A Filtering Laplace Transform Integration Scheme For Numerical Weather Prediction

Colm Clancy
and
Peter Lynch
Meteorology and Climate Centre, School of Mathematical Sciences
University College Dublin
Aim

- To develop a time-stepping scheme that filters high-frequency noise, based on Laplace transform theory

Overview

1: Laplace transform method: background theory

2: Eulerian shallow water model and Kelvin waves

3: Semi-Lagrangian model and orographic resonance
Definitions

Laplace transform: \( f(t), \ t \geq 0 \)

\[
\hat{f}(s) \equiv \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) \, dt
\]

Inverse Transform:

\[
f(t) \equiv \mathcal{L}^{-1}\{\hat{f}\} = \frac{1}{2\pi i} \int_C e^{st} \hat{f}(s) \, ds
\]
A Simple Filtering Example

Consider:

\[ f(t) = ae^{i\omega t} + Ae^{i\Omega t} \]

\[ |\omega| << |\Omega| \]

Laplace Transform:

\[ \hat{f}(s) = \frac{a}{s-i\omega} + \frac{A}{s-i\Omega} \]
Poles in the s-plane

\[ s - plane \]

\[ i \Omega \]

\[ i \omega \]

\[ C \]
Modify the Inversion Contour
Modified Inverse

- Define

\[ f^*(t) \equiv \mathcal{L}^* \{\hat{f}\} = \frac{1}{2\pi i} \oint_{c^*} e^{st} \hat{f}(s) ds \]

- Contribution from frequencies \( \omega < \gamma \) only

- Cauchy’s Integral Formula \( \Rightarrow f^*(t) = ae^{i\omega t} \)
Numerical Inversion
**Numerical Inversion Operator**

\[ f^*(t) \equiv \mathcal{L}^* \{ \hat{f} \} = \frac{1}{2\pi i} \oint_{C^*} e^{st} \hat{f}(s) ds \]

Approximated by:

\[ \mathcal{L}_N^* \{ \hat{f} \} \equiv \frac{1}{2\pi i} \sum_{n=1}^{N} e^{s_n t} \hat{f}(s_n) \Delta s_n \]
Numerical Inversion Operator

\[ \mathcal{L}_N^* \{ \hat{f} \} \equiv \frac{1}{2\pi i} \sum_{n=1}^{N} e^{sn} t \hat{f}(s_n) \Delta s_n \]
Numerical Inversion Operator

\[ \mathcal{L}_N^* \{ \hat{f} \} \equiv \frac{1}{2\pi i} \sum_{n=1}^{N} e^{s_n t} \hat{f}(s_n) \Delta s_n \]

Divide by correction factor:

\[ \kappa = \frac{\tan \frac{\pi}{N}}{\frac{\pi}{N}} \]

Use truncated exponential:

\[ e^{\frac{z}{N}} = \sum_{j=0}^{N-1} \frac{z^j}{j!} \]

\[ \mathcal{L}_N^* \{ \hat{f} \} \equiv \frac{1}{N} \sum_{n=1}^{N} e^{s_n t} \hat{f}(s_n) s_n \]
Numerical Inversion Operator Properties

Symmetry if \( f(t) \) is real:

\[
 f^*(t) \equiv \mathcal{L}_N^* \{ \hat{f} \} = \frac{2}{N} \sum_{n=1}^{N/2} \text{Re} \left\{ s_n \hat{f}(s_n) e^{sn t} \right\}
\]

Inversion is exact for constant function and powers of \( t < N \).
Laplace Transform Properties

Derivatives:

\[ \mathcal{L}\{f'(t)\} = s\hat{f}(s) - f(0) \]

Constants:

\[ \mathcal{L}\{a\} = \frac{a}{s} \]
Filtering a Dynamical System

\[ \frac{dX}{dt} + LX + N(X) = 0 \]

Take the Laplace transform over \([(\tau-1)\Delta t, (\tau+1)\Delta t]\):

\[ s\hat{X} - X^{\tau-1} + L\hat{X} + \frac{N_{\tau}}{s} = 0 \]

\[ \hat{X}(s) = (sI + L)^{-1}[X^{\tau-1} - \frac{N_{\tau}}{s}] \]
Filtered Forecast

\[ \hat{X}(s) = (sI + L)^{-1} \left[ X^{\tau-1} - N^\tau / s \right] \]

\[ X^{\tau+1} = \mathcal{L}_N^* \left\{ \hat{X}(s) \right\} \bigg|_{t=2\Delta t} \]
**Simple Oscillation Equation**

\[
\frac{dX}{dt} = i\omega X
\]

\[
\hat{X}(s) = (s - i\omega)^{-1} X^0
\]

\[
X(\Delta t) = \mathfrak{L}_N^* \{ \hat{X}(s) \} |_{t=\Delta t}
\]

\[
X(\Delta t) = X^0 H_N(\omega) e^{i\omega \Delta t}
\]
Filter Response

\[ X(\Delta t) = H_N(\omega) e^{i\omega \Delta t} \]

\[ H_N(\omega) = \frac{1}{1 + \left( \frac{i\omega}{\gamma} \right)^N} \]
Filter response $H_N = \frac{1}{1 + \omega^N}$

$N = 4$

$N = 8$

$N = 12$

$N = 16$
Stability

Lynch (1986):

\[ \Delta t \leq \frac{(N!)^{1/N}}{2\gamma} \]

Example: \( N = 8, \quad \tau_c = 6 \text{ hours} \Rightarrow \Delta t \leq 1.8 \text{ hours} \)
Applying the Laplace Transform Method to Shallow Water Models
Shallow Water Equations

• Suitable testing framework for numerical methods

• Standard test cases used by modelling community

• Two models: Eulerian and semi-Lagrangian
  Spectral models for efficient solution of Helmholtz equations

• Testing against reference semi-implicit method
Eulerian STSWM


- Updated by ICON - http://icon.enes.org/

- Spectral transform method

- Centred time-differencing, semi-implicit scheme

- Test cases proposed by Williamson et al (1992)
Spectral Solution

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} &= -\nabla . (\zeta + f) \mathbf{v} \\
\frac{\partial \delta}{\partial t} &= \mathbf{k} \cdot \nabla \times (\zeta + f) \mathbf{v} - \nabla^2 (\Phi + \frac{\mathbf{v} \cdot \mathbf{v}}{2}) \\
\frac{\partial \Phi^*}{\partial t} &= -\nabla . (\Phi^* \mathbf{v})
\end{align*}
\]

Expand each field: \( \zeta(\lambda, \mu, t) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \zeta^m_\ell(t) P^m_\ell(\mu) e^{im\lambda} \)
Spectral Solution

Get system of ODEs for the spectral coefficients:

\[
\frac{d}{dt} \eta^m_\ell = N^m_\ell
\]

\[
\frac{d}{dt} \delta^m_\ell = D^m_\ell + \frac{\ell(\ell + 1)}{a^2} \Phi^m_\ell
\]

\[
\frac{d}{dt} \Phi^m_\ell = F^m_\ell - \Phi^* \delta^m_\ell
\]
Reference Semi-Implicit Scheme

\[
\frac{\{\eta^m_\ell\}^{\tau+1} - \{\eta^m_\ell\}^{\tau-1}}{2 \Delta t} = \{N^m_\ell\}^\tau
\]

\[
\frac{\{\delta^m_\ell\}^{\tau+1} - \{\delta^m_\ell\}^{\tau-1}}{2 \Delta t} = \{D^m_\ell\}^\tau + \frac{\ell(\ell + 1)}{a^2} \frac{\{\Phi^m_\ell\}^{\tau+1} + \{\Phi^m_\ell\}^{\tau-1}}{2}
\]

\[
\frac{\{\Phi^m_\ell\}^{\tau+1} - \{\Phi^m_\ell\}^{\tau-1}}{2 \Delta t} = \{F^m_\ell\}^\tau - \bar{\Phi}^* \frac{\{\delta^m_\ell\}^{\tau+1} + \{\delta^m_\ell\}^{\tau-1}}{2}
\]
Laplace Transform Scheme

\[ s \hat{\eta}_\ell^m - \left\{ \eta_\ell^m \right\}^{\tau-1} = \frac{1}{s} \left\{ \mathcal{N}_\ell^m \right\}^\tau \]

\[ s \hat{\delta}_\ell^m - \left\{ \delta_\ell^m \right\}^{\tau-1} = \frac{1}{s} \left\{ \mathcal{D}_\ell^m \right\}^\tau + \frac{\ell(\ell + 1)}{a^2} \hat{\Phi}_\ell^m \]

\[ s \hat{\Phi}_\ell^m - \left\{ \Phi_\ell^m \right\}^{\tau-1} = \frac{1}{s} \left\{ \mathcal{F}_\ell^m \right\}^\tau - \Phi^* \delta_\ell^m \]
Comparing the Discretisations

\[ s \hat{\eta}_\ell^m = \{\eta_\ell^m\}^\tau - 1 + \frac{1}{s} \{N_\ell^m\}^\tau \]

\[ s \hat{\delta}_\ell^m = \left[1 + \hat{\Phi}_\ell^* \frac{\ell(\ell + 1)}{a^2} \frac{1}{s^2}\right]^{-1} \left(\mathcal{R}' + \frac{1}{s} \frac{\ell(\ell + 1)}{a^2} \mathcal{Q}'\right) \]

\[ s \hat{\Phi}_\ell^m = \left[1 + \hat{\Phi}_\ell^* \frac{\ell(\ell + 1)}{a^2} \frac{1}{s^2}\right]^{-1} \left(\mathcal{Q}' - \frac{1}{s} \hat{\Phi}_\ell^* \mathcal{R}'\right) \]

\[ \mathcal{R}' = \{\delta_\ell^m\}^\tau - 1 + \frac{1}{s} \{D_\ell^m\}^\tau \]

\[ \mathcal{Q}' = \{\Phi_\ell^m\}^\tau - 1 + \frac{1}{s} \{F_\ell^m\}^\tau \]

\[ \{\eta_\ell^m\}^{\tau + 1} = \{\eta_\ell^m\}^{\tau - 1} + 2 \Delta t \{N_\ell^m\}^\tau \]

\[ \{\delta_\ell^m\}^{\tau + 1} = \left[1 + \hat{\Phi}_\ell^* \frac{\ell(\ell + 1)}{a^2} \Delta t^2\right]^{-1} \left(\mathcal{R} + \mathcal{Q} \frac{\ell(\ell + 1)}{a^2} \Delta t\right) \]

\[ \{\Phi_\ell^m\}^{\tau + 1} = \left[1 + \hat{\Phi}_\ell^* \frac{\ell(\ell + 1)}{a^2} \Delta t^2\right]^{-1} \left(\mathcal{Q} - \mathcal{R} \hat{\Phi}_\ell^* \Delta t\right) \]

\[ \mathcal{R} = \{\delta_\ell^m\}^{\tau - 1} + 2 \Delta t \{D_\ell^m\}^\tau + \Delta t \frac{\ell(\ell + 1)}{a^2} \{\Phi_\ell^m\}^{\tau - 1} \]

\[ \mathcal{Q} = \{\Phi_\ell^m\}^{\tau - 1} + 2 \Delta t \{F_\ell^m\}^\tau + \Delta t \hat{\Phi}_\ell^* \{\delta_\ell^m\}^{\tau - 1} \]
Test Cases: Williamson et al (1992)

Case 1: Advection of a cosine bell by constant winds

Case 2: Steady zonal flow

Case 5: Flow over an isolated mountain

Case 6: Rossby-Haurwitz wave
Results

• Comparable accuracy and conservation with reference model

• In general, increasing number of points in inversion operator, $N$, from 8 to 16 did not significantly improve accuracy
Sample Results

Case 5
Flow Over a Mountain: Relative Errors

Case 5: T42 dt = 1200s

Max Error Norm

Hours

Reference
LT N=8
LT N=16

13 May 2011
Kelvin Waves

• Eigenfunctions of the linearised shallow water equations

• Dynamically important

• Semi-implicit methods slow down waves
Phase Error Analysis

Oscillation equation: \[
\frac{du}{dt} = i \nu u
\]

Look for: \[u^{\tau+1} = A u^\tau\]

Numerical phase: \[\theta = \tan^{-1} \left( \frac{\text{Im}(A)}{\text{Re}(A)} \right)\]

Relative Phase Change: \[R = \frac{\theta}{\nu \Delta t}\]

13 May 2011

RPN Seminar
**Phase Error Analysis**

\[
R_{SI} \approx 1 - \frac{(\nu \Delta t)^2}{12}
\]

\[
R_{LT} \approx 1 + \frac{N}{(N + 1)!} (\nu \Delta t)^N
\]
Kelvin Wave Relative Phase Error

\[ R \]

- SI \( m = 1 \)
- SI \( m = 5 \)
- LT \( m = 1 \)
- LT \( m = 5 \)
T63 $dt = 1800$ $tc = 3$
Semi-Lagrangian
Shallow Water Model
Semi-Lagrangian Laplace Transform

- Define the LT along a trajectory

\[ \hat{\zeta}(s) \equiv \mathcal{L}\{\zeta\} = \int e^{-st} \zeta \, dt \]

- Then

\[ \mathcal{L}\left\{ \frac{d\zeta}{dt} \right\} = s \hat{\zeta} - \zeta^n_D \]
Semi-Lagrangian Laplace Transform
SLLT

- Based on spectral SWEmodel (John Drake, ORNL)

- Compared with semi-Lagrangian semi-implicit SLSI
Shallow Water Equations

\[
\frac{d\zeta}{dt} + f\delta + \beta v = N_\zeta \\
\frac{d\delta}{dt} - f\zeta + \beta u + \nabla^2 \Phi = N_\delta \\
\frac{d\Phi}{dt} - \frac{d\Phi_s}{dt} + \Phi\delta = N_\Phi
\]
**SLLT Discretisation**

General evolution equation:

\[ \frac{dX}{dt} + L = N \]

SLLT:

\[ s \hat{X} - X^n_D + \hat{L} = \frac{1}{s} N^{n+\frac{1}{2}}_M \]
SLSI Discretisations

SLSI: \[ \frac{X_A^{n+1} - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} = N_M^{n+\frac{1}{2}} \]

SLSI SETTLLS, (Hortal, 2002):
\[ \frac{X_A^{n+1} - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} = \frac{1}{2} \left\{ (2N_D^n - N_D^{n-1}) + N_A^n \right\} \]
Departure Point Calculations

- Two time level scheme
- Trajectories calculated in spherical coordinates
  (Ritchie and Beaudoin, 1994)
- Bilinear interpolation when computing departure points
- Bicubic for model fields
- Extrapolation used for computing midpoint values
Stability

**SLSI:** Require $\Phi \geq \Phi_{\text{max}}$ (Côté and Staniforth, 1988)

**SLLT:** Stability not dependent on $\Phi$
Sample Results

Case 5
Flow Over a Mountain: Relative Errors

Case 5: T119

Graph showing relative errors over time for different cases and time steps.


Flow Over a Mountain: Relative Errors

Case 5: T119 dt = 3600s

Max Error Norm vs Hours for different methods:
- SLSI
- SLSI SETTLS
- SLLT
- SLLT SETTLS

Graph showing the maximum error norm over time for different methods in a mountain flow scenario.
Symmetry

Can use symmetry in inversion operator:

$$\mathcal{L}_N^* \{ \hat{f} \} \equiv \frac{1}{N} \sum_{n=1}^{N} e^{s_n t} \hat{f}(s_n) s_n$$

\[\downarrow\]

$$\mathcal{L}_N^* \{ \hat{f} \} = \frac{2}{N} \sum_{n=1}^{N/2} Re \left\{ s_n \hat{f}(s_n) e^{s_n t} \right\}$$
Efficiency

Relative Overhead of SLLT

\[ \frac{T(\text{SLLT})}{T(\text{SLSI})} \]

Spectral truncation
Orographic Resonance
Orographic Resonance

- Spurious resonance from coupling semi-Lagrangian and semi-implicit methods
  [reviewed in Lindberg & Alexeev (2000)]

- LT method has benefits over semi-implicit schemes

- Motivates investigating orographic resonance in SLLT model
Orographic Resonance Analysis

- Linear analysis of orographically forced stationary waves, following Ritchie & Tanguay (1996)

- Numerical simulations with shallow water SLLT

- Results consistently show benefits of SLLT scheme
Linear Analysis: $R = \frac{(\text{Numerical})}{(\text{Analytic})}$

Figure 4. The numerical response to orographic forcing divided by the physical response for (a) SLSI and (b) SLLT, with $T213$, $\Delta t = 7200$ s, $N = 16$ and $\tau_c = 3$ h. At $R = 1$ (dashed line), the numerical solution equals the analytic solution, and at $R = 0$ (dot-dashed line), the numerical solution is zero due to filtering. The extremes at $m\Delta \lambda / \pi \approx 0.04$ are artefacts, due to the vanishing of the physical solution.
Test Case with 500hPa Data

- Initial data:
  ERA-40 analysis of 12 UTC 12\textsuperscript{th} February 1979

- Used by Ritchie & Tanguay (1996) and Li & Bates (1996)

- Running at T119 resolution
SLSI SETTLS: $dt = 3600$: Height at 24 hours
SLLT: \( dt = 3600 \): Height at 24 hours
Summary

and

Conclusions
Summary and Conclusions

- LT method tested in the shallow water framework
- Comparable with reference semi-implicit schemes in terms of accuracy and stability
- Additional computational overhead, decreases with increasing resolution
- Advantages:
  - Accurate phase speed
  - No orographic resonance
Next?

- Implementation in a full spectral baroclinic model; filtering benefits may be fully exploited

- Alternative formulations?
Next?

- Non-spectral model?

\[
\hat{X}(s) = (sI + L)^{-1} \left[ X^{\tau-1} - N^{\tau}/s \right]
\]

\[
X^{\tau+1} = \mathcal{L}_N^* \left\{ \hat{X}(s) \right\}_{|t=2\Delta t}
\]