A Filtering Laplace Transform Integration Scheme For Numerical Weather Prediction

Colm Clancy

and

Peter Lynch

Meteorology and Climate Centre, School of Mathematical Sciences

University College Dublin

Aim

- **To develop a time-stepping scheme that filters high-frequency noise, based on Laplace transform theory**

- First used by Lynch (1985).
  Further work in Lynch (1986), (1991) and
  Van Isacker & Struylaert (1985), (1986)

Overview

1: Laplace transform method: background theory

2: Eulerian shallow water model and Kelvin waves

3: Semi-Lagrangian model and orographic resonance

Definitions

Laplace transform: $f(t), \ t \geq 0$

$$\widehat{f}(s) \equiv \mathfrak{L}\{f\} = \int_0^\infty e^{-st} f(t)dt$$

Inverse Transform:

$$f(t) \equiv \mathfrak{L}^{-1}\{\widehat{f}\} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{st} \widehat{f}(s)ds$$

A Simple Filtering Example

- Consider :

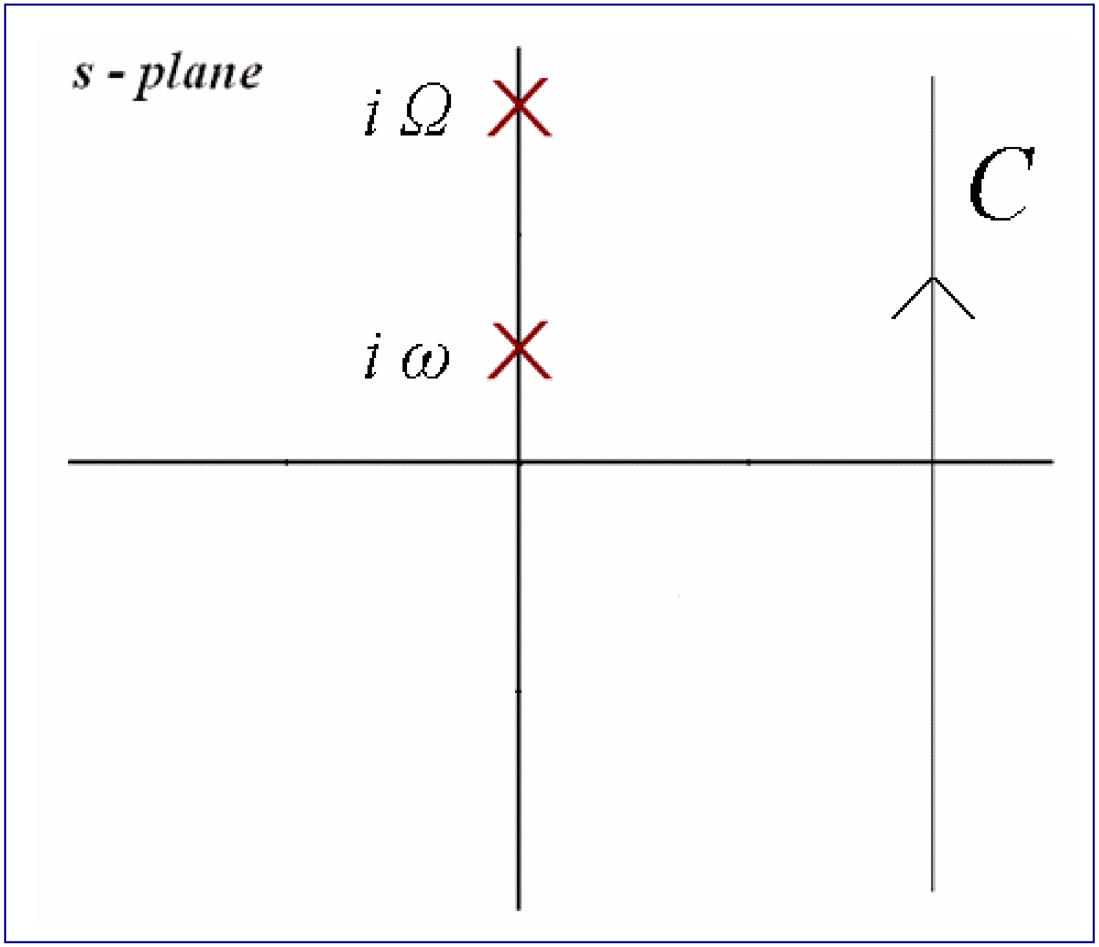
$$f(t) = ae^{i\omega t} + Ae^{i\Omega t}$$

$$|\omega| << |\Omega|$$

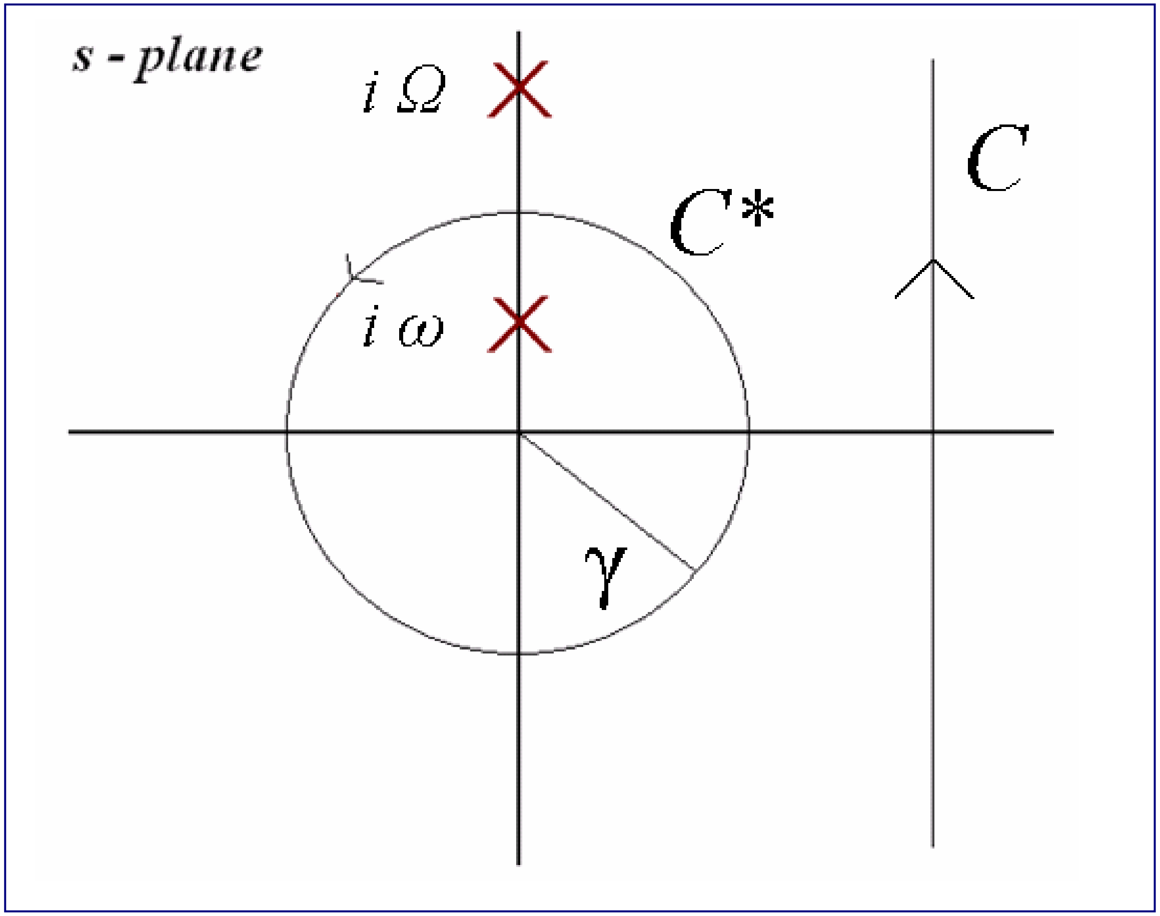
- Laplace Transform:

$$\hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}$$

Poles in the s-plane



Modify the Inversion Contour



Modified Inverse

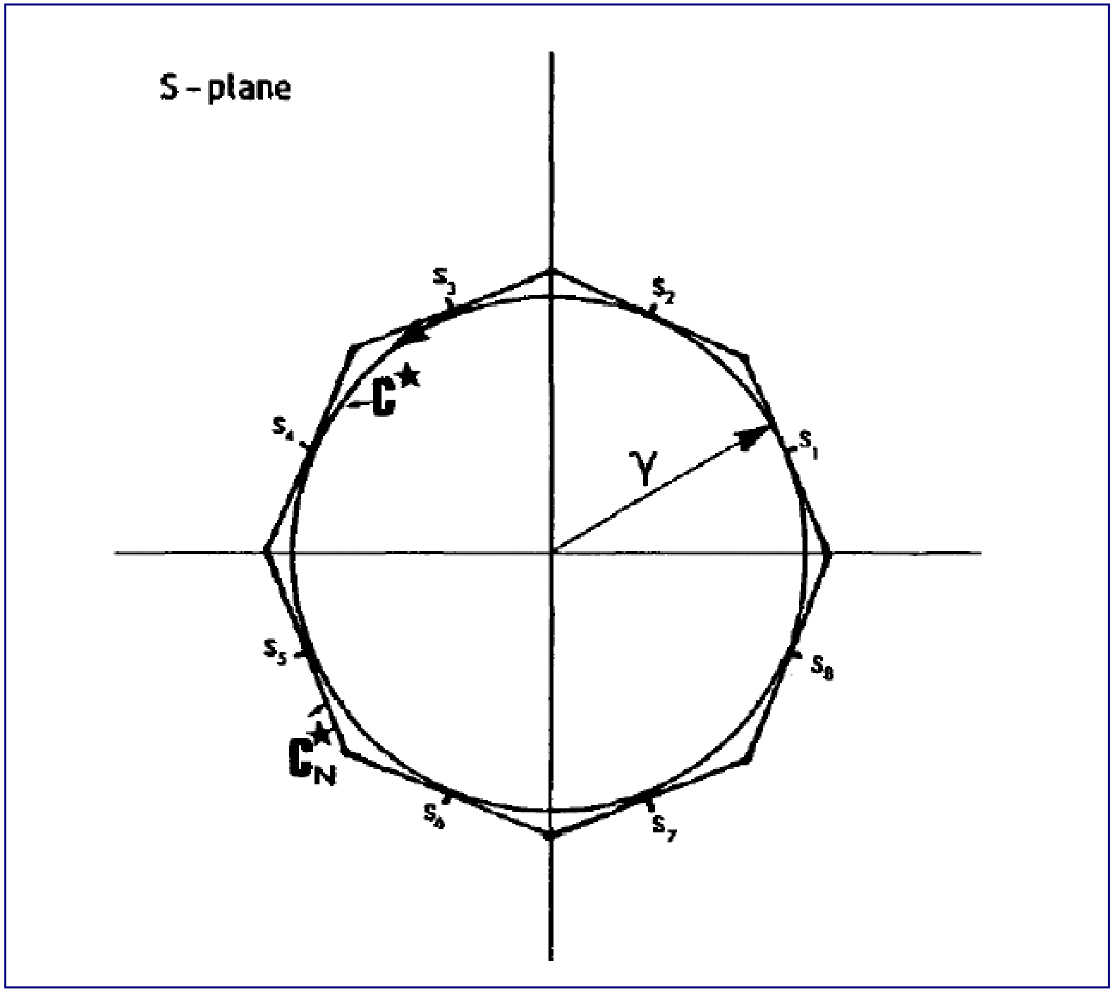
- Define

$$f^*(t) \equiv \mathfrak{L}^*\{\widehat{f}\} = \frac{1}{2\pi i}\oint_{\mathcal{C}^*} e^{st}\widehat{f}(s)ds$$

- Contribution from frequencies $\omega < \gamma$ only

- Cauchy's Integral Formula => $f^*(t) = ae^{i\omega t}$

Numerical Inversion



Numerical Inversion Operator

$$f^*(t) \equiv \mathfrak{L}^*\{\widehat{f}\} = \frac{1}{2\pi i} \oint_{\mathcal{C}^*} e^{st} \widehat{f}(s) ds$$

Approximated by:

$$\mathfrak{L}^*_N\{\widehat{f}\} \equiv \frac{1}{2\pi i} \sum_{n=1}^{N} e^{s_n t} \, \widehat{f}(s_n) \, \Delta s_n$$

Numerical Inversion Operator

$$\mathfrak{L}_N^*\{\widehat{f}\} \equiv \frac{1}{2\pi i}\sum_{n=1}^{N} e^{s_n\,t}\,\widehat{f}(s_n)\,\Delta s_n$$

Numerical Inversion Operator

$$\mathfrak{L}_N^*\{\widehat{f}\} \equiv \frac{1}{2\pi i}\sum_{n=1}^{N} e^{s_n t}\,\widehat{f}(s_n)\,\Delta s_n$$

Divide by correction factor: $\kappa = \dfrac{\tan\frac{\pi}{N}}{\frac{\pi}{N}}$

Use truncated exponential: $e_N^z = \displaystyle\sum_{j=0}^{N-1}\frac{z^j}{j!}$

$$\mathfrak{L}_N^*\{\,\widehat{f}\,\} \equiv \frac{1}{N}\sum_{n=1}^{N} e_N^{s_n t}\,\widehat{f}(s_n)\,s_n$$

Numerical Inversion Operator Properties

Symmetry if *f(t)* is real:

$$f^*(t) \equiv \mathfrak{L}^*_N\{\widehat{f}\} = \frac{2}{N}\sum_{n=1}^{N/2} Re\left\{s_n\,\widehat{f}(s_n)\,e^{\frac{s_n t}{N}}\right\}$$

Inversion is exact for constant function and powers of $t < N$.

Laplace Transform Properties

Derivatives:

$$\mathfrak{L}\{f'(t)\} \;=\; s\widehat{f}(s) - f(0)$$

Constants:

$$\mathfrak{L}\{a\} \;=\; \frac{a}{s}$$

Filtering a Dynamical System

$$\frac{d\mathbf{X}}{dt} + \mathbf{LX} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

Take the Laplace transform over $[(\tau\text{-}1)\Delta t\ ,(\tau+1)\Delta t\ ]$:

$$s\hat{\mathbf{X}} - \mathbf{X}^{\tau-1} + \mathbf{L}\hat{\mathbf{X}} + \frac{\mathbf{N}^{\tau}}{s} = \mathbf{0}$$

$$\hat{\mathbf{X}}(s) = (s\mathbf{I} + \mathbf{L})^{-1}[\mathbf{X}^{\tau-1} - \frac{\mathbf{N}^{\tau}}{s}]$$

Filtered Forecast

$$\widehat{\mathbf{X}}(s) \;\; = \;\; (s\mathbf{I} + \mathbf{L})^{-1}\left[\mathbf{X}^{\tau-1} - \mathbf{N}^{\tau}/s\right]$$

$$\mathbf{X}^{\tau+1} = \mathfrak{L}^{*}_{N}\{\;\widehat{\mathbf{X}}(s)\;\}\Big|_{t=2\Delta t}$$

Simple Oscillation Equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = i\omega X$$

$$\widehat{X}(s) = (s - i\omega)^{-1} X^0$$

$$X(\Delta t) = \mathfrak{L}^*_N \{\widehat{X}(s)\}|_{t=\Delta t}$$

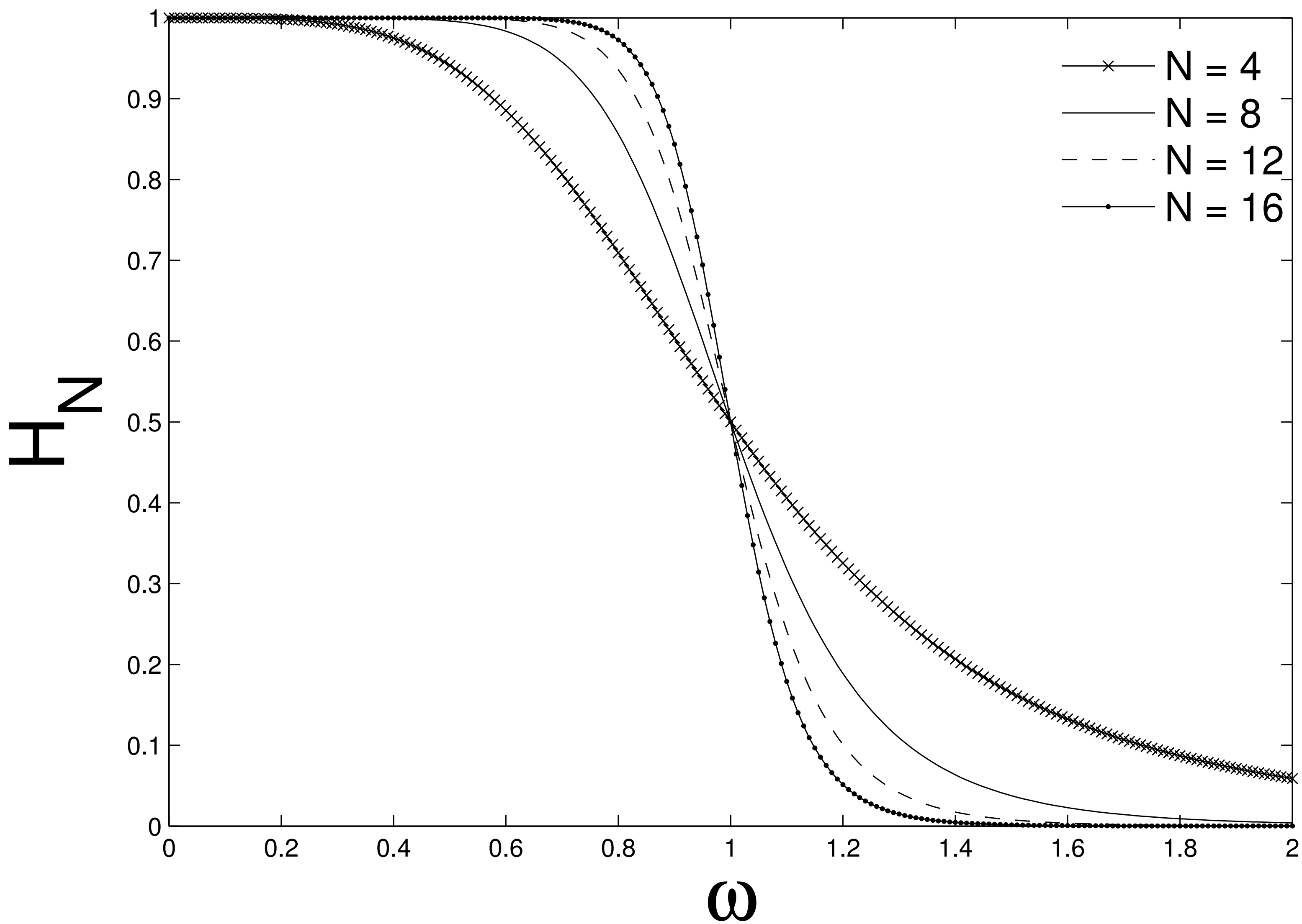
$$X(\Delta t) = X^0 \, H_N(\omega) \, e_N^{i\omega\Delta t}$$

Filter Response

$$X(\Delta t) \quad = \quad H_N(\omega)\, e_N^{i\omega\Delta t}$$

$$H_N(\omega) = \frac{1}{1 + \left(\frac{i\,\omega}{\gamma}\right)^N}$$

Filter response $H_N = 1/(1+\omega^N)$



Stability

Lynch (1986):

$$\Delta t \leq \frac{(N!)^{1/N}}{2\gamma}$$

$$Example: \quad N = 8, \;\; \tau_c = 6 \;\; hours \Rightarrow \Delta t \leq 1.8 \;\; hours$$

Applying the Laplace Transform Method to Shallow Water Models

Shallow Water Equations

- Suitable testing framework for numerical methods
- Standard test cases used by modelling community
- Two models: Eulerian and semi-Lagrangian
  Spectral models for efficient solution of Helmholtz equations
- Testing against reference semi-implicit method

Eulerian STSWM

- STSWM code: NCAR – Hack & Jakob (1992)
- Updated by ICON - http://icon.enes.org/
- Spectral transform method
- Centred time-differencing, semi-implicit scheme
- Test cases proposed by Williamson *et al* (1992)

Spectral Solution

$$\frac{\partial\zeta}{\partial t} = -\nabla.(\zeta + f)\mathbf{v}$$

$$\frac{\partial\delta}{\partial t} = \mathbf{k}.\nabla \times (\zeta + f)\mathbf{v} - \nabla^2(\Phi + \frac{\mathbf{v}.\mathbf{v}}{2})$$

$$\frac{\partial\Phi^*}{\partial t} = -\nabla.\left(\Phi^* \mathbf{v}\right)$$

Expand each field: $\zeta(\lambda, \mu, t) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \zeta_\ell^m(t) P_\ell^m(\mu) e^{im\lambda}$

Spectral Solution

Get system of ODEs for the spectral coefficients:

$$
\begin{aligned}
\frac{d}{dt}\eta_\ell^m &= \mathcal{N}_\ell^m \\
\frac{d}{dt}\delta_\ell^m &= \mathcal{D}_\ell^m + \frac{\ell(\ell+1)}{a^2}\,\Phi_\ell^m \\
\frac{d}{dt}\Phi_\ell^m &= \mathcal{F}_\ell^m - \bar{\Phi}^*\,\delta_\ell^m
\end{aligned}
$$

Reference Semi-Implicit Scheme

$$\frac{\{\eta_\ell^m\}^{\tau+1} - \{\eta_\ell^m\}^{\tau-1}}{2\,\Delta\,t} = \{\mathcal{N}_\ell^m\}^\tau$$

$$\frac{\{\delta_\ell^m\}^{\tau+1} - \{\delta_\ell^m\}^{\tau-1}}{2\,\Delta\,t} = \{\mathcal{D}_\ell^m\}^\tau + \frac{\ell(\ell+1)}{a^2}\,\frac{\{\Phi_\ell^m\}^{\tau+1} + \{\Phi_\ell^m\}^{\tau-1}}{2}$$

$$\frac{\{\Phi_\ell^m\}^{\tau+1} - \{\Phi_\ell^m\}^{\tau-1}}{2\,\Delta\,t} = \{\mathcal{F}_\ell^m\}^\tau - \bar{\Phi}^*\,\frac{\{\delta_\ell^m\}^{\tau+1} + \{\delta_\ell^m\}^{\tau-1}}{2}$$

Laplace Transform Scheme

$$
\begin{aligned}
s\,\widehat{\eta_\ell^m} - \{\eta_\ell^m\}^{\tau-1} &= \frac{1}{s}\,\{\mathcal{N}_\ell^m\}^{\tau} \\
s\,\widehat{\delta_\ell^m} - \{\delta_\ell^m\}^{\tau-1} &= \frac{1}{s}\,\{\mathcal{D}_\ell^m\}^{\tau} + \frac{\ell(\ell+1)}{a^2}\,\widehat{\Phi_\ell^m} \\
s\,\widehat{\Phi_\ell^m} - \{\Phi_\ell^m\}^{\tau-1} &= \frac{1}{s}\,\{\mathcal{F}_\ell^m\}^{\tau} - \bar{\Phi}^*\,\widehat{\delta_\ell^m}
\end{aligned}
$$

Comparing the Discretisations

LT

$$s\,\widehat{\eta_\ell^m} = \{\eta_\ell^m\}^{\tau-1} + \frac{1}{s}\,\{\mathcal{N}_\ell^m\}^\tau$$

$$s\,\widehat{\delta_\ell^m} = \left[1 + \bar{\Phi}^*\,\frac{\ell(\ell+1)}{a^2}\,\frac{1}{s^2}\right]^{-1}\left(\mathcal{R}' + \frac{1}{s}\,\frac{\ell(\ell+1)}{a^2}\,\mathcal{Q}'\right)$$

$$s\,\widehat{\Phi_\ell^m} = \left[1 + \bar{\Phi}^*\,\frac{\ell(\ell+1)}{a^2}\,\frac{1}{s^2}\right]^{-1}\left(\mathcal{Q}' - \frac{1}{s}\,\bar{\Phi}^*\,\mathcal{R}'\right)$$

$$\mathcal{R}' = \{\delta_\ell^m\}^{\tau-1} + \frac{1}{s}\,\{\mathcal{D}_\ell^m\}^\tau$$

$$\mathcal{Q}' = \{\Phi_\ell^m\}^{\tau-1} + \frac{1}{s}\,\{\mathcal{F}_\ell^m\}^\tau$$

SI

$$\{\eta_\ell^m\}^{\tau+1} = \{\eta_\ell^m\}^{\tau-1} + 2\,\Delta t\,\{\mathcal{N}_\ell^m\}^\tau$$

$$\{\delta_\ell^m\}^{\tau+1} = \left[1 + \bar{\Phi}^*\,\frac{\ell(\ell+1)}{a^2}\,\Delta t^2\right]^{-1}\left(\mathcal{R} + \mathcal{Q}\,\frac{\ell(\ell+1)}{a^2}\,\Delta t\right)$$

$$\{\Phi_\ell^m\}^{\tau+1} = \left[1 + \bar{\Phi}^*\,\frac{\ell(\ell+1)}{a^2}\,\Delta t^2\right]^{-1}\left(\mathcal{Q} - \mathcal{R}\,\bar{\Phi}^*\,\Delta t\right)$$

$$\mathcal{R} = \{\delta_\ell^m\}^{\tau-1} + 2\,\Delta t\{\mathcal{D}_\ell^m\}^\tau + \Delta t\,\frac{\ell(\ell+1)}{a^2}\,\{\Phi_\ell^m\}^{\tau-1}$$

$$\mathcal{Q} = \{\Phi_\ell^m\}^{\tau-1} + 2\,\Delta t\{\mathcal{F}_\ell^m\}^\tau + \Delta t\,\bar{\Phi}^*\,\{\delta_\ell^m\}^{\tau-1}$$

Test Cases: Williamson et al (1992)

Case 1: Advection of a cosine bell by constant winds

Case 2: Steady zonal flow

Case 5: Flow over an isolated mountain

Case 6: Rossby-Haurwitz wave

Results

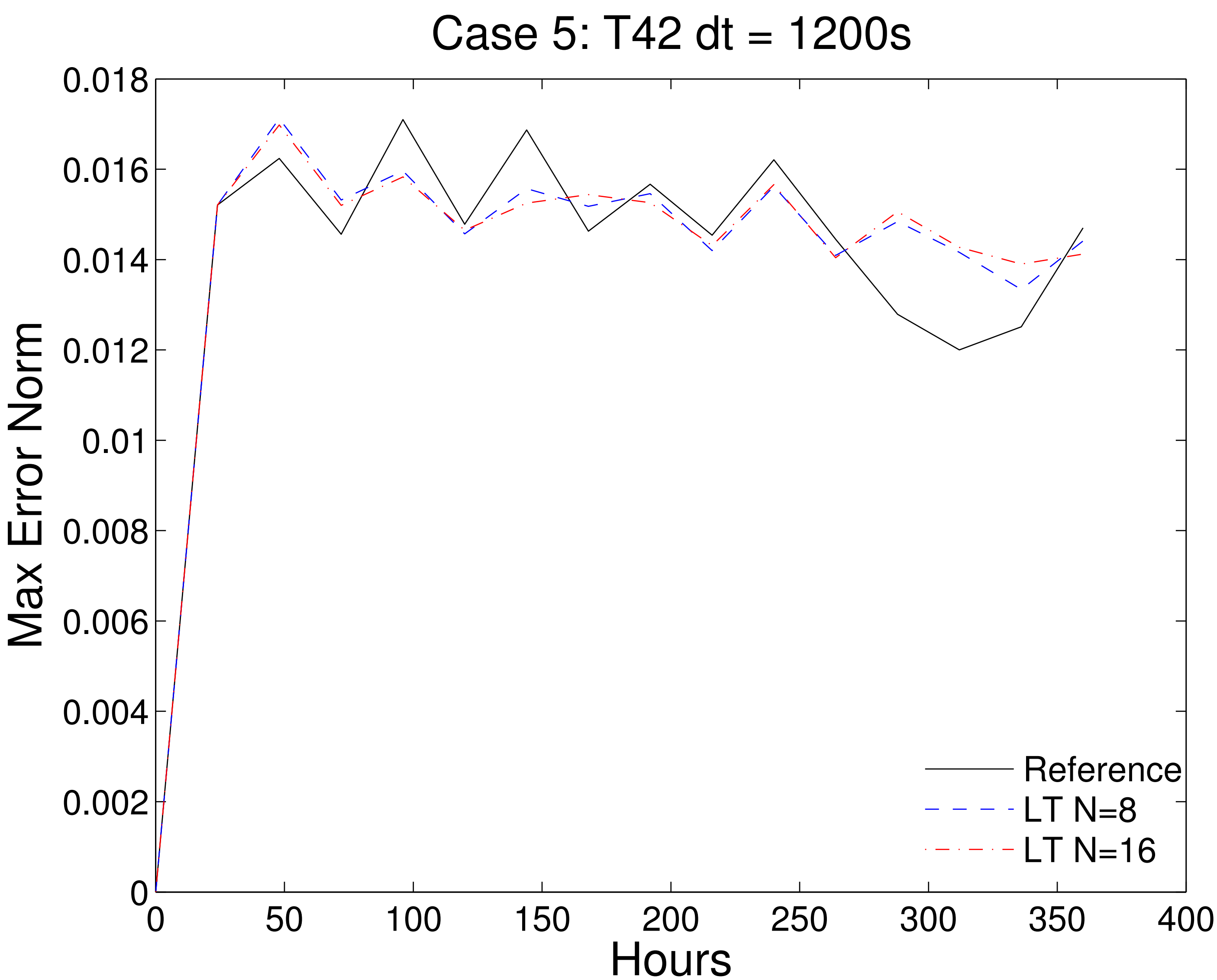
•Comparable accuracy and conservation with reference model

•In general, increasing number of points in inversion operator, *N*, from 8 to 16 did not significantly improve accuracy

Sample Results

Case 5

Flow Over a Mountain: Relative Errors



Kelvin Waves

- Eigenfunctions of the linearised shallow water equations
- Dynamically important
- Semi-implicit methods slow down waves

Phase Error Analysis

Oscillation equation: $$\frac{du}{dt} = i\,\nu\,u$$

Look for: $$u^{\tau+1} = A\,u^{\tau}$$

Numerical phase: $$\theta = \tan^{-1}\left(\frac{Im(A)}{Re(A)}\right)$$

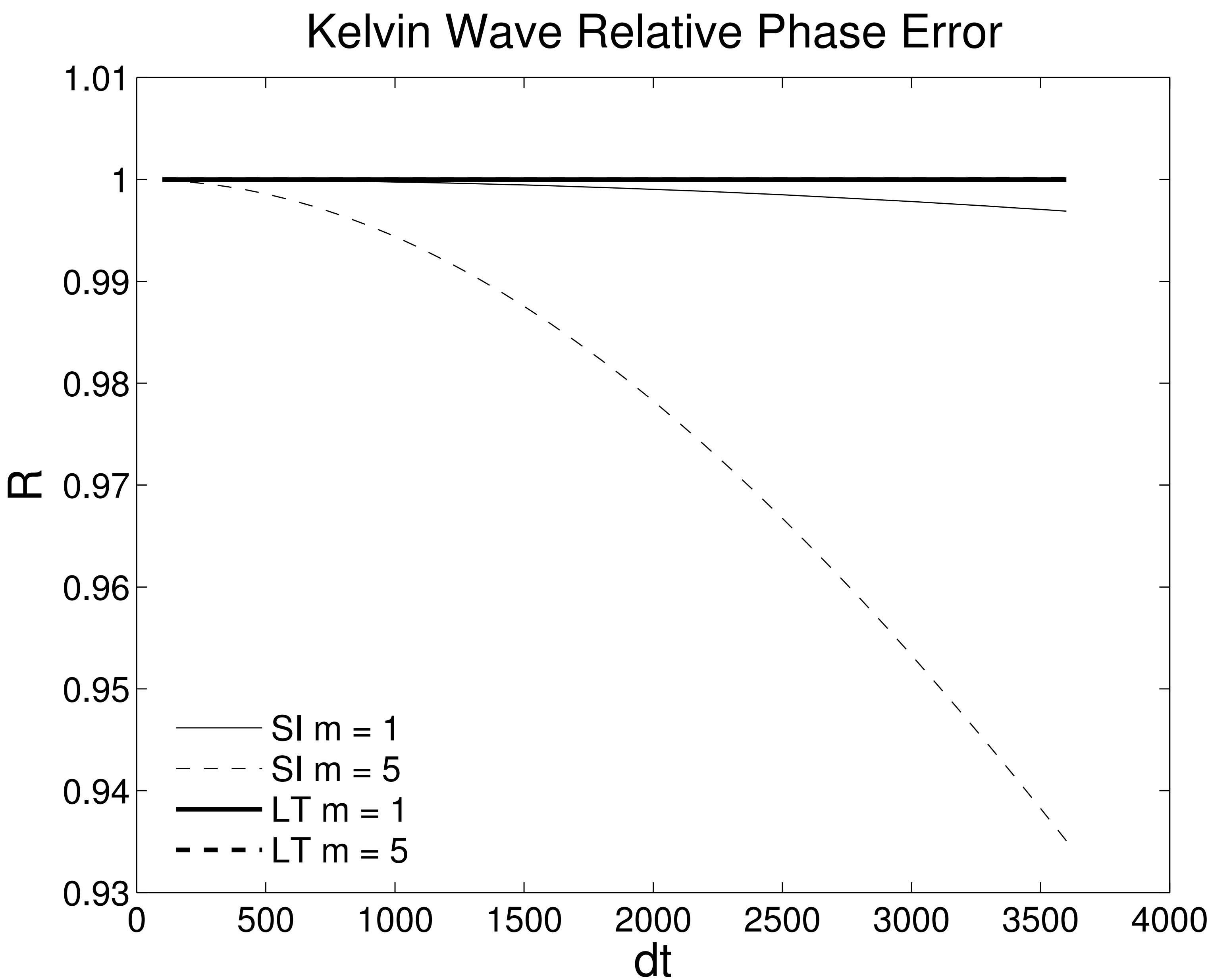
Relative Phase Change:
R = (numerical) / (actual)

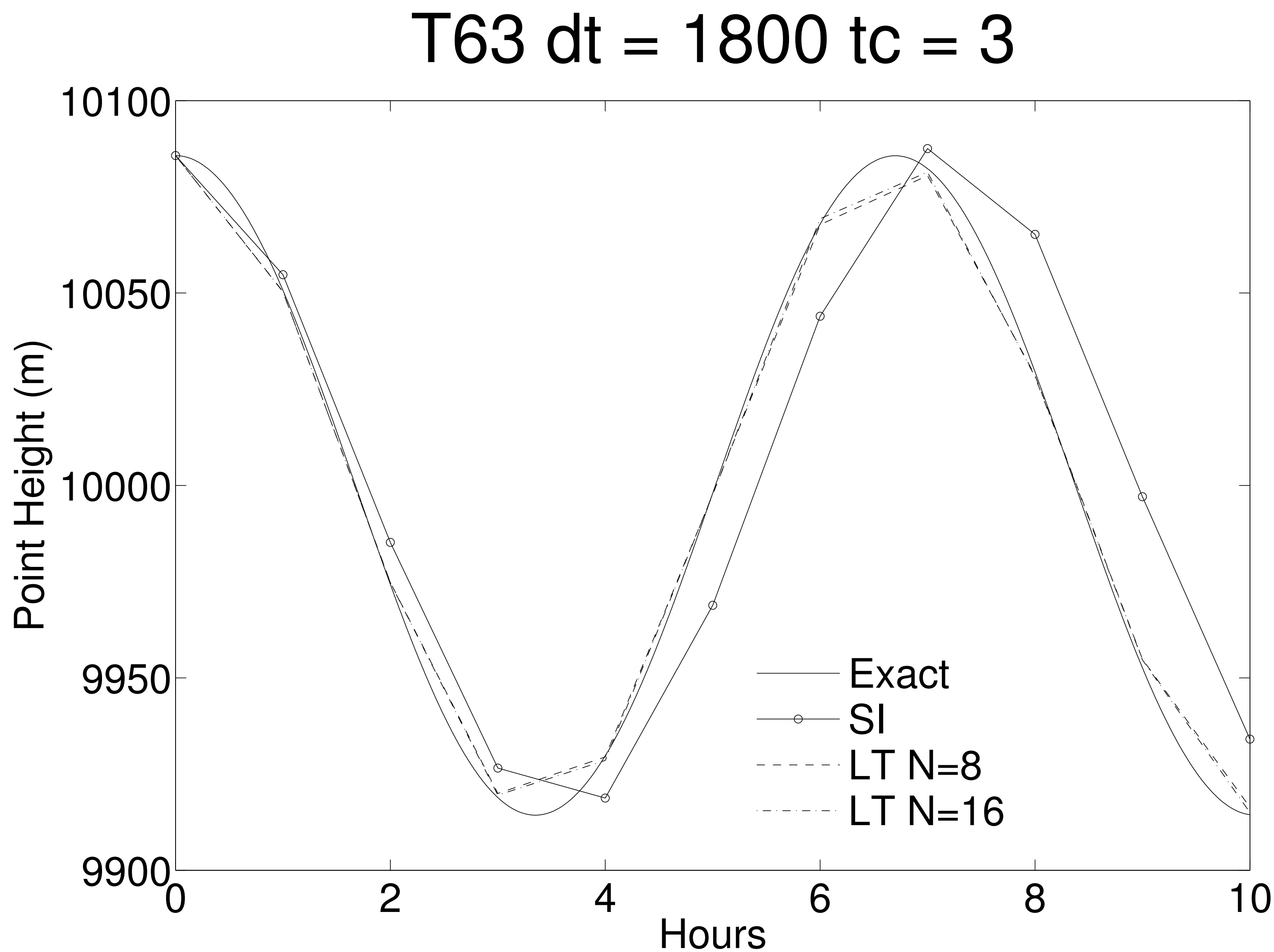
$$R = \frac{\theta}{\nu\,\Delta t}$$

Phase Error Analysis

$$R_{SI} \approx 1 - \frac{(\nu\,\Delta t)^2}{12}$$

$$R_{LT} \approx 1 + \frac{N}{(N+1)!}\,(\nu\,\Delta t)^N$$


Kelvin Wave Relative Phase Error
R
dt
1.01
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0
500
1000
1500
2000
2500
3000
3500
4000
SI m = 1
SI m = 5
LT m = 1
LT m = 5

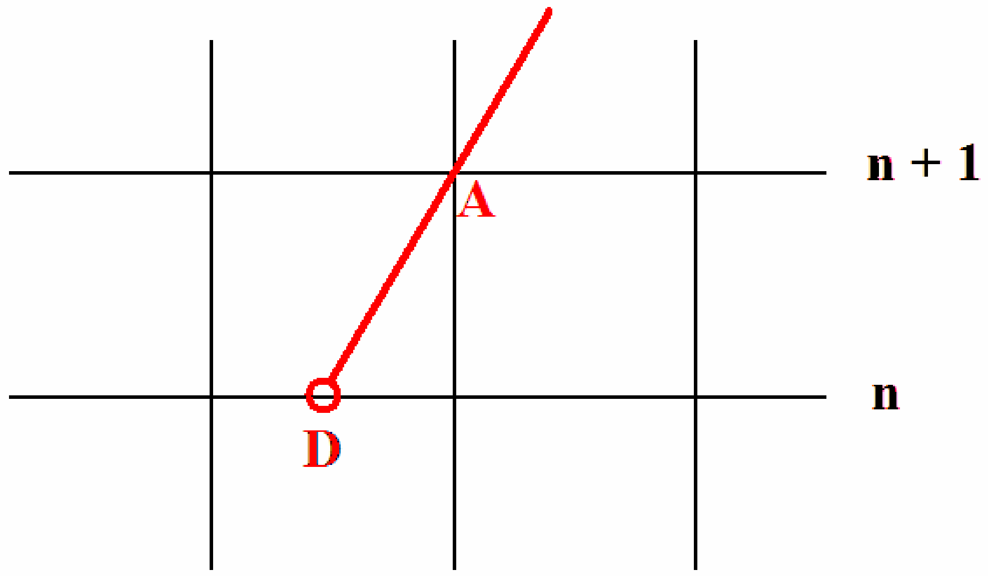

T63 dt = 1800 tc = 3
Point Height (m)
10100
10050
10000
9950
9900
0
2
4
6
8
10
Hours
Exact
SI
LT N=8
LT N=16

Semi-Lagrangian

Shallow Water Model

Semi-Lagrangian Laplace Transform

- Define the LT along a trajectory



$$\widehat{\zeta}(s) \equiv \mathfrak{L}\{\zeta\} = \int_{\text{Along Trajectory}} e^{-st}\,\zeta\,dt$$

- Then $\mathfrak{L}\left\{\frac{d\zeta}{dt}\right\} = s\,\widehat{\zeta} - \zeta_D^n$

Semi-Lagrangian Laplace Transform SLLT

- Based on spectral SWEmodel (John Drake, ORNL)
- Compared with semi-Lagrangian semi-implicit SLSI

Shallow Water Equations

$$\frac{d\zeta}{dt} + f\delta + \beta v = N_\zeta$$

$$\frac{d\delta}{dt} - f\zeta + \beta u + \nabla^2\Phi = N_\delta$$

$$\frac{d\Phi}{dt} - \frac{d\Phi_s}{dt} + \bar{\Phi}\delta = N_\Phi$$

SLLT Discretisation

General evolution equation:

$$\frac{dX}{dt} + L = N$$

SLLT:

$$s\,\widehat{X} - X_D^n + \widehat{L} = \frac{1}{s}\,N_M^{n+\frac{1}{2}}$$

SLSI Discretisations

SLSI:

$$\frac{X_A^{n+1} - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} = N_M^{n+\frac{1}{2}}$$

SLSI SETTLS, (Hortal, 2002):

$$\frac{X_A^{n+1} - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} = \frac{1}{2} \left\{ \left(2\, N_D^n - N_D^{n-1}\right) + N_A^n \right\}$$

Departure Point Calculations

- Two time level scheme
- Trajectories calculated in spherical coordinates
  (Ritchie and Beaudoin, 1994)
- Bilinear interpolation when computing departure points
- Bicubic for model fields
- Extrapolation used for computing midpoint values

Stability

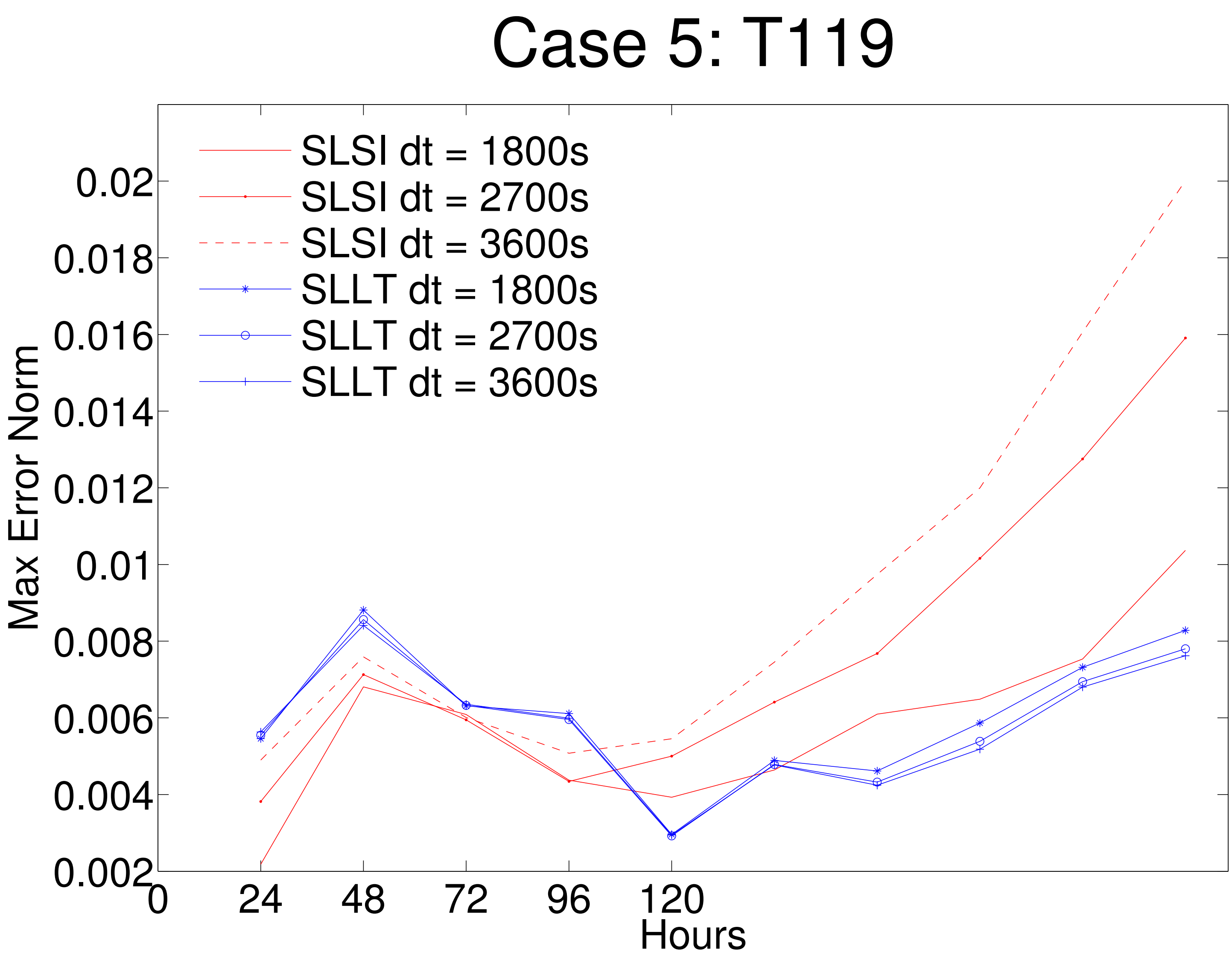
SLSI: Require $\bar{\Phi} \geq \Phi_{\max}$ (Côté and Staniforth, 1988)

SLLT: Stability not dependent on $\bar{\Phi}$

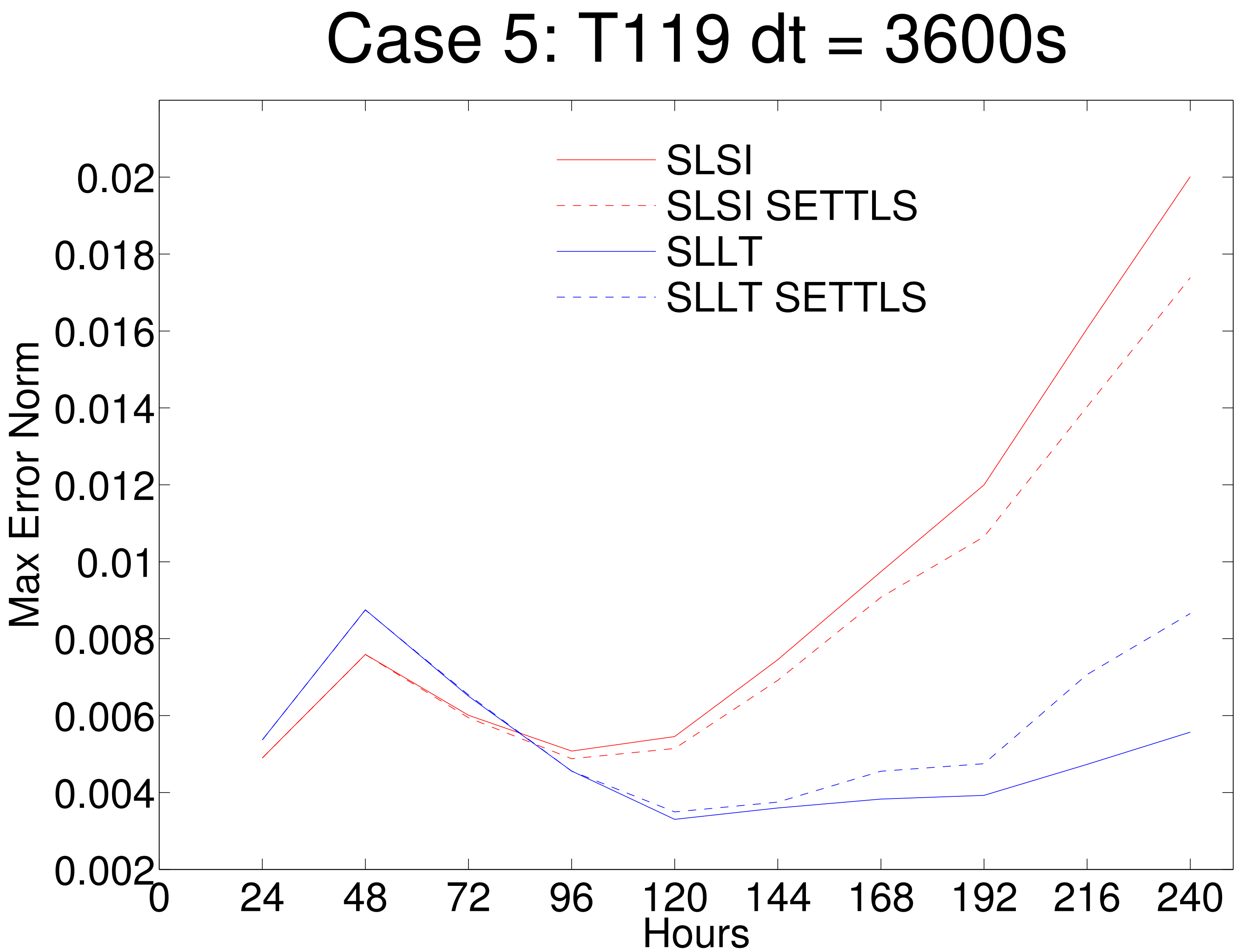
Sample Results

Case 5

Flow Over a Mountain: Relative Errors



Flow Over a Mountain: Relative Errors



Symmetry

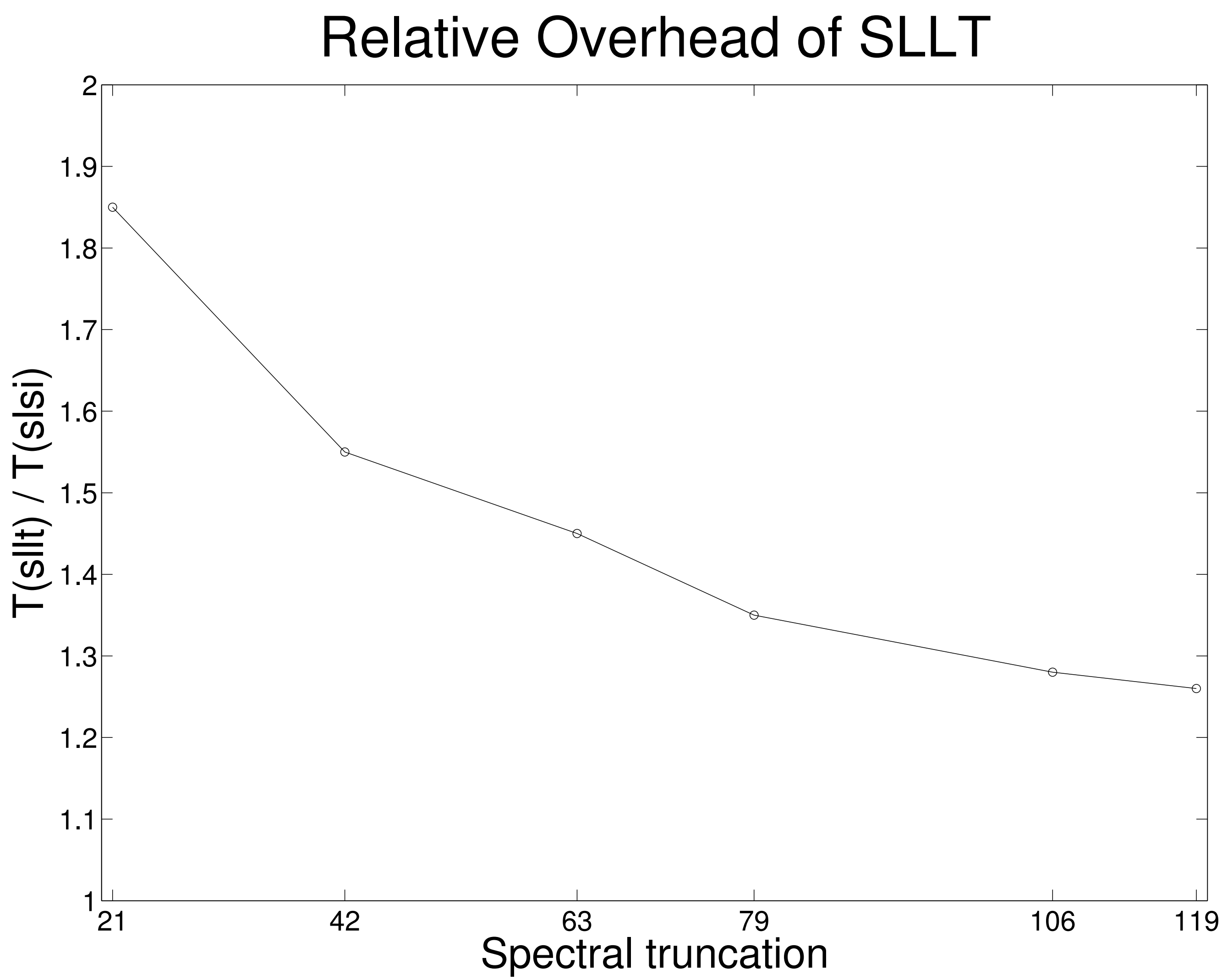
Can use symmetry in inversion operator:

$$\mathfrak{L}_N^*\{\,\widehat{f}\,\} \equiv \frac{1}{N}\sum_{n=1}^{N} e_N^{s_n\,t}\,\widehat{f}(s_n)\,s_n$$

$$\downarrow$$

$$\mathfrak{L}_N^*\{\widehat{f}\} = \frac{2}{N}\sum_{n=1}^{N/2} Re\left\{s_n\,\widehat{f}(s_n)\,e_N^{s_n t}\right\}$$

Efficiency


Relative Overhead of SLLT
T(sllt) / T(slsi)
Spectral truncation
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
21
42
63
79
106
119

Orographic Resonance

Orographic Resonance

- Spurious resonance from coupling semi-Lagrangian and semi-implicit methods

  [reviewed in Lindberg & Alexeev (2000)]

- LT method has benefits over semi-implicit schemes

- Motivates investigating orographic resonance in SLLT model

Orographic Resonance Analysis

- Linear analysis of orographically forced stationary waves, following Ritchie & Tanguay (1996)

- Numerical simulations with shallow water SLLT

- Results consistently show benefits of SLLT scheme

Linear Analysis: R = (Numerical)/(Analytic)

SLSI SLLT

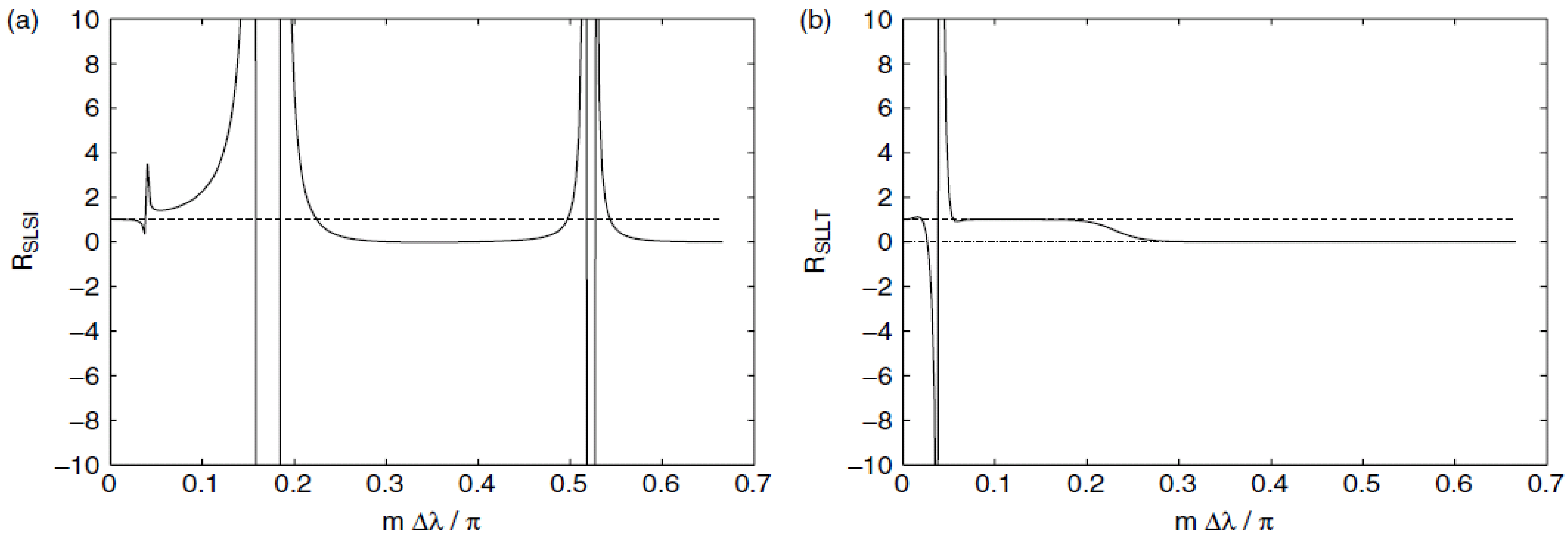


Figure 4. The numerical response to orographic forcing divided by the physical response for (a) SLSI and (b) SLLT, with T213, $\Delta t = 7200$ s, $N = 16$ and $\tau_c = 3$ h. At $R = 1$ (dashed line), the numerical solution equals the analytic solution, and at $R = 0$ (dot-dashed line), the numerical solution is zero due to filtering. The extremes at $m\Delta\lambda/\pi \approx 0.04$ are artefacts, due to the vanishing of the physical solution.

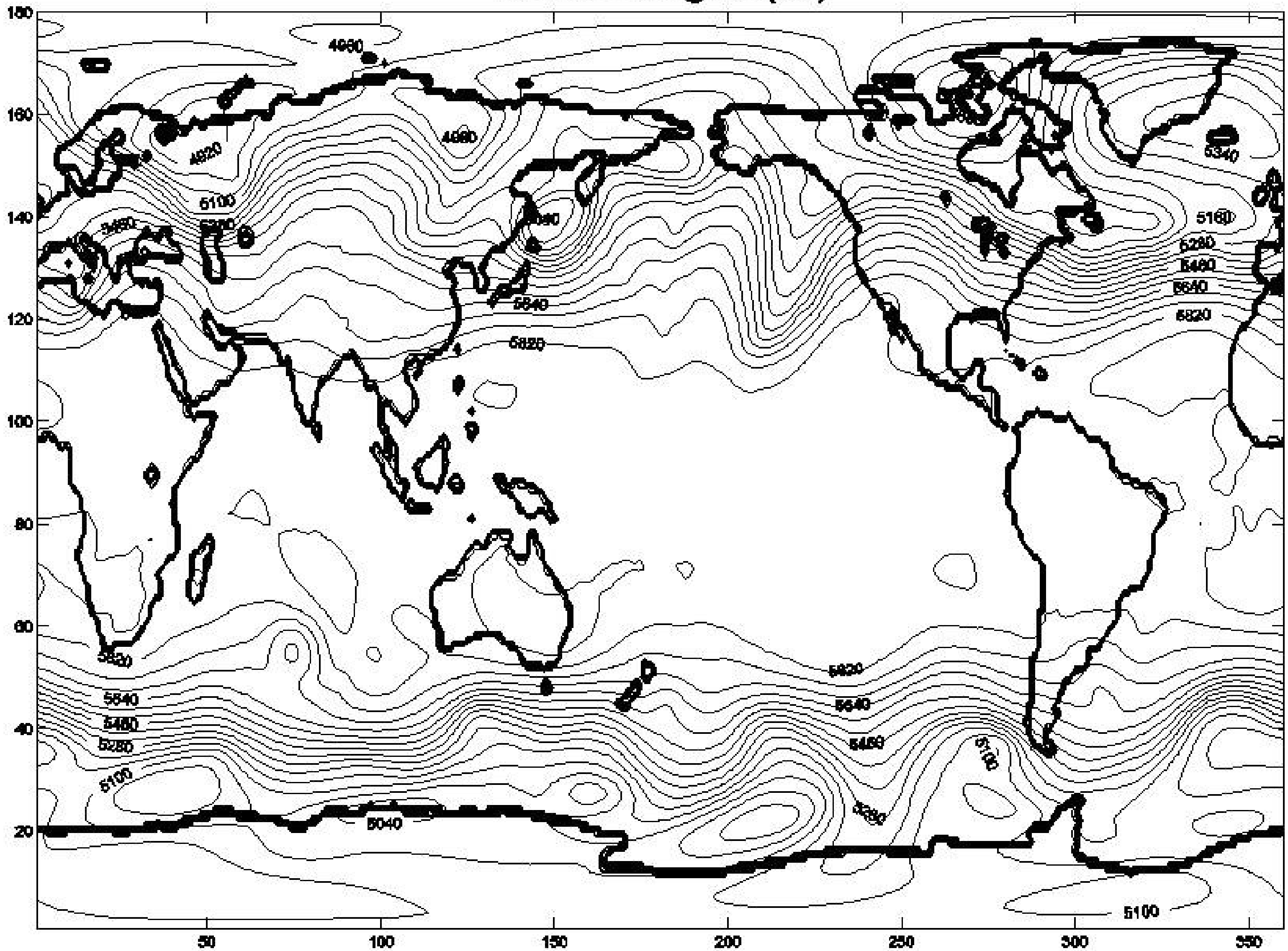
Test Case with 500hPa Data

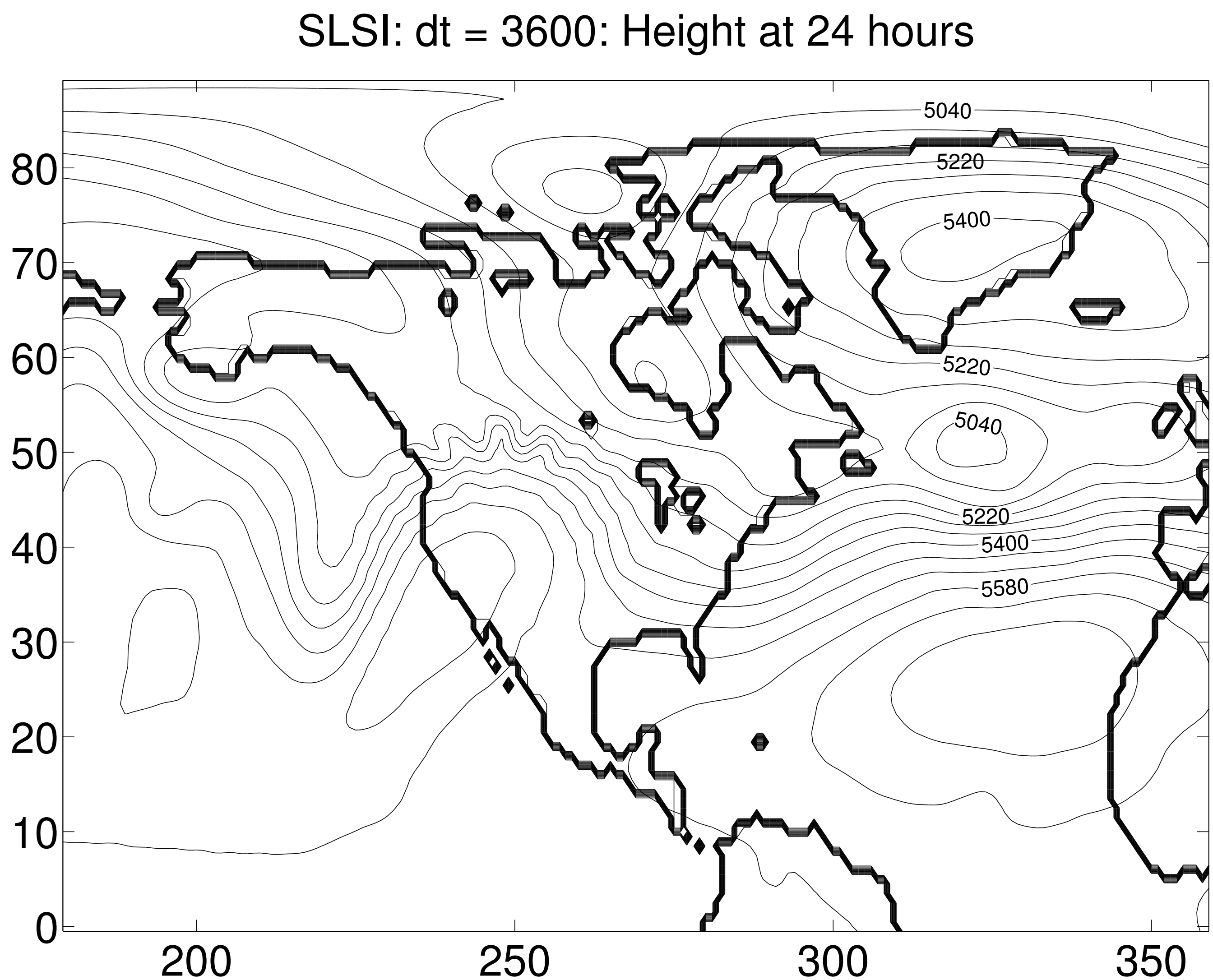
- Initial data:
  ERA-40 analysis of 12 UTC 12th February 1979

- Used by Ritchie & Tanguay (1996) and Li & Bates (1996)

- Running at T119 resolution

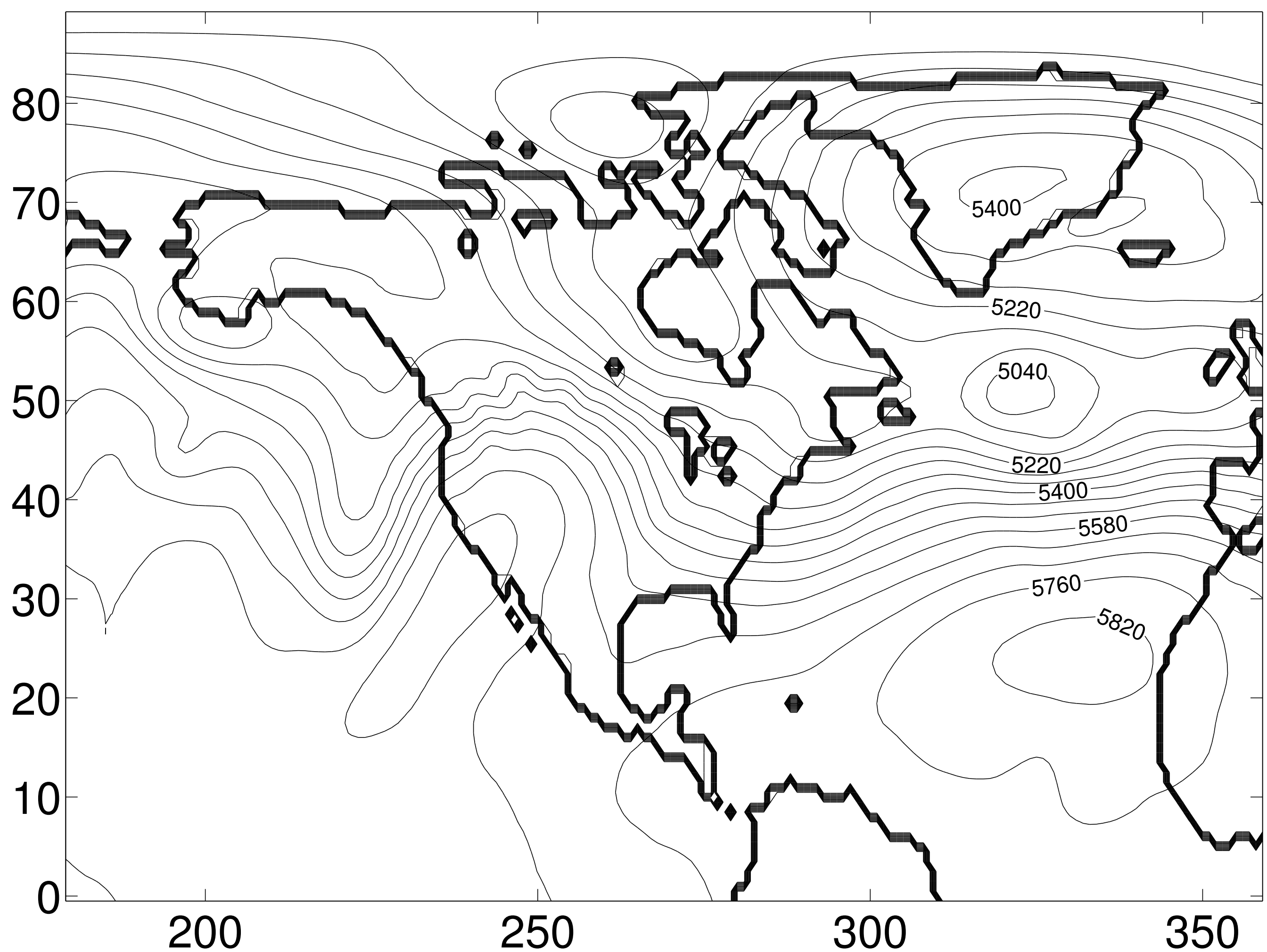
Initial Height (m)



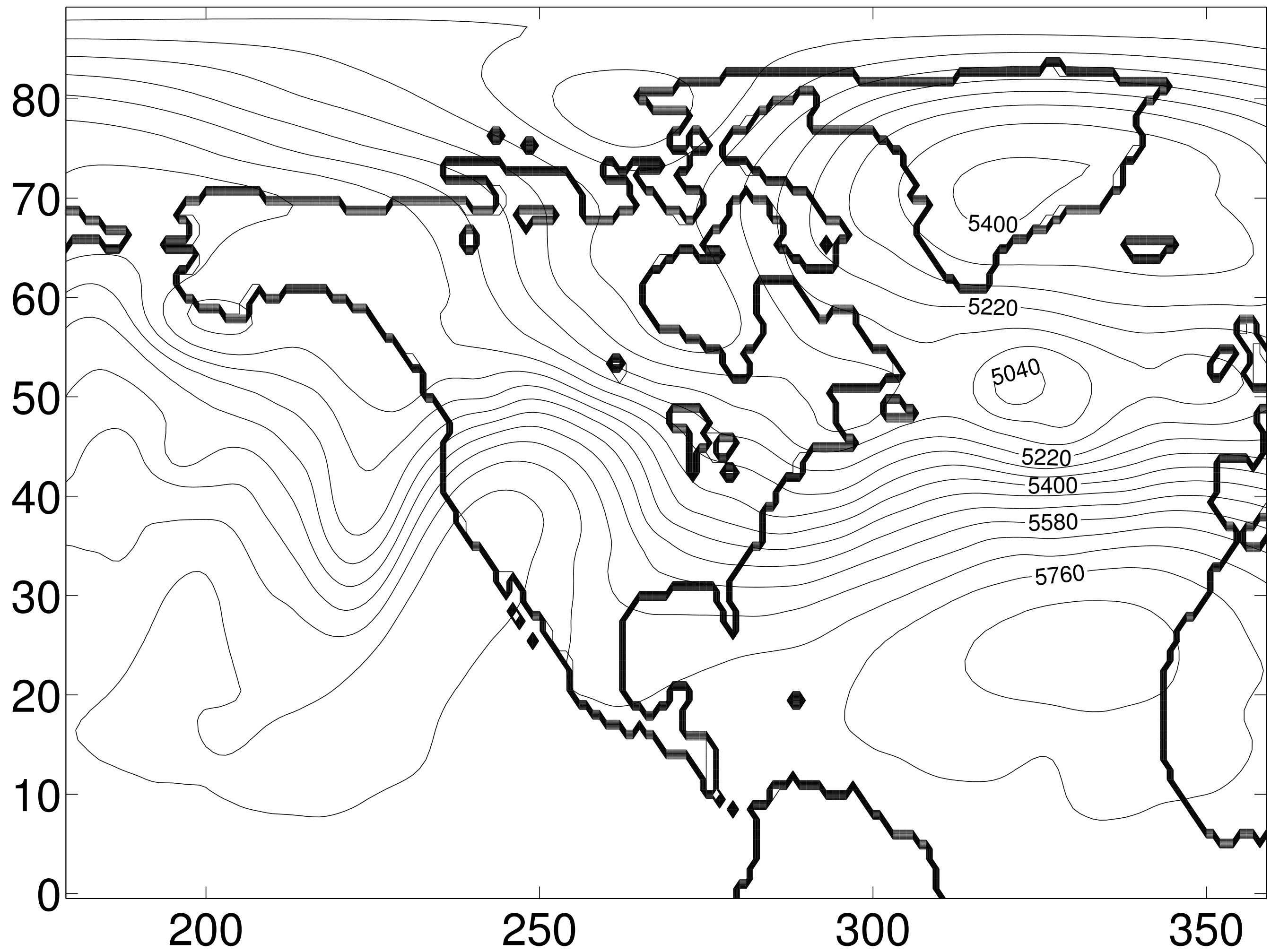


SLSI: dt = 3600: Height at 24 hours
5040
5220
5400
5220
5040
5220
5400
5580
80
70
60
50
40
30
20
10
0
200
250
300
350

SLSI SETTLS: dt = 3600: Height at 24 hours



SLLT: dt = 3600: Height at 24 hours



Summary

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Conclusions

Summary and Conclusions

- LT method tested in the shallow water framework
- Comparable with reference semi-implicit schemes in terms of accuracy and stability
- Additional computational overhead, decreases with increasing resolution
- Advantages:
  - Accurate phase speed
  - No orographic resonance

Next?

- Implementation in a full spectral baroclinic model; filtering benefits may be fully exploited
- Alternative formulations?

Next?

- Non-spectral model?

$$\widehat{\mathbf{X}}(s) \quad = \quad (s\mathbf{I}+\mathbf{L})^{-1}\left[\mathbf{X}^{\tau-1}-\mathbf{N}^{\tau}/s\right]$$

$$\mathbf{X}^{\tau+1} = \mathfrak{L}_N^*\{\ \widehat{\mathbf{X}}(s)\ \}\big|_{t=2\Delta t}$$