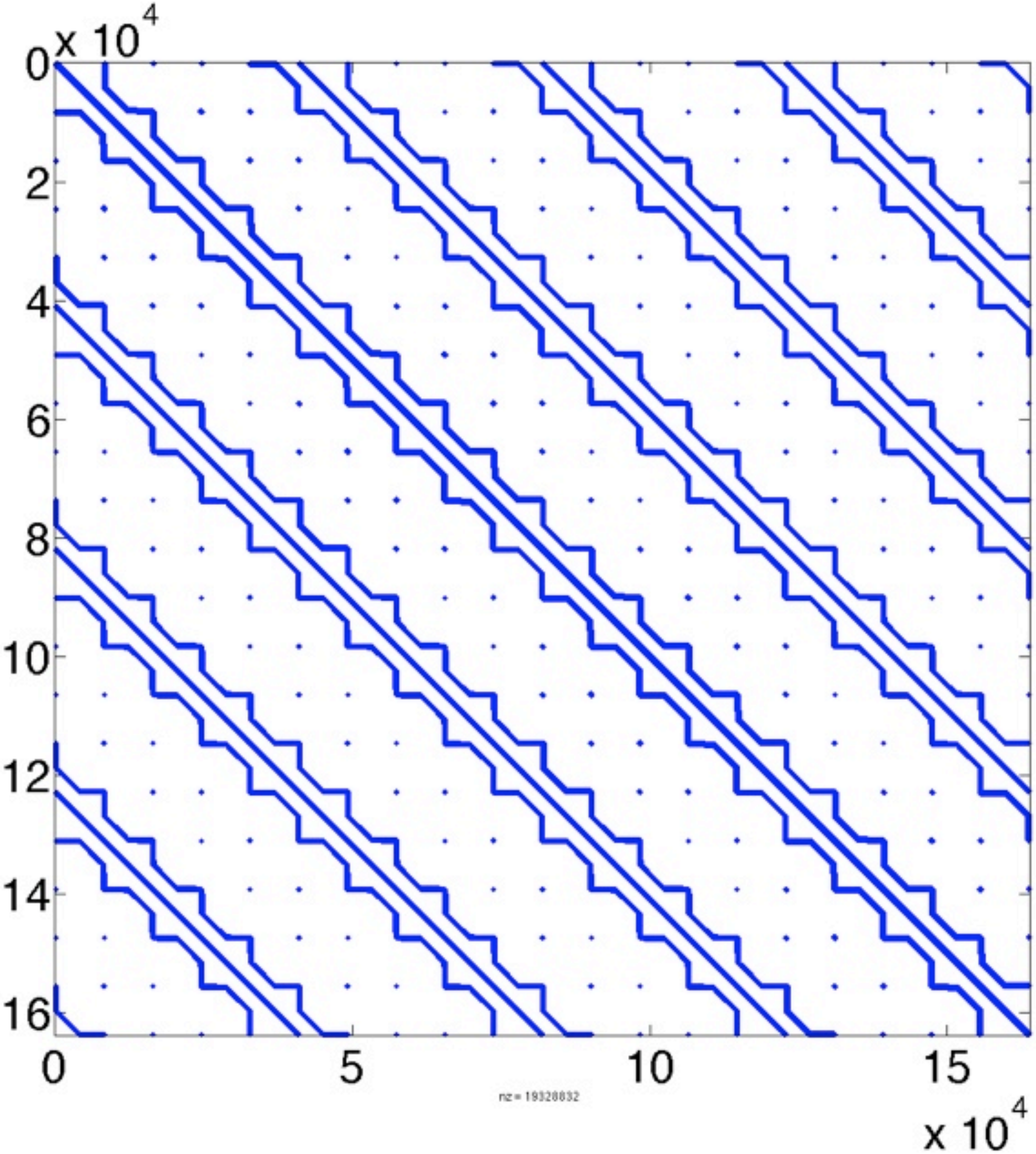
$$R * (\mathbf{IT} * \mathbf{G}_z)\{qp\} + 2(\mathbf{\Omega} \times \{u\})_z + n_z * gt_1 * \{T_p\}$$



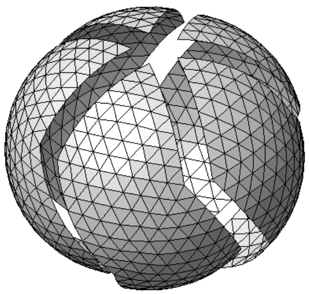
$$F_{qp} = -\mathbf{T}_q\{qp\} - k_1\{\mathrm{Div}\} + gt2\{w\}$$

$$F_{tp} = -\mathbf{T}_t\{T_p\} - k_2(\{T_p\} + T^\star). * \{\mathrm{Div}\}$$

$$gt1 = g/T^\star,\ gt2 = g/(R * T^\star),\ k_1 = 1/(1-\kappa),\ k_2 = \kappa/(1-\kappa)$$



The sparsity pattern of the Jacobi operator for the system of shallow water equations operator calculated as on icosahedral grid number 6.



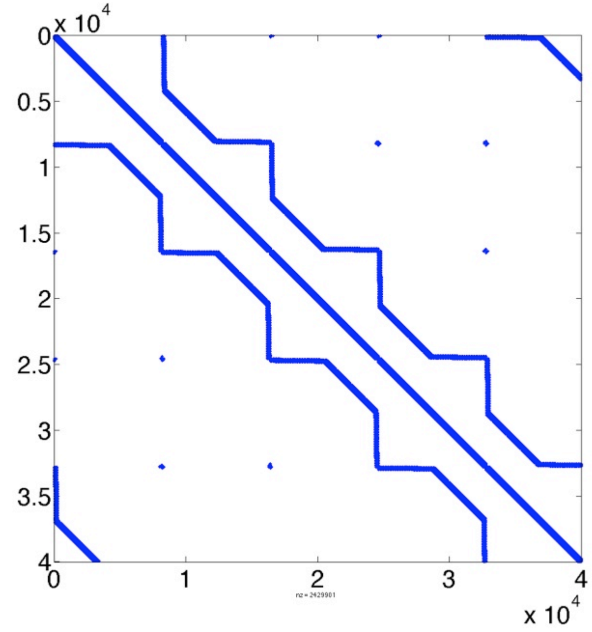
$$\frac{\partial(F_x,\, F_y,\, F_z,\, G)^T}{\partial(u_x,\, u_y,\, u_z,\, h)}$$

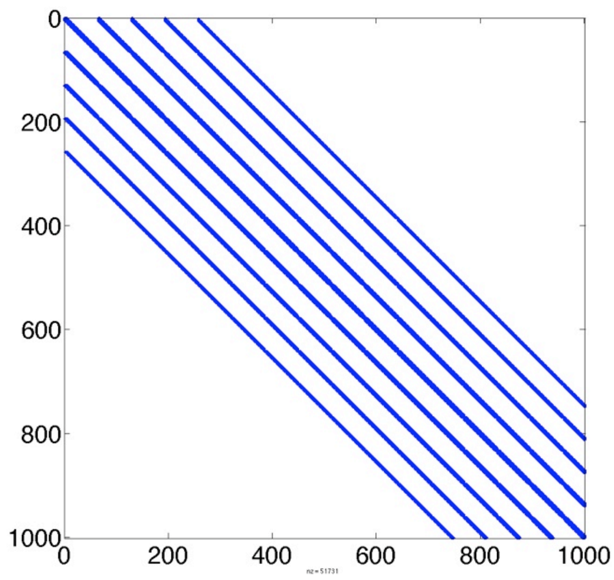
$$\begin{cases}
\frac{d}{dt}\{u^x\} = -\mathbf{W}^x - \mathbf{GS}_x\{f\} \\
\frac{d}{dt}\{u^y\} = -\mathbf{W}^y - \mathbf{GS}_y\{f\} \\
\frac{d}{dt}\{u^z\} = -\mathbf{W}^z - \mathbf{GS}_z\{f\} \\
\frac{d}{dt}\{h\} = -\mathbf{D}_x\{u^x h\} - \mathbf{D}_y\{u^y h\} - \mathbf{D}_z\{u^z h\},
\end{cases}$$

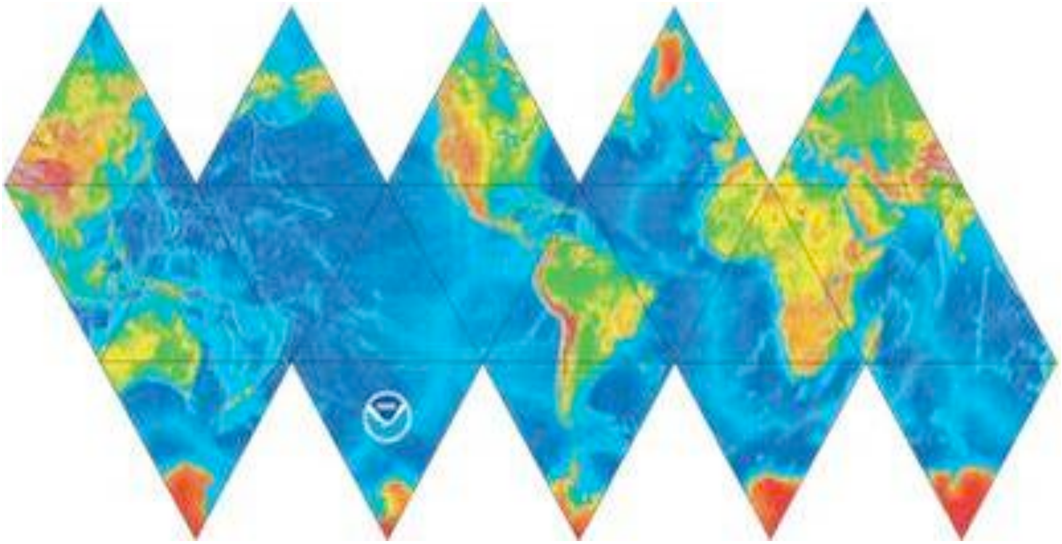
The use of Krylov subspace projection methods in the 1980's provided a method to evaluate "phi functions" for large matrices.

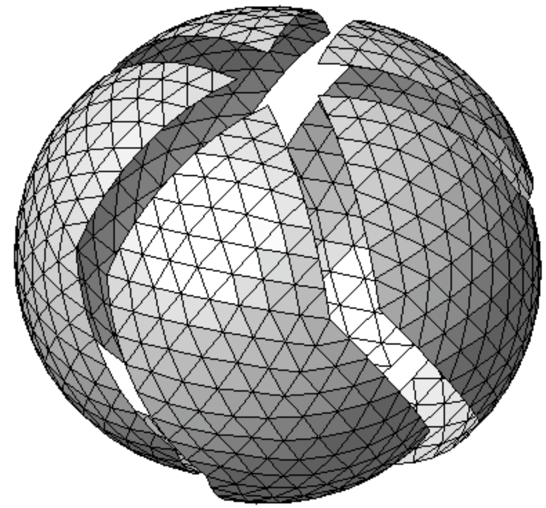
The Krylov subspace projection method is based on projecting a matrix onto a smaller dimension Krylov subspace.

The exponential function is evaluated with the smaller projected matrix as the argument, then the result is lifted back to the original space.

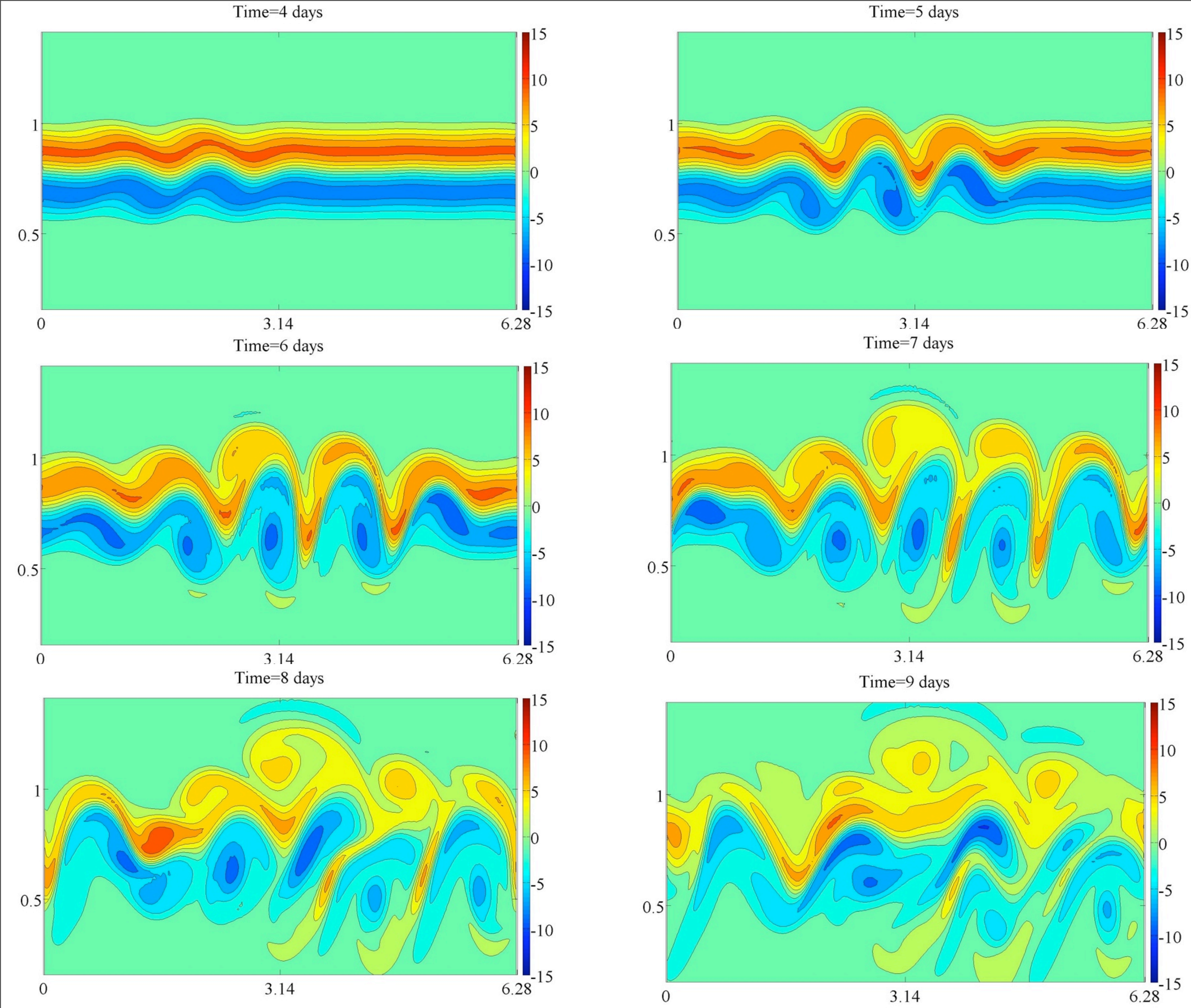


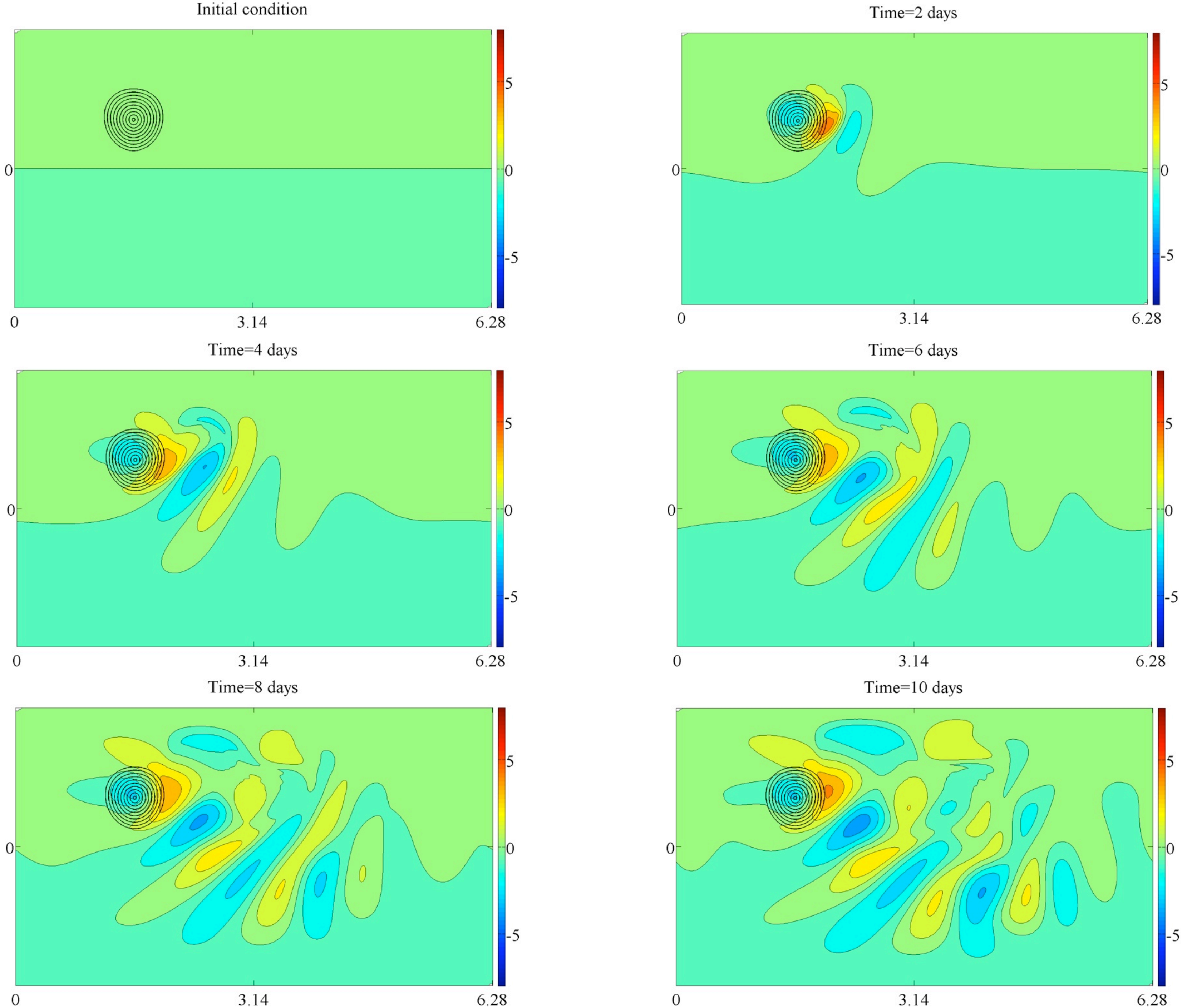




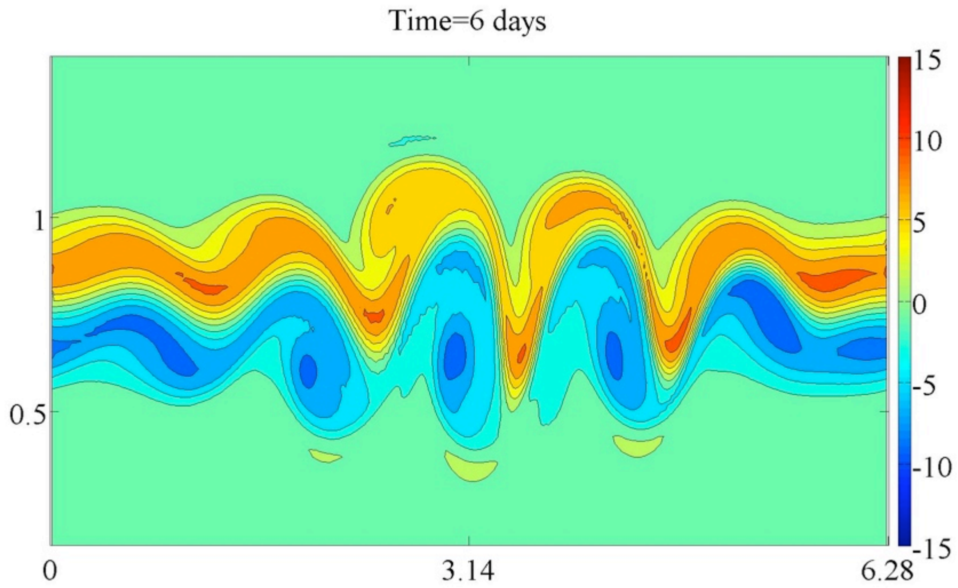


Time step = 7200 sec instead of 240 sec, grid number 6
EPI3 exponential propagation scheme
IOM2 instead of Arnoldi iteration in phipm


Time=4 days
Time=5 days
Time=6 days
Time=7 days
Time=8 days
Time=9 days
0
3.14
6.28
0.5
1
15
10
5
0
-5
-10
-15


Initial condition
Time=2 days
Time=4 days
Time=6 days
Time=8 days
Time=10 days
0
3.14
6.28
5
0
-5

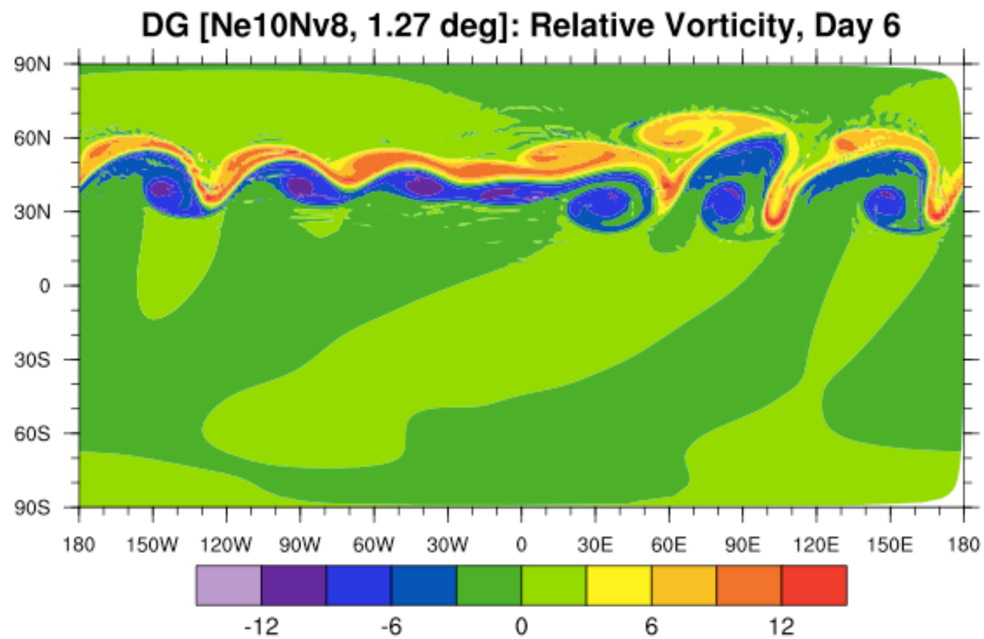
The main conclusion from the tests



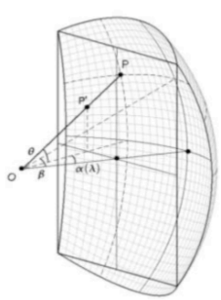
The icosahedral solution on grid 6 (40 962 points) with time step of the order of 7200 sec converges to the solution published by Galewsky et al. (2004) obtained with the spectral model T341 with time step of 30 sec.

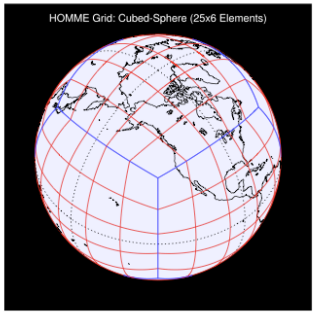
Efficiency of the method is becoming comparable to the semi-implicit method but the scheme is much more accurate.

Unstable jet on the cubed sphere (Nair 2008)

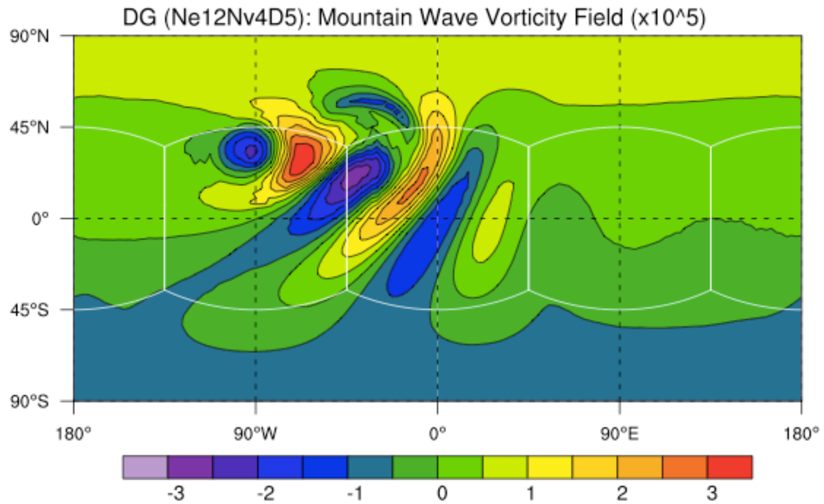


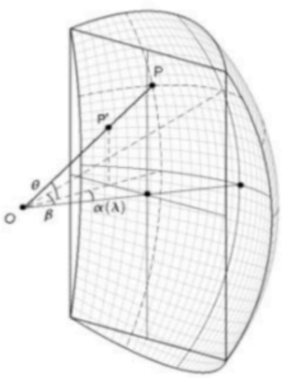
Discontinuous Galerkin method

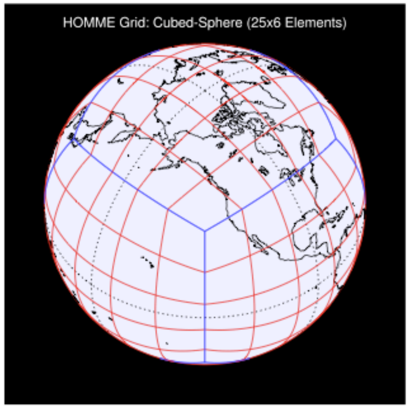




Mountain wave on the cubed sphere (Nair 2008)

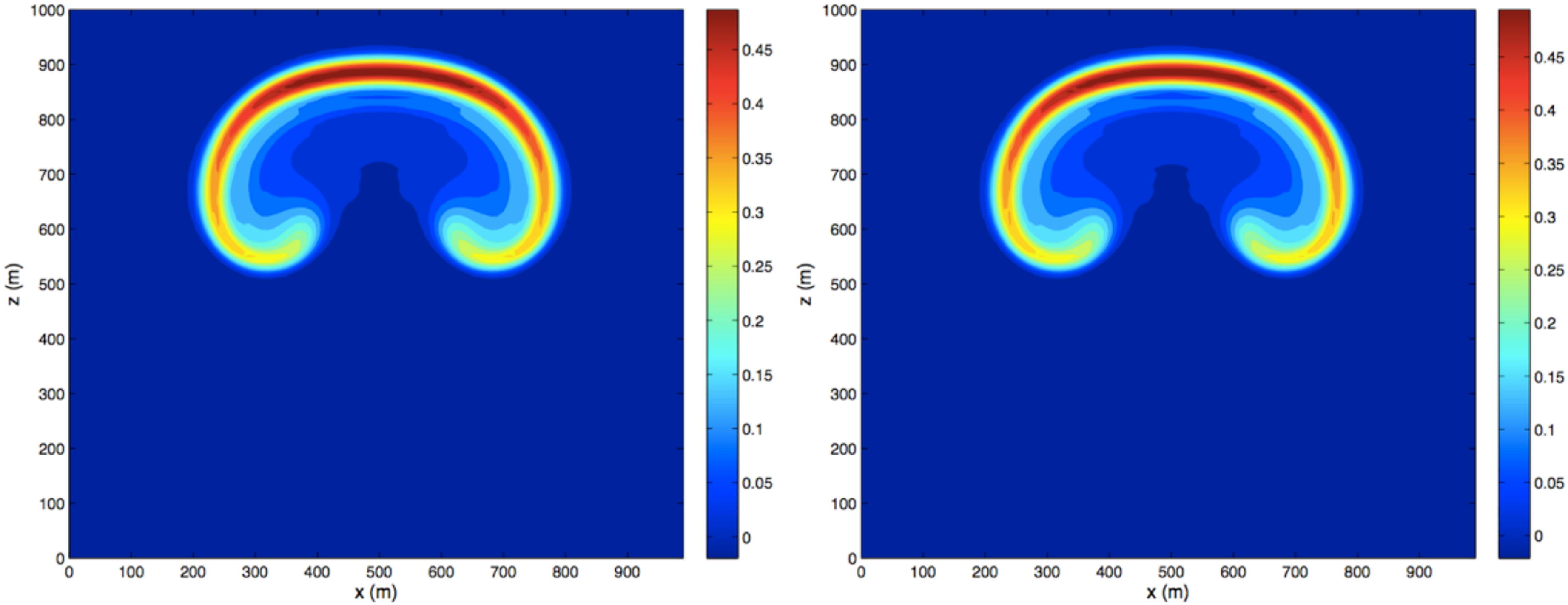






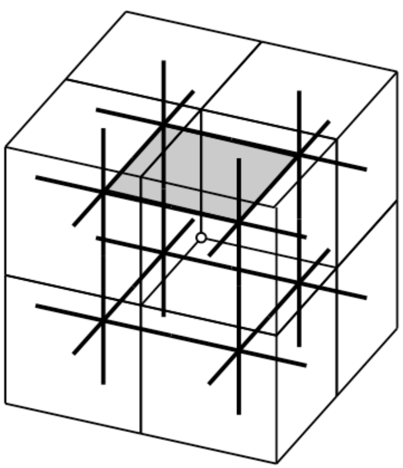
We consider vertical slab, the code is obtained by redefinition of sparse matrices describing operators

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - c_p\left(\bar{\theta}+\theta'\right)\frac{\partial \pi'}{\partial x} \\
\frac{\partial w}{\partial t} &= -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - c_p\left(\bar{\theta}+\theta'\right)\frac{\partial \pi'}{\partial z} + g\frac{\theta'}{\bar{\theta}} \\
\frac{\partial \pi'}{\partial t} &= -u\frac{\partial \pi'}{\partial x} - w\frac{\partial \pi'}{\partial z} - \frac{R}{c_v}\left(\bar{\pi}+\pi'\right)\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{gw}{c_p\bar{\bar{\theta}}} \\
\frac{\partial \theta'}{\partial t} &= -u\frac{\partial \theta'}{\partial x} - w\frac{\partial \theta'}{\partial z} - w\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}z}
\end{aligned}
$$



Left: RK4 with $\Delta t = 0.03$s. Right: EPI2 with $\Delta t = 10$s.

dx=10m, dt=10 sec, Acoustic waves Courant number=300



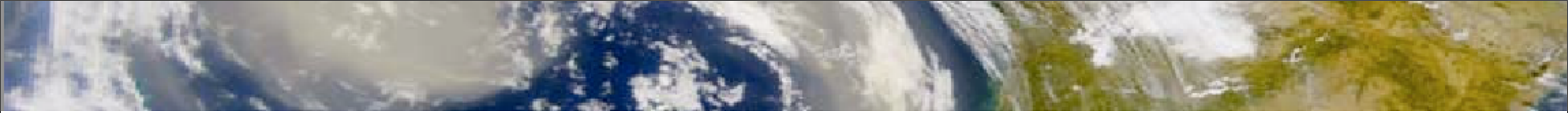
Cartesian grid is used...

Future work

1) Introduce monotonic correction to the advective part of the Jacobian

2) Investigate interaction between acoustic, gravity and Rossby waves

3) Include moist thermodynamics in the Jacobian



Thank you