Elliptic problem in GEM model and in INMI method

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Elliptic problem in GEM and INMI

- Two and three dimensional elliptic problem plays a major role in many weather and climate numerical models.
- In GEM model, Implicit time discretization gives an 3D elliptic problem for P to solve at each time-step.
- The implicit normal mode initialisation (INMI) method gives many 2D elliptic problems to solve at each INMI's iteration.

outline

- GEM Model equations
- Elliptic PDE in implicit time stepping
- Direct and iterative elliptic solvers in GEM
- Scaling Results
- Equations and elliptic problems in INMI
- Preliminary results

Model equations

$$(\mathbf{V}_h,\ w,\ T \text{ or } T',\ q,\ (\dot{\zeta},\ s),\ \phi',\ \mu)$$

$$T_* = const$$

$$\phi' = \phi - \phi_*$$

$$\ln p = \ln \pi + q$$

$$\ln \pi = \zeta + Bs$$

$$\phi_*(\zeta) = -RT_*(\zeta - \zeta_S)$$

$$\Rightarrow$$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs + q) + (1+\mu)\nabla_\zeta\phi' = 0$$

$$\frac{dw}{dt} - g\mu = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(Bs + q)\right] - \kappa\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \left(\frac{\partial}{\partial\zeta} + 1\right)\dot{\zeta} = 0$$

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

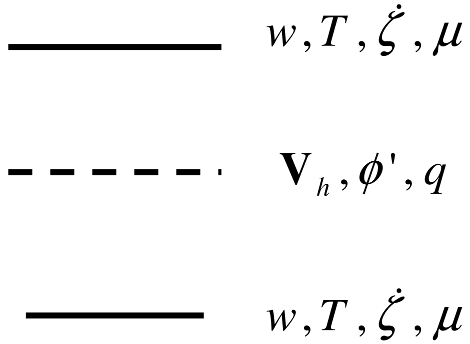
$$1 + \mu - e^q\left(1 + \frac{\partial q}{\partial(\zeta + Bs)}\right) = 0$$

$$\frac{T}{T_*} - e^q\left(1 - \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta + Bs)}\right) = 0$$

Spatial discretization

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta h}\nabla_\zeta(Bs+q) + \left(1+\overline{\mu}^{\zeta h}\right)\nabla_\zeta\phi' = 0$$

Charney Phillips Grid

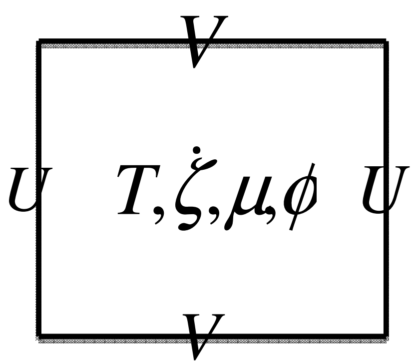


$$\frac{dw}{dt} - g\mu = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa\overline{(Bs+q)}^{\zeta}\right] - \kappa\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1+\delta_\zeta\overline{B}^{\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta} = 0$$

Arakawa C Grid



$$\frac{d\overline{\phi'}^{\zeta}}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$1+\mu - e^{\overline{q}^{\zeta}}\left[1 + \frac{\delta_\zeta q}{\delta_\zeta(\zeta+Bs)}\right] = 0$$

$$\frac{T}{T_*} - e^{\overline{q}^{\zeta}}\left[1 - \frac{\delta_\zeta(\phi'/RT_* + Bs)}{\delta_\zeta(\zeta+Bs)}\right] = 0$$

2 time level Semi-Lagrangian Implicit time discretization

$$\frac{dF_i}{dt} + G_i = 0 \quad ; \quad \text{8 equations}$$

$$\mathbf{F}_h \equiv \mathbf{V}_h \qquad \mathbf{G}_h \equiv f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta h}\nabla_\zeta(Bs+q) + \left(1+\overline{\mu}^{\zeta h}\right)\nabla_\zeta \phi'$$

$$F_w \equiv w \qquad G_w \equiv -g\mu$$

$$F_\theta \equiv \ln\left(\frac{T}{T_*}\right) - \kappa\overline{(Bs+q)}^{\zeta} \qquad G_\theta \equiv -\kappa\dot{\zeta}$$

$$F_C \equiv Bs + \ln\left(1+\delta_\zeta \overline{B}^{\zeta} s\right) \qquad G_C \equiv \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}$$

$$F_\phi \equiv \overline{\phi'}^{\zeta} \qquad G_\phi \equiv -RT_*\dot{\zeta} - gw$$

$$F_\mu \equiv 0 \qquad G_\mu \equiv 1 + \mu - e^{\overline{q}^{\zeta}}\left[1 + \frac{\delta_\zeta q}{\delta_\zeta(\zeta + Bs)}\right] = 0$$

$$F_H \equiv 0 \qquad G_H \equiv \frac{T}{T_*} - e^{\overline{q}^{\zeta}}\left[1 - \frac{\delta_\zeta(\phi'/RT_* + Bs)}{\delta_\zeta(\zeta + Bs)}\right] = 0$$

Time Discretization: time-differencing and averaging the dynamical forcings along trajectory

$$\frac{dF_i}{dt} + G_i = 0$$

$$\frac{dF_i}{dt} \approx \frac{F_i^A - F_i^D}{\Delta t}; \quad G_i \approx b^A G_i^A + \left(1 - b^A\right) G_i^D; \qquad 0.5 \le b^A \le 0.6 \quad \text{(off-centering)}$$

$$\frac{F_i^A - F_i^D}{\Delta t} + b^A G_i^A + \left(1 - b^A\right) G_i^D = 0$$

$A: \ (\mathbf{r}, t)$ Arrival

$D: \ (\mathbf{r} - \Delta\mathbf{r}, t - \Delta t)$ Departure

Trajectory= Approximate solution to:

$$\frac{d\mathbf{r}_h}{dt} = \mathbf{V}_h \qquad \frac{d^2\mathbf{r}_h}{dt^2} = -\mathbf{r}_h \frac{\mathbf{V}_h^{\,2}}{a^2}$$

$$\frac{d\zeta}{dt} = \dot{\zeta} \qquad \frac{d^2\zeta}{dt^2} = 0$$

Next Monique Tanguay seminar

Mid-point approximation : Jean Côté

Trapezoidal approximation: Claude Girard

GEM- Temporal Iterative Solver

$$\frac{F_i^A}{\tau} + G_i^A = \frac{F_i^D}{\tau} - \beta G_i^D \equiv \mathrm{R}_i\,;\; i = 1, Neq$$

Linearisation

$$L_i \equiv \left(\frac{F_i^A}{\tau} + G_i^A \right)_{lin} ; \qquad N_i \equiv \frac{F_i^A}{\tau} + G_i^A - \left(\frac{F_i^A}{\tau} + G_i^A \right)$$

Do jter=1,2   (Crank Nicholson)
Do iter=1,2 (Non-linear)

$$\left(L_i\right)^{iter\,,\,jter} = \left(R_i\right)^{jter} - \left(N_i\right)^{iter-1,\,jter}; \qquad \left(N_i\right)^{0,1} = N_i\left(\mathbf{r}, t - \Delta t\right)$$

$$i = 1, \ldots, Neq$$

end do
end do

Linear terms

$$\mathbf{L}_h = \frac{\mathbf{V}_h}{\tau} + \nabla_\zeta [\phi' + RT_*(Bs + q)]$$

$$L_w = \frac{w}{\tau} - g\mu$$

$$L_\theta = \frac{T'}{\tau T_*} - \kappa \left(\dot{\zeta} + \frac{\overline{(Bs+q)}^\zeta}{\tau} \right)$$

$$L_C = \frac{1}{\tau}\left[\overline{B}^{\zeta\zeta} s + \delta_\zeta \overline{B}^\zeta s\right] + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^\zeta$$

$$L_\phi = \frac{\overline{\phi'}^\zeta}{\tau} - RT_* \dot{\zeta} - gw$$

$$L_\mu = \mu - \left(\delta_\zeta q + \overline{q}^\zeta\right) \neq 0$$

$$L_H = \frac{T'}{T_*} - \overline{q}^\zeta + \frac{\delta_\zeta (\phi' + RT_* Bs)}{RT_*} \neq 0$$

Non-linear terms

$$\mathbf{N}_h = f\mathbf{k}\times\mathbf{V}_h + R\overline{T'}^{\zeta}\nabla_{\zeta}(Bs+q) + \overline{\mu}^{\zeta}\nabla_{\zeta}\phi'$$

$$N_w = 0$$

$$N_\theta = \frac{1}{\tau}\left[\ln\left(\frac{T}{T_*}\right) - \frac{T'}{T_*}\right]$$

$$N_C = \frac{1}{\tau}\left[Bs + \ln\left(1+\delta_\zeta \overline{B}^{\zeta} s\right) - \overline{B}^{\zeta\zeta} s - \delta_\zeta \overline{B}^{\zeta} s\right]$$

$$N_\phi = 0$$

$$N_\mu = -\left(\mu - \delta_\zeta q - \overline{q}^{\zeta}\right) = -L_\mu$$

$$N_H = -\left(\frac{T'}{T_*} - \overline{q}^{\zeta} + \frac{\delta_\zeta\left(\phi' + RT_* Bs\right)}{RT_*}\right) = -L_H$$

Elliptic problem for P by elimination

$$P \equiv \phi' + RT_*\left(Bs + q\right) \qquad X = \dot{\zeta} + \frac{\overline{\left(Bs + q\right)}^{\zeta}}{\tau}$$

$$\mathbf{L}_h = \frac{\mathbf{V}_h}{\tau} + \nabla_\zeta P$$

$$L'_w = \frac{w}{\tau} - g\left(\delta_\zeta q + \overline{q}^{\zeta}\right)$$

$$L'_\theta = \frac{1}{\tau}\left(\delta_\zeta q + \overline{q}^{\zeta}\right) - \frac{\delta_\zeta P}{\tau R T_*} - \kappa X$$

$$L_C = -\frac{1}{\tau}\overline{\left(\delta_\zeta q + \overline{q}^{\zeta}\right)}^{\zeta} + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta X + \overline{X}^{\zeta}$$

$$L_\phi = \frac{\overline{P}^{\zeta}}{\tau} - RT_* X - gw$$

Elliptic problem by elimination

$$\nabla_\zeta \cdot \mathbf{L}_h - \frac{1}{\tau}\left(L_C - \frac{\varepsilon\tau}{H_*}\overline{L'_w}^{\zeta}\right) \equiv L''_C = \nabla_\zeta^2 P - \frac{1}{\tau}\left(\delta_\zeta X + \overline{X}^{\zeta}\right) + \varepsilon \frac{\overline{w}^{\zeta}}{\tau w_*}$$

$$\frac{\gamma}{\kappa\tau}\left(L'_\theta + \frac{\varepsilon\tau}{H_*}L'_w + \frac{\varepsilon}{RT_*}L_\phi\right) \equiv L''_\theta = -\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon\overline{P}^{\zeta}\right) - \frac{X}{\tau}$$

$$\frac{\gamma}{\kappa\tau}\left(L'_\theta + \frac{\varepsilon\tau}{H_*}L'_w - \frac{\kappa}{RT_*}L_\phi\right) \equiv L''_\phi = -\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa\overline{P}^{\zeta}\right) + \frac{w}{\tau H_*}$$

$$L''_C - \left(\delta_\zeta L''_\theta + \overline{L''_\theta}^{\zeta}\right) - \varepsilon\overline{L''_\phi}^{\zeta} \equiv L_P = \nabla_\zeta^2 P + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta^2 P + \overline{\delta_\zeta P}^{\zeta} - \varepsilon(1-\kappa)\overline{P}^{\zeta\zeta}\right)$$

Elliptic problem

$$\nabla_\zeta^2 P + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta^2 P + \overline{\delta_\zeta P}^{\zeta} - \varepsilon(1-\kappa)\overline{P}^{\zeta\zeta}\right) = R$$

Vertical boundary conditions:

$$\left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon\overline{P}^{\zeta}\right)\right]_T = -\left(L''_\theta\right)_T \qquad \left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa\overline{P}^{\zeta}\right)\right]_S = -\left(L''_\theta\right)_S + \frac{\phi_S}{\tau^2 RT_*}$$

Horizontal boundary conditions:

LAM : Newman
Yin-Yang: Dirichlet
Lat_Lon: periodic-Newman

Back substitution

$$\mathbf{V}_h: \qquad \frac{\mathbf{V}_h}{\tau} = \left[\mathbf{R}_h - \mathbf{N}_h - \nabla_\zeta P\right]$$

$$w: \qquad \frac{w}{\tau H_*} = \left[R''_\phi - N''_\phi + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa \overline{P}^\zeta\right)\right]$$

$$q: \quad \delta_\zeta q + \overline{q}^\zeta = -\frac{\varepsilon\tau^2}{H_*}\left[R_w - N'_w - \frac{w}{\tau}\right]; \qquad q_T = 0$$

$$s: \qquad s = \frac{P_S - \phi_S}{RT_*} - q_S$$

$$\dot{\zeta}: \qquad \frac{\dot{\zeta}}{\tau} = -\left[R''_\theta - N''_\theta + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon \overline{P}^\zeta\right)\right] - \frac{\overline{(Bs+q)}^\zeta}{\tau^2}; \quad \dot{\zeta}_T = \dot{\zeta}_S = 0$$

$$\phi': \qquad \phi' = P - RT_*(q + Bs)$$

Summary

- Elliptic problem for the variable P, is the kernel of the time iterative solver. It is called 4 times per time step.
- Once P computed all the other variables are calculated by back-substitution
- References

- Girard et al. 2013 MWR
- Côté and Staniforth 1988 MWR
- Tanguay , Robert and Laprise 1990 MWR
- Gravel and Staniforth 1993 MWR

3D Direct Elliptic Solver

$$\mathbf{P}\nabla_{\zeta}^{2}P+\frac{\gamma}{\kappa\tau^{2}RT_{*}}\left(\mathbf{P}_{\delta\delta}+\mathbf{P}_{\delta\mu}-\varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)P=\mathrm{R}$$

$$\text{with: } \kappa=R/c_{p},\varepsilon=RT_{*}/(g\tau)^{2},\gamma=\kappa/(\kappa+\varepsilon)$$

Vertical Separation: $\left(\mathbf{P}_{\delta\delta}+\mathbf{P}_{\delta\mu}-\varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)\mathrm{Z}=\Lambda\mathbf{P}\mathrm{Z}$

$$Y^{T}\left(\mathbf{P}_{\delta\delta}+\mathbf{P}_{\delta\mu}-\varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)=\Lambda Y^{T}\mathbf{P}$$

$$\Lambda=diag\,(\eta_{1,}\eta_{2,\ldots,}\eta_{\mathrm{Nk}}),\ \ [\mathrm{Z}_{1},\mathrm{Z}_{2},\ldots.\mathrm{Z}_{\mathrm{NK}}]$$

$$[\mathrm{Y}_{1},\mathrm{Y}_{2},\ldots.\mathrm{Y}_{\mathrm{NK}}],\qquad \mathrm{Y}_{\mathrm{k}}^{\mathrm{T}}\mathbf{P}\mathrm{Z}_{\mathrm{k}}=\mathrm{I}$$

NK 2D Helmholtz problems to solve:

$$\nabla_{\zeta}^{2}\tilde{P}_{k}+\frac{\gamma}{\kappa\tau^{2}RT_{*}}\eta\,\tilde{P}_{k}=\tilde{R}_{k}\ ,\quad k=1,NK$$

Vertical Nk x NK operators

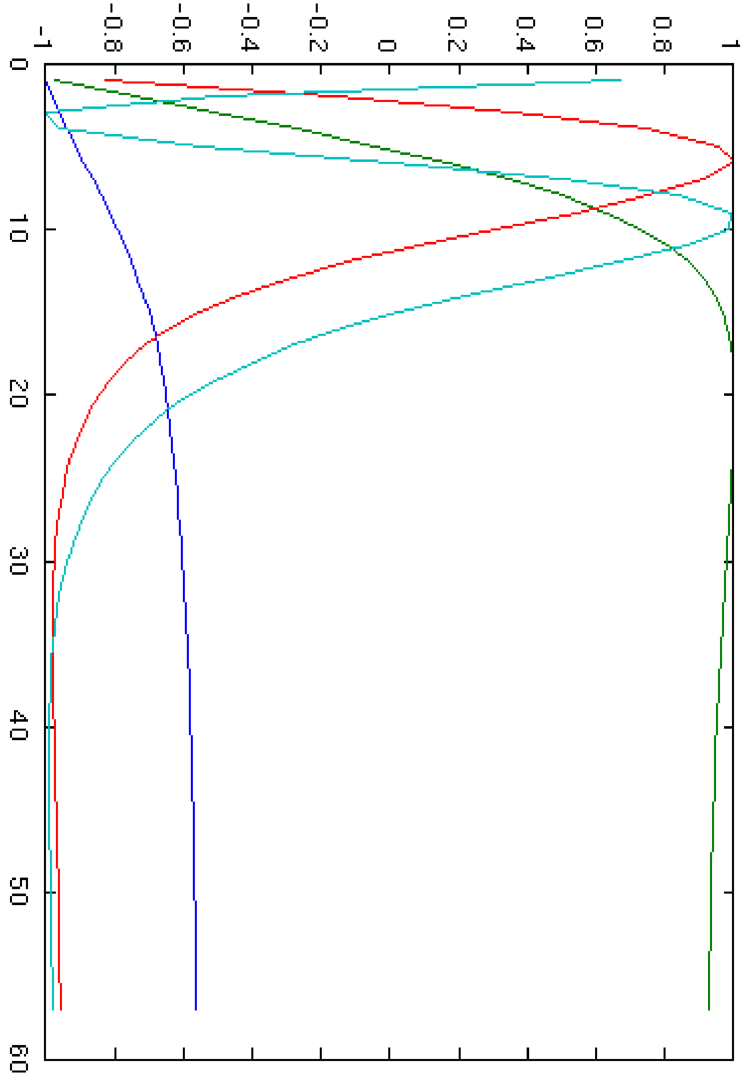
$$\mathbf{P}=\begin{vmatrix}\Delta\zeta_{k_0} & 0 & 0\\ 0 & \Delta\zeta_k & 0\\ 0 & 0 & \Delta\zeta_N\end{vmatrix}$$

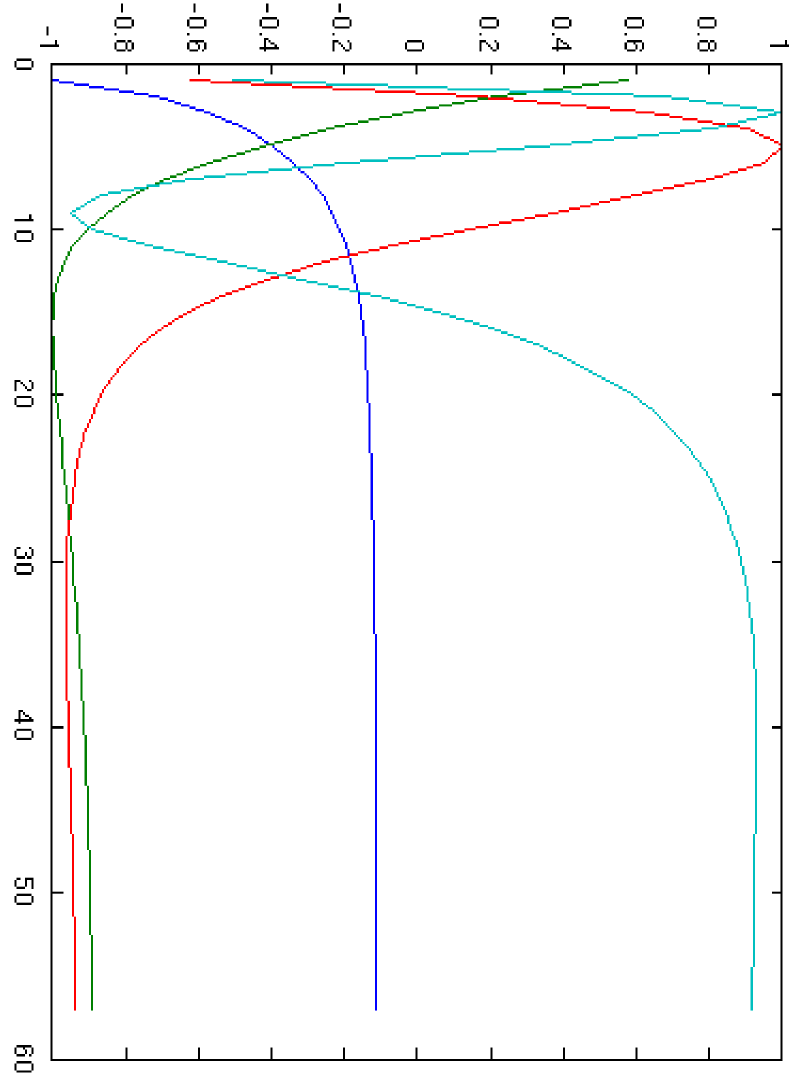
$$\mathbf{P}_{\delta\delta}=\begin{vmatrix}-\left(\frac{1-\alpha_T}{\Delta\zeta_{k_0-\frac{1}{2}}}+\frac{1}{\Delta\zeta_{k_0+\frac{1}{2}}}\right) & \frac{1}{\Delta\zeta_{k_0+\frac{1}{2}}} & 0\\ \frac{1}{\Delta\zeta_{k-\frac{1}{2}}} & -\left(\frac{1}{\Delta\zeta_{k-\frac{1}{2}}}+\frac{1}{\Delta\zeta_{k+\frac{1}{2}}}\right) & \frac{1}{\Delta\zeta_{k+\frac{1}{2}}}\\ 0 & \frac{1}{\Delta\zeta_{N-\frac{1}{2}}} & -\left(\frac{1}{\Delta\zeta_{N-\frac{1}{2}}}+\frac{1-\alpha_S}{\Delta\zeta_{N+\frac{1}{2}}}\right)\end{vmatrix}$$

$$\mathbf{P}_{\delta\mu}=\begin{vmatrix}\left(\frac{(1-\alpha_T)\varpi^-_{k_0}}{\Delta\zeta_{k_0-\frac{1}{2}}}-\frac{\varpi^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}\right)\Delta\zeta_{k_0} & \frac{\varpi^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}\Delta\zeta_{k_0} & 0\\ -\frac{\varpi^-_k}{\Delta\zeta_{k-\frac{1}{2}}}\Delta\zeta_k & \left(\frac{\varpi^-_k}{\Delta\zeta_{k-\frac{1}{2}}}-\frac{\varpi^+_k}{\Delta\zeta_{k+\frac{1}{2}}}\right)\Delta\zeta_k & \frac{\varpi^+_k}{\Delta\zeta_{k+\frac{1}{2}}}\Delta\zeta_k\\ 0 & -\frac{\varpi^-_N}{\Delta\zeta_{N-\frac{1}{2}}}\Delta\zeta_N & \left(\frac{\varpi^-_N}{\Delta\zeta_{N-\frac{1}{2}}}-\frac{(1-\alpha_S)\varpi^+_N}{\Delta\zeta_{N+\frac{1}{2}}}\right)\Delta\zeta_N\end{vmatrix}$$

$$\mathbf{P}_{\mu\mu}=\begin{vmatrix}\left[\varpi^-_{k_0}\left(\alpha_T\varpi^-_{k_0-\frac{1}{2}}+\varpi^+_{k_0-\frac{1}{2}}\right)+\varpi^+_{k_0}\varpi^-_{k_0+\frac{1}{2}}\right]\Delta\zeta_{k_0} & \varpi^+_{k_0}\varpi^+_{k_0+\frac{1}{2}}\Delta\zeta_{k_0} & 0\\ \varpi^-_k\varpi^-_{k-\frac{1}{2}}\Delta\zeta_k & \left(\varpi^-_k\varpi^+_{k-\frac{1}{2}}+\varpi^+_k\varpi^-_{k+\frac{1}{2}}\right)\Delta\zeta_k & \varpi^+_k\varpi^+_{k+\frac{1}{2}}\Delta\zeta_k\\ 0 & \varpi^-_N\varpi^-_{N-\frac{1}{2}}\Delta\zeta_N & \left[\varpi^-_N\varpi^+_{N-\frac{1}{2}}+\varpi^+_N\left(\varpi^-_{N+\frac{1}{2}}+\varpi^+_{N+\frac{1}{2}}\alpha_S\right)\right]\Delta\zeta_N\end{vmatrix}$$

The first four vertical modes





250METRES HORIZ. RESOLUTION WITH 57 VERTICAL LEVELS AND TIME STEP =30SEC LEFT: HYDRO RIGHT: NON_HYDRO

Vertical separation: Non-symetric matrix ; imaginary eigenvalue

$$\left(\mathbf{P}_{\delta\delta} + \mathbf{P}_{\delta\mu} - \varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)\mathrm{Z} = \Lambda\mathbf{P}\mathrm{Z}$$

$$\Lambda = diag\,(\eta_{1}, \eta_{2,...,}\eta_{\mathrm{Nk}}), [\mathrm{Z}_1, \mathrm{Z}_2, ...\mathrm{Z}_{\mathrm{NK}}]$$

$$\mathrm{Z}_{\mathrm{k}}^{\mathrm{T}}\mathbf{P}\mathrm{Z}_{\mathrm{k}} = \mathrm{I}$$

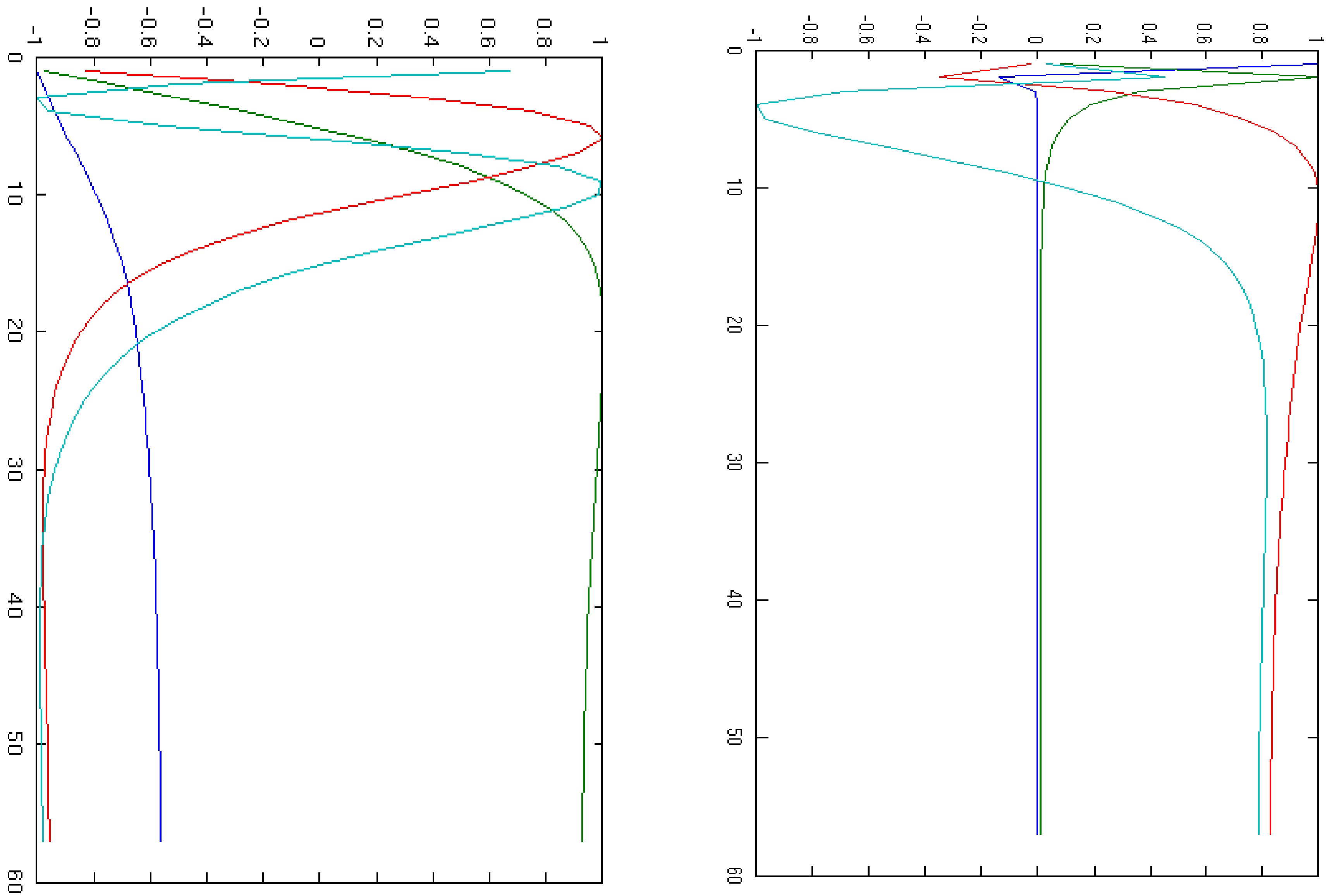
Constraint: Similar to tri-diagonal symmetric matrix

$$\mathrm{S} = \mathrm{D}^{-1}\left(\mathbf{P}_{\delta\delta} + \mathbf{P}_{\delta\mu} - \varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)\mathrm{D}$$

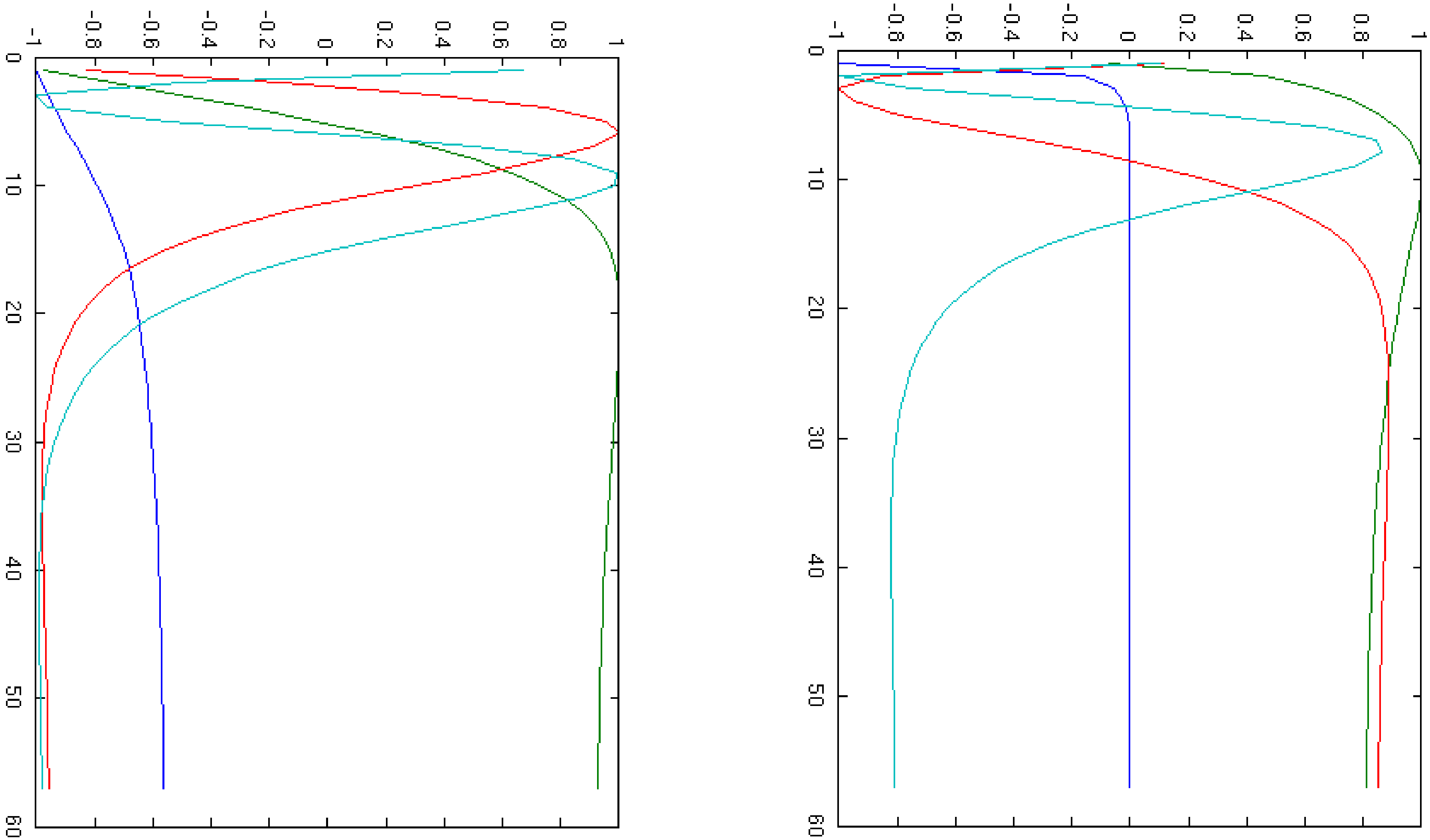
$$\mathrm{D} = \mathrm{diag}(\mathrm{d}_1, \mathrm{d}_2, ...\mathrm{d}_{\mathrm{Nk}})\,; \left(\mathbf{P}_{\delta\delta} + \mathbf{P}_{\delta\mu} - \varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right) = tridiag\,(a, b, c, {}_{NK})$$

$$\mathrm{d}_1 = 1,\ \mathrm{d}_{\mathrm{k}} = \sqrt{\frac{a_k}{c_{k-1}}}\mathrm{d}_{\mathrm{k\text{-}1}}\ , k = 2, {}_{NK} \ \Rightarrow \frac{a_k}{c_{k-1}} \rangle 0$$

Subroutine PreverIn: Vertical discretization is not compatible with time step. The solver will not work



250M 57 LEVEL TIME STEP 10SEC LEFT: HYDRO RIGHT: NON_HYDRO



250M 57 LEVEL TIME STEP 12SEC LEFT: HYDRO RIGHT: NON_HYDRO

Direct elliptic Solver

- Fourier Analysis of the right-hand sides
- Solve Nk 2D Helmholtz problems of dimension NI*NK:

$$\nabla^2_{\zeta}\widetilde{P}_k + \eta_k \widetilde{P}_k = \widetilde{R}_k \;, \eta \langle 0, \; k = 1, NK$$

a- we solve 2D Helmholtz problem by a direct method

b- we solve 2D Helmholtz problem by iterative GMRES

- Fourier synthesis of the solutions

Helmholtz direct solver

$$\nabla_{\zeta}^{2}\tilde{P}_{k}+\eta_{k}\,\tilde{P}_{k}=\tilde{R}_{k}\quad,\eta_{k}\langle 0,\ k=1,NK$$

Separation along each latitude

$$\left(\mathbf{P}_{\lambda\lambda}\otimes\mathbf{P}_{\theta}^{\prime}+\mathbf{P}_{\lambda}\otimes\mathbf{P}_{\theta\theta}+\eta\,\mathbf{P}_{\lambda}\otimes\mathbf{P}_{\theta}\right)\tilde{P}_{k}=(\mathbf{P}_{\lambda}\otimes\mathbf{P}_{\theta})\tilde{R}_{k}$$

$$\mathbf{P}_{\lambda\lambda}\mathrm{X}=\Lambda\mathbf{P}_{\lambda}\mathrm{X}\ ,\ \mathrm{X}_{\mathrm{i}}^{\mathrm{t}}\mathbf{P}_{\lambda}\mathrm{X}_{\mathrm{k}}=\mathrm{I}$$

$$\Lambda=diag\,(\Lambda_{1},\Lambda_{2,\ldots,}\Lambda_{\mathrm{Ni}}),\ [\mathrm{X}_{1},\mathrm{X}_{2},\ldots.\mathrm{X}_{\mathrm{NK}}]$$

Ni xNk Tridiagonal problems to solve (Gauss elimination without pivoting)

$$\left(\mathbf{P}_{\theta\theta}+\Lambda_{i}\mathbf{P}_{\theta}^{\prime}+\eta_{k}\,\mathbf{P}_{\theta}\right)\tilde{P}_{k,i}=\mathbf{P}_{\theta}\tilde{R}_{k,i}\ ,\quad k=1,NK\ ,\ i=1,NI$$

Horizontal operators

$$\tilde{\lambda}_i = (\lambda_i + \lambda_{i+1})/2, \quad \tilde{\theta}_j = (\theta_j + \theta_{j+1})/2$$

$$\mathbf{P}_{\lambda\lambda} = \begin{vmatrix} -\frac{1}{\Delta\lambda_1} & \frac{1}{\Delta\lambda_1} & 0 \\ \frac{1}{\Delta\lambda_1} & \left(-\frac{1}{\Delta\lambda_1} - \frac{1}{\Delta\lambda_2}\right) & \frac{1}{\Delta\lambda_2} \\ 0 & \frac{1}{\Delta\lambda_{Ni-1}} & -\frac{1}{\Delta\lambda_{Ni-1}} \end{vmatrix}$$

$$\mathbf{P}_{\lambda} = \begin{vmatrix} \Delta\tilde{\lambda}_1 & 0 & 0 \\ 0 & \Delta\tilde{\lambda}_2 & 0 \\ 0 & 0 & \Delta\tilde{\lambda}_{Ni} \end{vmatrix}$$

$$\mathbf{P}_{\theta} = \begin{vmatrix} \Delta\sin(\tilde{\theta}_0) & 0 & 0 \\ 0 & \Delta\sin(\tilde{\theta}_1) & 0 \\ 0 & 0 & \Delta\sin(\tilde{\theta}_{Nj\text{-}1}) \end{vmatrix}$$

$$\mathbf{P}_{\theta} = \begin{vmatrix} \Delta\sin(\tilde{\theta}_0)/\cos^2(\theta_1) & 0 & 0 \\ 0 & \Delta\sin(\tilde{\theta}_1)/\cos^2(\theta_2) & 0 \\ 0 & 0 & \Delta\sin(\tilde{\theta}_{Nj\text{-}1})/\cos^2(\theta_{Nj}) \end{vmatrix}$$

$$\mathbf{P}_{\theta} = \begin{vmatrix} -\cos^2(\tilde{\theta}_1)/\Delta\sin(\theta_1) & \cos^2(\tilde{\theta}_1)/\Delta\sin(\theta_1) & 0 \\ \cos^2(\tilde{\theta}_1)/\Delta\sin(\theta_1) & -\cos^2(\tilde{\theta}_1)/\Delta\sin(\theta_1) - \cos^2(\tilde{\theta}_2)/\Delta\sin(\theta_2) & \cos^2(\tilde{\theta}_2)/\Delta\sin(\theta_2) \\ 0 & \cos^2(\tilde{\theta}_{Nj\text{-}1})/\Delta\sin(\theta_{Nj-1}) & -\cos^2(\tilde{\theta}_{Nj\text{-}1})/\Delta\sin(\theta_{Nj-1}) \end{vmatrix}$$

Direct Helmholtz solver

- Transp1:(Ni/Npex,Nj/Npey,Nk) → (Nj/Npey,Nk/Npex,Ni)
- Fourier Analysis of the right-hand sides
- Transp2:(Nj/Npey,Nk/Npex,Ni) → (Nk/Npex,Ni/Npey,Nj)
- Solve Nk x NI Tridiagonal problems (NJ)

$$\left(\mathbf{P}_{\theta\theta} + \Lambda_i \mathbf{P}_\theta' + \eta_k \mathbf{P}_\theta\right)\tilde{P}_{k,i} = \mathbf{P}_\theta \tilde{R}_{k,i}\,, \quad k = 1, NK\,,\ i = 1, NI$$

- Transp3:(Nk/Npex,Ni/Npey,Nj) → (Nj/Npey,Nk/Npex,Ni)
- Fourier synthesis of the solutions
- Transp4:(Nj/Npey,Nk/Npex,Ni) → (Ni/Npex,Nj/Npey,Nk)

Slow/Fast Direct Helmholtz Solver

- Fourier Analysis of the right-hand sides; Slow: Matrix-Matrix multiply: cost per grid point increases linearly with NI; Fast : FFT: cost per grid point increases logarithmically with NI
- Solve Nk x NI Tridiagonal problems of dimension NJ
- Fourier synthesis of the solutions; Slow: Matrix-Matrix multiply ; Fast: FFT

Longitudinal boundary conditions and Fourier series functions:

LAM	: Newman	Shifted cosines functions
Yin-Yang:	Dirichlet	sines functions
Lat_Lon:	periodic	cosines and sines functions

Iterative 2D Helmholtz solver Krylov method

- Iterative and projection method for our problem of dimension n
- Seeks solution on the Krylov subspace K of dimension m<<n, by constraining the residual to be orthogonal to a subspace L of dimension m.
- .

$$K = [r, Ar, ..., A^{m-1} r]$$

$$Ax = b, \quad r = b - Ax$$

Iterative Projection method

$1. Until\ convergence$

$2. Select\ a\ pair\ of\ subspaces\ K\ and\ L$

$3. Choose\ bases\ V=[v_1,\ldots,v_m]\ and\ W=[w_1,\ldots,w_m]$

$4. r=b-Ax$

$5. y=(W^T AV)^{-1}W^T r$

$6. x:=x+Vy$

$7. enddo$

$W^T AV$: nonsingular

1. A positive definie and $L=K \rightarrow x$ minimizes $A-$ norm error

2. A is nonsingular and $L=AK \rightarrow x$ min $l2$ norm of the residual

Preconditioning- 2 Basic principles

$$\mathrm{Ax} = \mathrm{b}; \quad \mathrm{M}^{-1}\mathrm{A}\,\mathrm{x} = \mathrm{M}^{-1}\mathrm{b}$$

1-The matrix M approximate well A; $\mathrm{M}^{-1}\mathrm{A}$ is better conditioned than A

2- Inversion of matrix M is Cheap,

- In GEM model, the matrix M is the bloc-diagonals of A; local solve is done by the direct slow solver: no MPI communications

$$M^{-1} = \begin{vmatrix} A_1^{-1} & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & A_{Ndom}^{-1} \end{vmatrix}$$

Preconditioned GMRES Method

$$Ax = b; \quad M^{-1}A\,x = M^{-1}b$$

1 . *Compute* $r_o = M^{-1}(b - Ax_o)$, $\beta = \|r_o\|$ *and* $v_1 = \frac{r_o}{\beta}$ ← MPI_allreduce

2 . *For* $j = 1, \ldots, m$ *Do*

3 . *Compute* $w := M^{-1} A\, v_j$

4 . *for* $i = 1, \ldots, j$

5 . $h_{i,j} := (w, v_i)$

6 . $w = w - h_{i,j} v_i$

7 . *enddo*

8 . *Compute* $h_{j+1,j} = \|w\|_2$ *and* $v_{j+1} = \frac{w}{h_{j+1,j}}$ ← MPI_allreduce, Dot product operation

9 . *enddo*

10 . *Define* $v_m = [v_1, \ldots, v_m]$; $\overline{H}_m = \{h_{i,j}\}_{1 \le i \le j+1, 1 \le j \le m}$

11 . *Compute* $y_m = \arg\min_y \|\beta e_1 - \overline{H}_m y\|_2$ *and* $x_m = x_o + v_m y_m$

12 . *if satisfied stop, else set* $x_o = x_m$ *and GoTo* 1

MPI_Exch_halo
Matrix-vector operation (→ steps 2 and 3)

Test Problems: 2.5 Km GEM_LAM with 93 levels

Nodes	PxQ NixNj	# of MPI PEs	FFT seconds	MXMA Sec	Iterative seconds	# itera
1	4x4 400x400	16	109	126	812	11
4	8x8 800x800	64	111	196	794	11
16	16x16 1600x1600	256	115	333	810	11
64	32x32 3200x3200	1024	124	633	936	12
138	47x47 4800x4800	2209	170	956	1000	12

Weak scalibility MPI test run(number of solver calls is 800)
Comparison of timings between using the FFT, MXMA, and iterative

Transp1:(Ni/Npex,Nj/Npey,Nk) → (Nj/Npey,Nk/Npex,Ni)

Why 3D-iterative elliptic solver

- Vertical separation is not possible in some cases.
- To Solve numerical instability; introduction of more linear terms in implicit scheme.
- Example to fix abrupt mountain's instability (with Girard)

$$\mathbf{P}\nabla_{\zeta}^{2}P+\frac{\gamma}{\kappa\tau^{2}RT_{*}}\left(\mathbf{P}_{\delta\delta}+\mathbf{P}_{\delta\mu}-\varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)P+\frac{B}{RT_{*}}\nabla_{\zeta}.(\overline{\overline{\delta P}}^{h\zeta}\nabla_{\zeta}\phi_{s})=\mathrm{R}$$

Weak scalability MPI test run (the number of solver calls is 800).
Comparison for timings **between using 3D_PGMRES and direct with 2D_PGMRES Problems: 2.5 Km GEM_LAM with 93 vertical- levels**

nodes	PxQ NixNj	#PEs MPI	#iteration 2D/3D	Exec. Time Seconds 2D/3D
1	4x4 400x400	16	11/16	812/2000
4	8x8 800x800	64	11/16	794/1982
16	16x16 1600x1600	64	11/16	810/1914
64	32x32 3200x3200	1024	12/17	936/2120
128	64x64 6400x6400	4096	17/24	1500/3200

MC2 model with GMRES(10) with ADI_V and a horizontal resolution 50-70 KM
-60 iterations for a problem 119x119x31
-70 iterations for a problem 511x539x31
-Delta_z=806m

REFERENCES:

Qaddouri, Côté and Valin 2000 High performance computing

Qaddouri et al. 2001 ECMWF workshop book

Qaddouri and Côté 2003 VECPAR

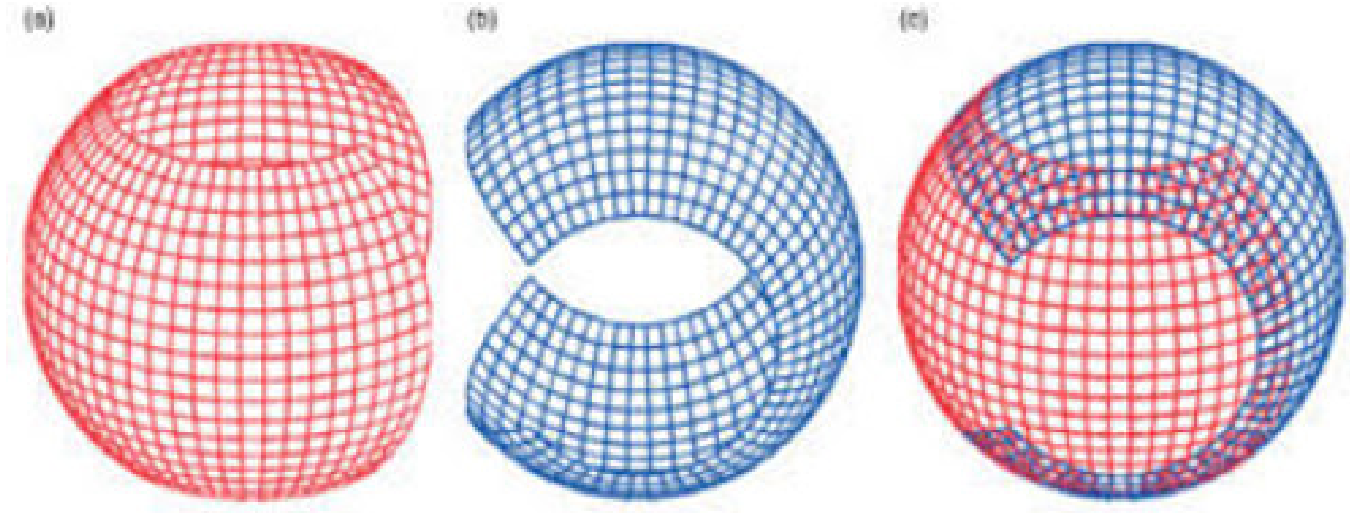
Qaddouri an Côté 2004   MWR

Qaddouri and Lee 2010   Modeling Simulation and Optimization

Thomas et al. 1997 Parallel Comput.

Yin-Yang Grid for global Forecast

Yin Yan Yin-Yang



A two-way coupling method between two-limited area models; Qaddouri and Lee QJRMS 2011

Advantages of Yin-Yang Grid

- No poles; global quasi-uniform grid : this simplify numerical schemes when used with Yin-Yang grid
- Semi-Lagragian scheme without considering fluid parcel trajectory as great circle.
- Explicit numerical diffusion solver
- More balanced computational load for scalability purpose compared to Lat-Lon.

Iterative Schwarz method for the elliptic problem

Global problem on circle

$$(\eta - \partial_{\lambda\lambda}) P = R \quad \text{on} \quad \Omega = [0 \;\; 2\pi], \;\; \eta \; \rangle 0$$

Two overlapped subdomains

$$\Omega_1 = [0 \;\; \pi + \delta], \quad \Omega_2 = [\pi \;\; 2\pi + \delta], \quad \delta \rangle 0$$

$$(\eta - \partial_{\lambda\lambda}) P_{1,k} = R_1 \;\; \text{on} \quad \Omega_1$$

$$BP_{1,k}(0) = BP_{2,k-1}(2\pi)$$

$$BP_{1,k}(\pi + \delta) = BP_{2,k-1}(\pi + \delta)$$

$$(\eta - \partial_{\lambda\lambda}) P_{2,k} = R_2 \;\; \text{on} \quad \Omega_2$$

$$BP_{2,k}(\pi) = BP_{1,k-1}(\pi)$$

$$BP_{2,k}(2\pi + \delta) = BP_{1,k-1}(\delta)$$

Schwarz method's convergence rate

- Classical Schwarz $B = Id$

$$\rho_{dirichlet} = \frac{\left(e^{\sqrt{\eta}\pi} + e^{\sqrt{\eta}\delta}\right)^2}{\left(1 + e^{\sqrt{\eta}(\pi+\delta)}\right)^2}$$

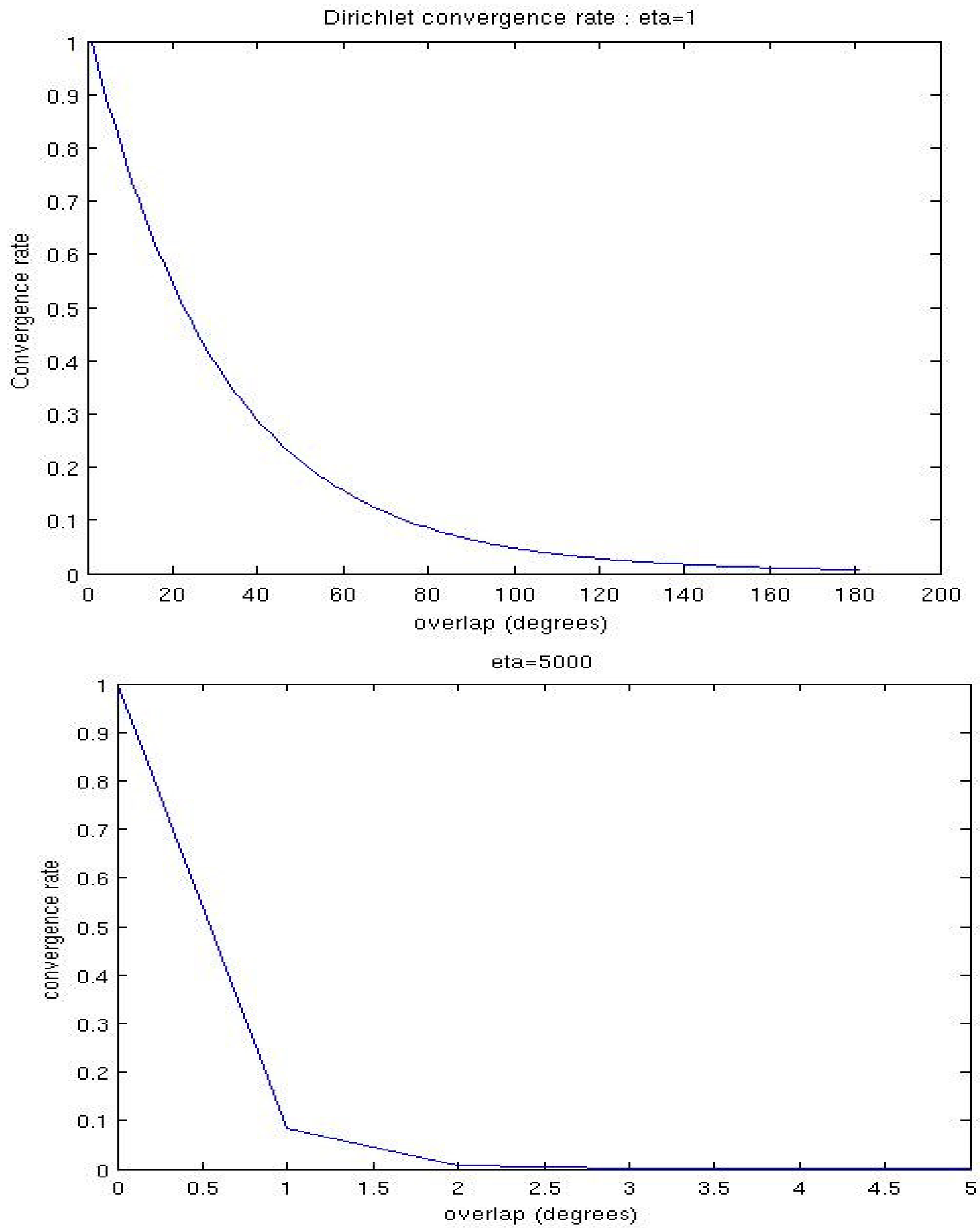
- Optimized Schwarz method

$$B = \partial_\lambda + \beta$$

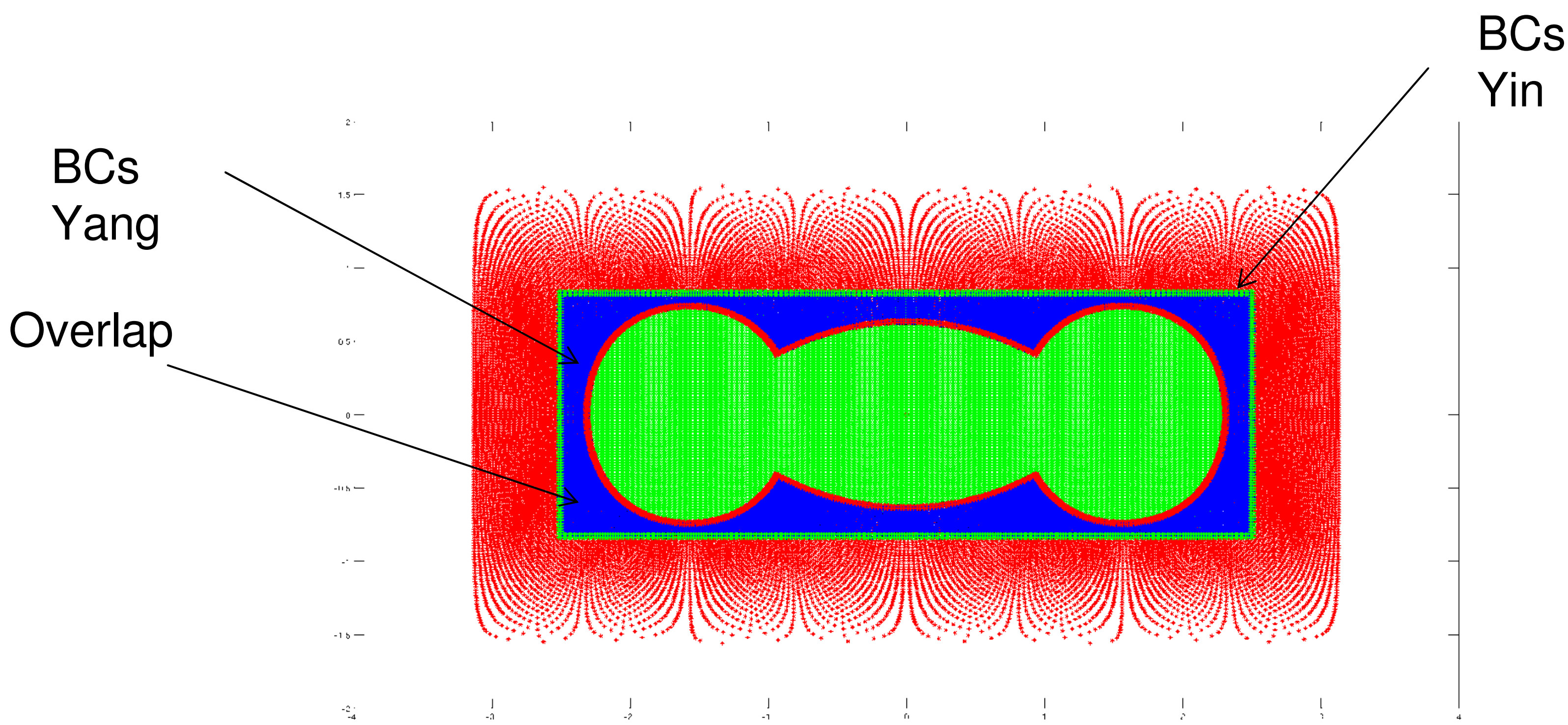
$$\rho_{Oschwaz} = \min \rho(A(\sqrt{\eta}, \delta, \beta_1^1, \beta_1^2, \beta_2^1, \beta_2^2)$$

$\beta_1^1, \beta_1^2, \beta_2^1, \beta_2^2$

Qaddouri et al. Applied Numerical Mathematics 2008


Dirichlet convergence rate : eta=1
Convergence rate
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
20
40
60
80
100
120
140
160
180
200
overlap (degrees)
eta=5000
convergence rate
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
overlap (degrees)

Iterative Schwarz method :Yin-Yang grid



GEM_Yin-Yang 15KM : 3 iterations for convergence 10^{-4} with 2 degrees overlap

Yin-Yang Classical Schwarz iterations (error 10^{-4})

NixNjxNk	Horz.res (degrees)	Time-step (seconds)	Overlap (degrees)	Yin-Yang # of iterations
270x90x57x2	1.04	3600	2	8
540x180x57x2	0.52	3600	2	8
1080x360x57x2	0.26	3600	2	8

NIxNjxNK	Horz.res	Time-step (seconds)	Overlap (degrees)	Yin-Yang # of iterations
270x90x57x2	1.04	3600	2	8
540x180x57x2	0.52	1800	2	6
1080x360x57x2	0.26	900	2	4

Matlab test run for Yin-Yang 300x100 x2

With : $\delta = 1\Delta\lambda$ $\eta = 1$

Boundary conditions	# iterations
Dirichlet	289
Robin	28
Second order	19

With Robin and second-order condition we can’t use FFT !

Implicit Normal mode initialisation (INMI)

work with Luc Fillion

- NMI is an initialisation technique for NWP models; Goal : Adjust initial conditions to avoid spurious oscillations.
- The horizontal normal modes (HM) in spherical coordinates are explicitly computed when the horizontal equations are separable .
- INMI can be implemented without knowing the HM. The horizontal separability is not necessary. We have to solve some elliptic problems.

Systems of non-separable horizontal equations

$$\frac{\partial \nabla^2 \psi}{\partial t} = -\mathcal{F}\chi + \mathcal{B}\psi$$

$$\frac{\partial \nabla^2 \chi}{\partial t} = \mathcal{F}\psi + \mathcal{B}\chi - \nabla^2 P$$

$$\frac{\partial P}{\partial t} = -\Phi_n \nabla^2 \chi$$

With the following :

$$U = \frac{1}{a^2}\left(\frac{\partial \chi}{\partial \lambda} - (1-\mu^2)\frac{\partial \psi}{\partial \mu} \right) \quad ; \qquad \zeta = \nabla^2 \psi$$

$$V = \frac{1}{a^2}\left(\frac{\partial \psi}{\partial \lambda} + (1-\mu^2)\frac{\partial \chi}{\partial \mu} \right) \quad ; \qquad D = \nabla^2 \chi$$

$$\mathcal{F} \equiv \frac{1}{a^2}\left[\frac{1}{(1-\mu^2)}\frac{\partial}{\partial \lambda}\left(f\frac{\partial}{\partial \lambda} \right) + \frac{\partial}{\partial \mu}\left(f(1-\mu^2)\frac{\partial}{\partial \mu} \right) \right]$$

$$\mathcal{B} \equiv \frac{1}{a^2}\left[\frac{\partial}{\partial \lambda}\left(f\frac{\partial}{\partial \mu} \right) - \frac{\partial}{\partial \mu}\left(f\frac{\partial}{\partial \lambda} \right) \right]$$

Elliptic solvers in one INMI's iteration

$$M\left(\frac{\partial P_G}{\partial t}\right) = \nabla^2\left(\frac{\partial P_o}{\partial t}\right) - \mathcal{F}_s\left(\frac{\partial \psi_o}{\partial t}\right)$$

$$\Phi_n \nabla_\chi^2(\Delta\chi) = \frac{\partial P_G}{\partial t}$$

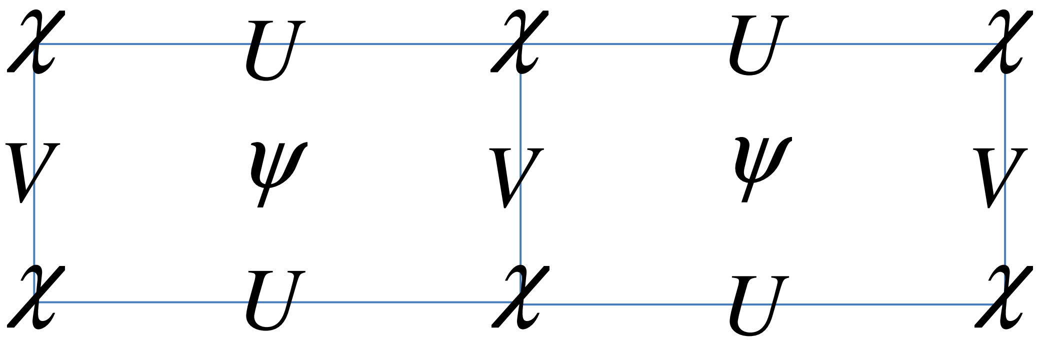
$$M(\Delta P) = \frac{\partial D_o}{\partial t} + \mathcal{B}_s(\Delta\chi)$$

$$\nabla_\psi^2(\Delta\psi) = \frac{1}{\Phi}\mathcal{F}^s \nabla_\chi^{-2}(\Delta P)$$

Where the Operator M:

$$M \equiv \nabla^2 - \frac{1}{\Phi}\mathcal{F}_s \nabla_\psi^{-2} \mathcal{F}^s \nabla_\chi^{-2}$$

Horizontal distribution of variables



P Same position as χ

$$U = \frac{1}{a^2}\left(\frac{\partial\chi}{\partial\lambda} - (1-\mu^2)\frac{\partial\psi}{\partial\mu}\right)$$

$$V = \frac{1}{a^2}\left(\frac{\partial\psi}{\partial\lambda} + (1-\mu^2)\frac{\partial\chi}{\partial\mu}\right)$$

PGMRES Iterative Solver

$$M\left(\frac{\partial P_G}{\partial t}\right) = \nabla^2\left(\frac{\partial P_o}{\partial t}\right) - \mathcal{F}_s\left(\frac{\partial \psi_o}{\partial t}\right)$$

2 Poisson problems

$$M \equiv \nabla^2 - \frac{1}{\Phi}\mathcal{F}_s \nabla_\psi^{-2} \mathcal{F}^s \nabla_\chi^{-2}$$

$$\mathcal{F} \equiv \frac{1}{a^2}\left[\frac{1}{(1-\mu^2)}\frac{\partial}{\partial\lambda}\left(f\frac{\partial}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(f(1-\mu^2)\frac{\partial}{\partial\mu}\right)\right]$$

preconditioner (Block- Jacobi)

$$(\nabla^2 - (min(f) + max(f))/2\ I\ )X = R$$

Poisson equations

$$\Phi_n \nabla_\chi^2 (\Delta \chi) = \frac{\partial P_G}{\partial t}$$

$$\nabla_\psi^2 (\Delta \psi) = \frac{1}{\Phi} \mathcal{F}^s \nabla_\chi^{-2} (\Delta P)$$

Gauss Constraints

$$\int_{sphere} \Delta \psi \quad d\lambda \; d\sin(\theta) = 0$$

$$\int_{sphere} \Delta \chi \quad d\lambda \; d\sin(\theta) = 0$$

Direct 2D Poisson Solver

- Fourier Analysis of the right-hand sides; for mode zero removes the field average.
- Solve NI Tridiagonal problems of dimension NJ
- Fourier synthesis of the solutions

Preliminary result

- From a given vector-solution we construct a vector-right hand side , and solve. We try to retrieve the vector solution, 3 iterations for 10-5

$$M^{-1} A x = M^{-1} R$$

$$A \equiv \nabla^2 - \frac{1}{\Phi} \mathcal{F}_s \nabla_\psi^{-2} \mathcal{F}^s \nabla_\chi^{-2}$$

$$M = \nabla^2 - (\min(f) + \max(f))/2$$

REFERENCES:

Fillion and Temperton 1989 MWR
Temperton and Roch   1990 MWR
Temperton 1987 MWR

$$\frac{\partial \mathbf{u}}{\partial t} - f\mathbf{v} + \frac{1}{a\cos\theta}\frac{\partial \mathbf{P}}{\partial \lambda} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + f\mathbf{u} + \frac{1}{a}\frac{\partial \mathbf{P}}{\partial \theta} = 0$$

$$\frac{\partial \mathbf{P}}{\partial t} = -\,\mathbf{Cd}$$

Projection on normal modes space

$$\frac{\partial U}{\partial t} = fV - \frac{\partial P}{\partial \lambda}$$

$$\frac{\partial V}{\partial t} = -fU - \frac{(1-\mu^2)}{a^2}\frac{\partial P}{\partial \mu}$$

$$\frac{\partial P}{\partial t} = -\,\Phi D$$