A mass-conserving semi-implicit semi-Lagrangian scheme for the shallow water equations on the sphere

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Overview

- Motivations/Aims
- Coupling conservative scheme to momentum equations
- Other aspects of the SW model
- Results
- Conclusions

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Motivations (1)

- **Mass conservation considered important for climate simulations**
- **Inherent conservation more accurate than *ad hoc* corrections**
- **Standard semi-Lagrangian (SL) schemes non-conservative**

Motivations (2)

Currently the Unified Model (the New Dynamics)

- **Uses *Eulerian flux* form of continuity equation**
- **Mixed SL/Eulerian approach**
- **Inconsistent treatment of equations**

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Aims of the work

Primary Aim = Develop a global shallow-water model that:

- **Inherently conserves mass;**
- **Stable (unconditional (on Δt) stability);**
- **Consistent treatment of all equations;**
- **At least as accurate as present SISL scheme;**
- **Not a major departure from our existing model (evolution not revolution)**

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Continuous Continuity equations

Standard (Lagrangian) shallow-water continuity equation:

$$\frac{D\Phi}{Dt} + \Phi\nabla \cdot \mathbf{u} = 0$$

We use the more primitive integral (conservative) form:

$$\frac{D}{Dt}\left(\int_{\delta\mathcal{A}} \Phi d\mathcal{A}\right) = 0$$

Now apply SL scheme but key to stability is to couple directly to momentum equations

Reference "profile"

Define a reference depth $\Phi^{\text{ref}} = \text{constant}$ **and rewrite as:**

$$\frac{D}{Dt}\left[\int_{\delta\mathcal{A}} \left(\Phi - \Phi^{\text{ref}}\right) d\mathcal{A}\right] + \frac{D}{Dt}\left(\int_{\delta\mathcal{A}} \Phi^{\text{ref}} d\mathcal{A}\right) = 0$$

or as

$$\frac{D}{Dt}\left[\int_{\delta\mathcal{A}} \left(\Phi - \Phi^{\text{ref}}\right) d\mathcal{A}\right] + \Phi^{\text{ref}}\left(\int_{\delta\mathcal{A}} \nabla\cdot\mathbf{u}\, d\mathcal{A}\right) = 0$$

Key aspect is that have now pulled off a divergence term to be treated semi-implicitly.

Temporal discretization (1)

Integrating along the trajectory gives the *exact* result

$$\left[\int_{\delta\mathcal{A}} \left(\Phi - \Phi^{\mathrm{ref}}\right) d\mathcal{A}\right]_A^{n+1} - \left[\int_{\delta\mathcal{A}} \left(\Phi - \Phi^{\mathrm{ref}}\right) d\mathcal{A}\right]_D^{n}$$

$$+ \ \Delta t \Phi^{\mathrm{ref}} \int_{\mathrm{traj}} \left(\int_{\delta\mathcal{A}} \nabla \cdot \mathbf{u} d\mathcal{A}\right) dt = 0$$

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Temporal discretization (2)

Approximating the time integral along trajectory using weighted averages and grouping terms, gives:

$$\left\{ \int_{\delta\mathcal{A}} \left[\left(\Phi - \Phi^{\mathrm{ref}} \right) + \alpha \Delta t \Phi^{\mathrm{ref}} \nabla \cdot \mathbf{u} \right] d\mathcal{A} \right\}_{A}^{n+1} =$$

$$\left\{ \int_{\delta\mathcal{A}} \left[\left(\Phi - \Phi^{\mathrm{ref}} \right) - \beta \Delta t \Phi^{\mathrm{ref}} \nabla \cdot \mathbf{u} \right] d\mathcal{A} \right\}_{D}^{n}$$

Temporal discretization (3)

Using a control volume approach $\left\{\int_{\delta\mathcal{A}} \Phi d\mathcal{A}\right\}_A^n \simeq \Phi_A^n \Delta\mathcal{A}$ **(**$\Delta\mathcal{A} \equiv$ **arrival area):**

$$\left[\left(\Phi - \Phi^{\text{ref}}\right) + \alpha\Delta t\Phi^{\text{ref}}\nabla\cdot\mathbf{u}\right]_A^{n+1} = \frac{1}{\Delta\mathcal{A}}\left\{\int_{\delta\mathcal{A}}\left[\left(\Phi - \Phi^{\text{ref}}\right) - \beta\Delta t\Phi^{\text{ref}}\nabla\cdot\mathbf{u}\right] d\mathcal{A}\right\}_D^n$$

Conservation assured if RHS integration is conservative

(here use SLICE)

Compare with Non-conservative SL implementation

Starting point:

$$\frac{D\left(\Phi - \Phi^{\text{ref}}\right)}{Dt} + \Phi^{\text{ref}}\nabla \cdot \mathbf{u} = -\left(\Phi - \Phi^{\text{ref}}\right)\nabla \cdot \mathbf{u}$$

Integrate along trajectory and approximate time integrals:

$$\left[\left(\Phi - \Phi^{\text{ref}}\right) + \alpha\Delta t\Phi^{\text{ref}}\nabla \cdot \mathbf{u}\right]_A^{n+1} = \left[\left(\Phi - \Phi^{\text{ref}}\right) - \beta\Delta t\Phi\nabla \cdot \mathbf{u}\right]_D^{n} - \alpha\Delta t\left[\left(\Phi - \Phi^{\text{ref}}\right)\nabla \cdot \mathbf{u}\right]_A^{n+1}$$

Other aspects of model

- **Velocity handled via (shallow) rotation matrix**

$$\left(\mathbf{u} - \alpha \Delta t \mathbf{\Psi}\right)_A^{n+1} = \left[\mathbf{\Lambda}\left(\mathbf{u} + \beta\,\Delta t\,\mathbf{\Psi}\right)_{D_L}^{n}\right]$$

- **Coriolis terms handled as in (Thuburn & Staniforth, 2004), eg:**

$$-2\left(\mathbf{\Omega}_r \times \mathbf{u}\right)_u = \frac{1}{a\ \cos\phi\Delta\lambda}\left\langle \langle\widetilde{v}\rangle^{\phi}\frac{f}{\Phi}\right\rangle^{\lambda}$$

where

$$\widetilde{v} \equiv a\ \cos\phi\Delta\lambda\,\overline{\Phi}^{\phi} v$$

is a mass flux variable

$$f \equiv 2\Omega\ \sin\phi$$

Departure points calculations

$$\frac{D\mathbf{x}}{Dt} = \mathbf{u}$$

discretized as

$$(\mathbf{x} + \beta_x \Delta t\, \mathbf{u})_D^n = (\mathbf{x} - \alpha_x \Delta t\, \mathbf{u})_A^{n+1}$$

and solved as

$$X_D^{(k)} = -\gamma \Delta t \left[\alpha_x U_A^{n+1} + \beta_x U^n \left(\lambda_D^{(k-1)}, \phi_D^{(k-1)} \right) \right]$$

$$Y_D^{(k)} = -\gamma \Delta t \left[\alpha_x V_A^{n+1} + \beta_x V^n \left(\lambda_D^{(k-1)}, \phi_D^{(k-1)} \right) \right]$$

where

$$\begin{pmatrix} U \\ V \end{pmatrix} = \mathbf{\Lambda} \begin{pmatrix} u \\ v \end{pmatrix}$$

Unified implementation (conservative or not)

For non-(or) conservative options, the discrete equations are:

$$\left[\left(\Phi - \Phi^{\mathrm{ref}}\right) + \alpha \Delta t \Phi^{\mathrm{ref}} \nabla \cdot \mathbf{u}\right]^{n+1} = R_{\Phi} \qquad (1)$$

$$\left[\mathbf{u} - \alpha \Delta t \boldsymbol{\Psi}\right]^{n+1} = \boldsymbol{\Lambda}\left(\mathbf{u} + \beta\, \Delta t\, \boldsymbol{\Psi}\right)^{n}_{D_L} \qquad (2)$$

where,

$$R_{\Phi} = \begin{cases} \frac{1}{\Delta \mathcal{A}} \left\{ \int_{\delta \mathcal{A}} \left[\left(\Phi - \Phi^{\mathrm{ref}}\right) - \beta \Delta t \Phi^{\mathrm{ref}} \nabla \cdot \mathbf{u}\right] d\mathcal{A} \right\}^{n}_{D}, & \text{for SLICE} \\ \left[\left(\Phi - \Phi^{\mathrm{ref}}\right) - \beta \Delta t \Phi \nabla \cdot \mathbf{u}\right]^{n}_{D} - \alpha \Delta t \left[\left(\Phi - \Phi^{\mathrm{ref}}\right) \nabla \cdot \mathbf{u}\right]^{*}_{A}, & \text{for SL} \end{cases}$$

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The Helmholtz problem

Irrespective of the options (conservative or not), the discrete equations are:

$$\left[\mathbf{u} - \alpha \Delta t \mathbf{\Psi}\right]^{n+1} = R_u$$

$$\left[\left(\Phi - \Phi^{\text{ref}}\right) + \alpha \Delta t \Phi^{\text{ref}} \nabla \cdot \mathbf{u}\right]^{n+1} = R_\Phi$$

which are combined to obtain the Helmholtz problem for the geopotential perturbation $\Phi' = \Phi - \Phi^{\text{ref}}$**,**

$$h_1 \delta_\lambda \left(h_2 \delta_\lambda \Phi'\right) + h_1 \delta_\phi \left(h_3 \delta_\phi \Phi'\right) - \Phi' = RHS(R_u, R_\Phi) \qquad (3)$$

where $h_{j=1,3} = h_j\left(\alpha,\ \Delta t,\ a,\ \cos(\phi),\ \Phi^{\text{ref}}\right)$ **are invariant coefficients for a fixed grid in time. Eq. (3) is solved using the BiCGstab (Van der Vorst, SIAM J. Sci. Stat. Comput., 1992).**

The iterative solution procedure

Do time-step loop:

- **given** $(u, v, \Phi)^n$ **at level** n

  Do outer-loop iteration (departure loop):

  - **compute** (λ_D^n, ϕ_D^n) **using** $(u, v)^n$ **and latest** $(u, v)^{n+1}$
  - **evaluate the departure terms** R_D^n

    Do inner-loop iteration (Coriolis loop):

    - **evaluate Coriolis and nonlinear terms** $R^* = R^{n+1}$
    - **solve Helmholtz problem for** Φ^{n+1}
    - **update** $(u, v)^{n+1}$

    Enddo

  Enddo

Enddo

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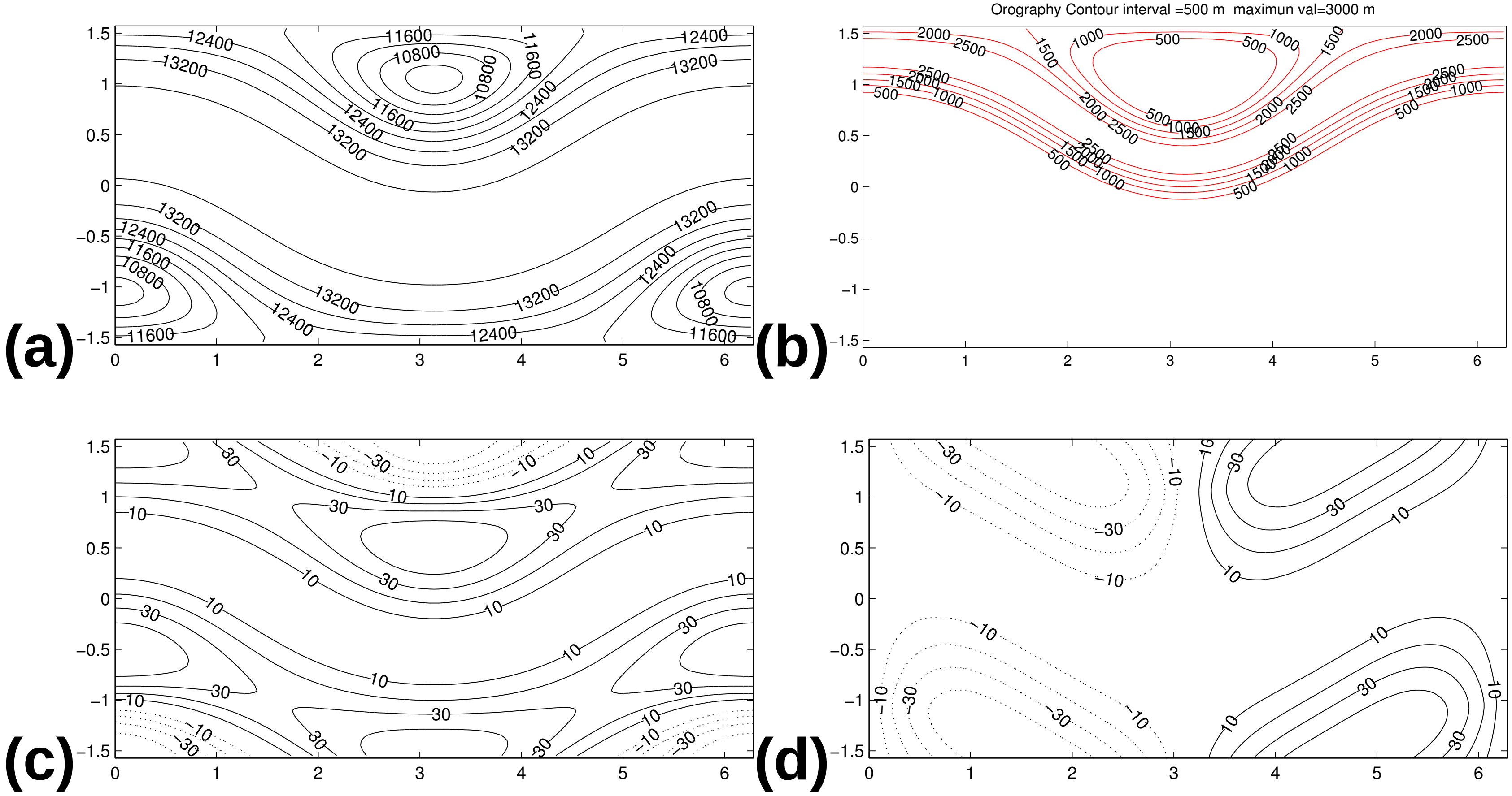
Test cases

1. Stationary jets over zonal orography (Staniforth & White, 2007)
2. Exact unsteady flow (Lauter et al., 2005)
3. Flow over an isolated mountain; test case 5 of (Williamson et al., 1992)

Stationary jets over zonal orography - Initial Fields



(a) $\left(\Phi + \Phi^S\right)/g\ [400\,\mathrm{m}]$; **(b)** $\Phi^S/g\ [500\,\mathrm{m}]$; **(c)** $u\ \left[10\,\mathrm{m\,s^{-1}}\right]$; **(d)** $v\ \left[10\,\mathrm{m\,s^{-1}}\right]$.

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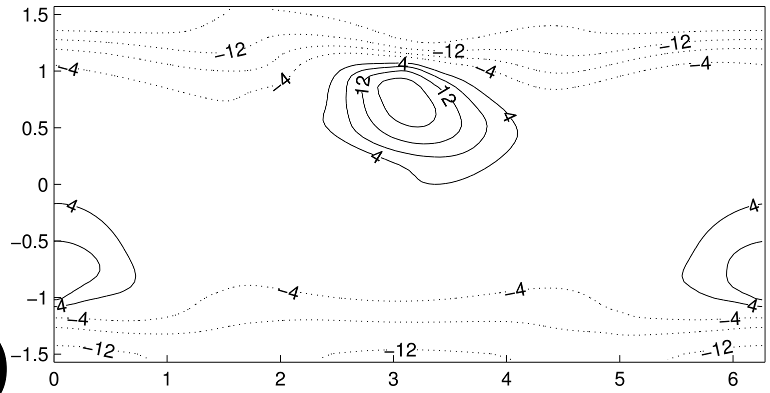
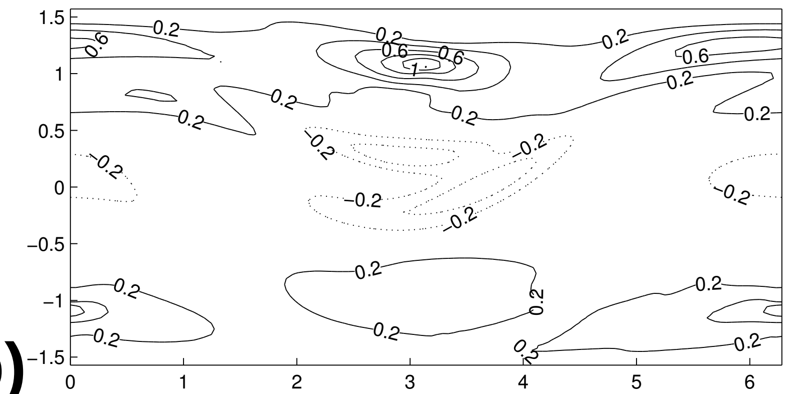
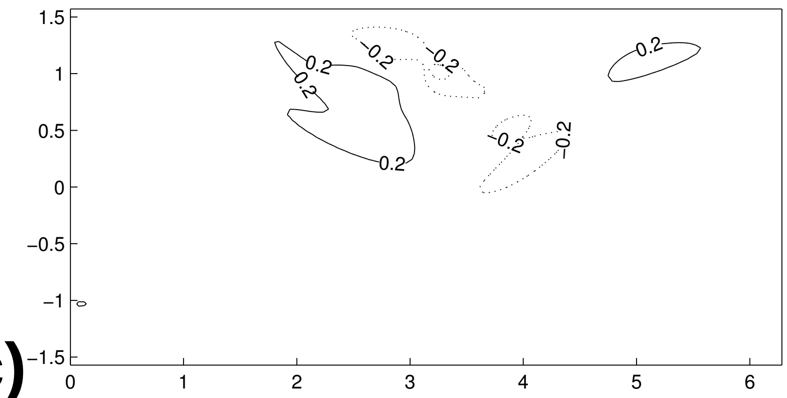
Stationary jets - Error Fields after 5 days


(a)

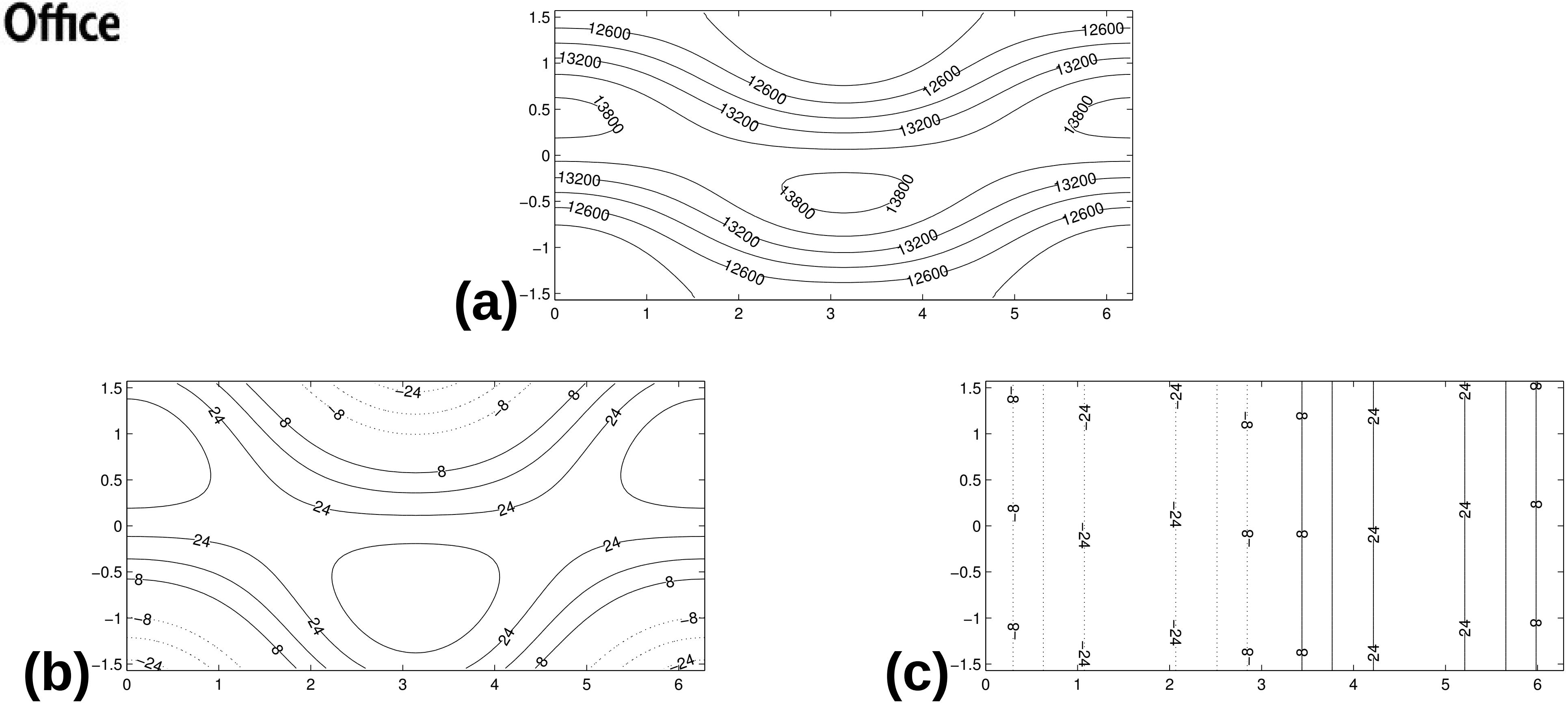

(b)


(c)

Resolution 128x64; $\Delta t =$1h;

(a) $\left(\Phi + \Phi^S\right)/g\ [4\,\mathrm{m}]$**; (b) & (c)** u **&** v $\left[0.2\,\mathrm{m\,s^{-1}}\right]$

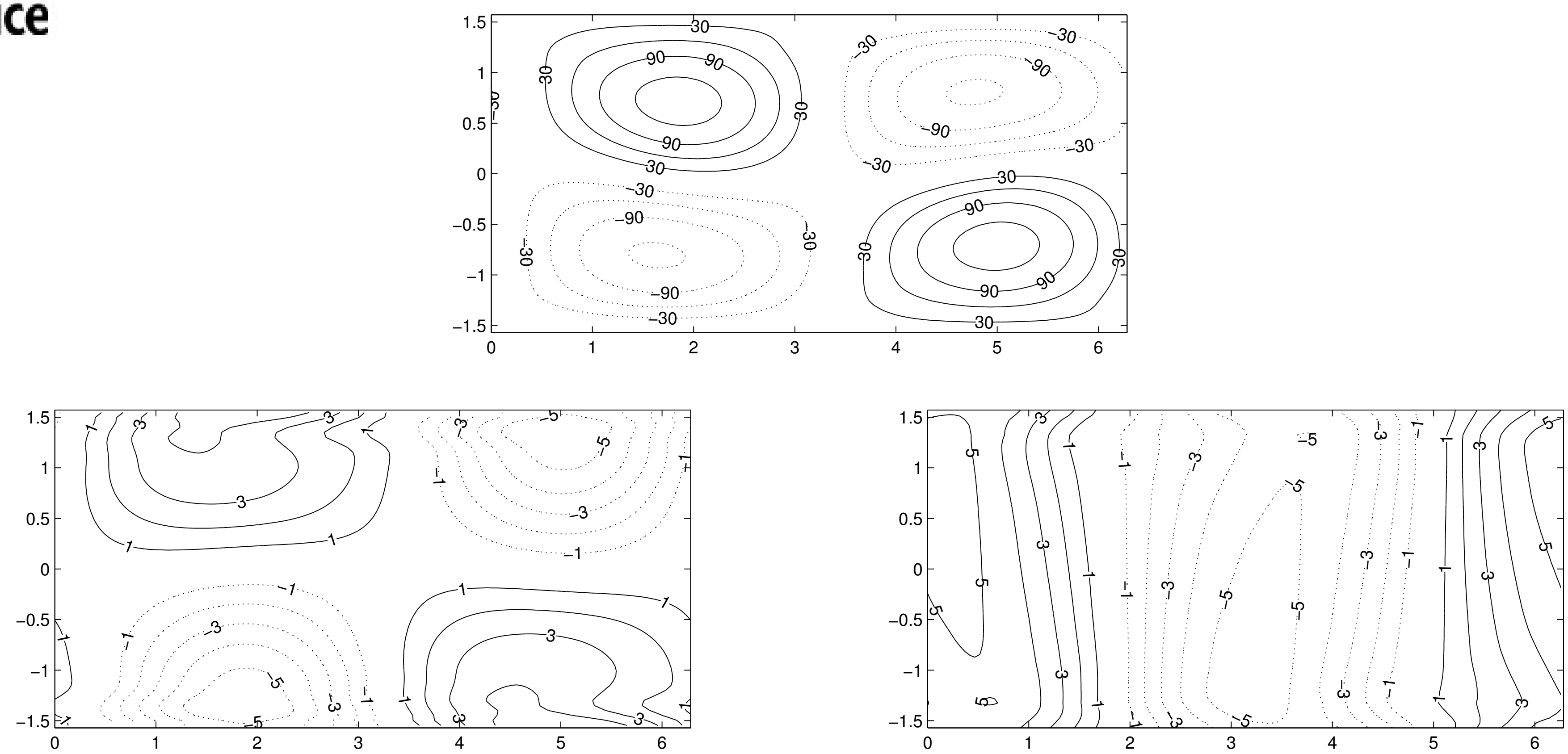
Exact Unsteady Flow - Initial Fields



(a) $\left(\Phi + \Phi^S\right)/g\ [400\,\mathrm{m}]$;

(b) $u\ \left[10\,\mathrm{m\,s^{-1}}\right]$; **(c)** $v\ \left[10\,\mathrm{m\,s^{-1}}\right]$.

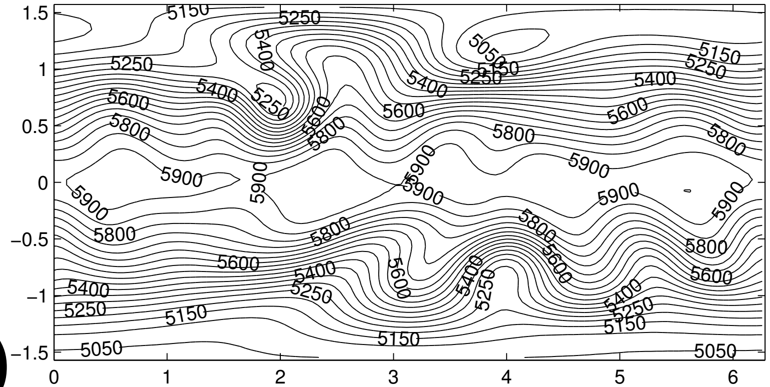
Exact Unsteady Flow - Error Fields 5 Days



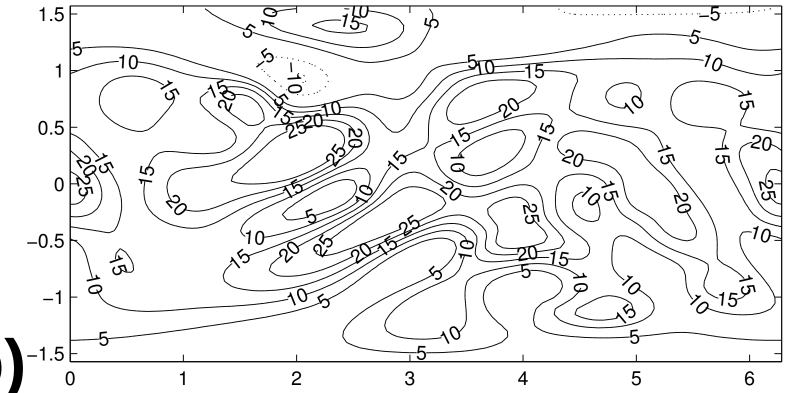
$$\Delta t = 1\,\mathrm{h}\text{; } (I = 128, J = 64)\text{.}$$

Contour intervals: $30\,\mathrm{m}$ **for** $\left(\Phi + \Phi^S\right)/g$**;** $1\,\mathrm{m\,s^{-1}}$ **for** u **and** v**.**

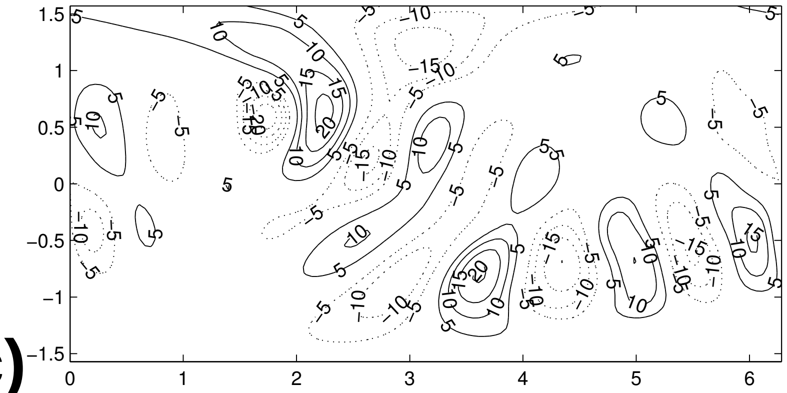
Williamson Test Case 5 - Fields 15 Days



(a)



(b)



(c)

Mass-conserving $\Delta t = 600$ **s;**(128×64)**: (a)** $\left(\Phi + \Phi^S\right)/g\ [50\,\mathrm{m}]$**;**

(b) $u\ \left[5\,\mathrm{m\,s^{-1}}\right]$**; (c)** $v\ \left[5\,\mathrm{m\,s^{-1}}\right]$**.**

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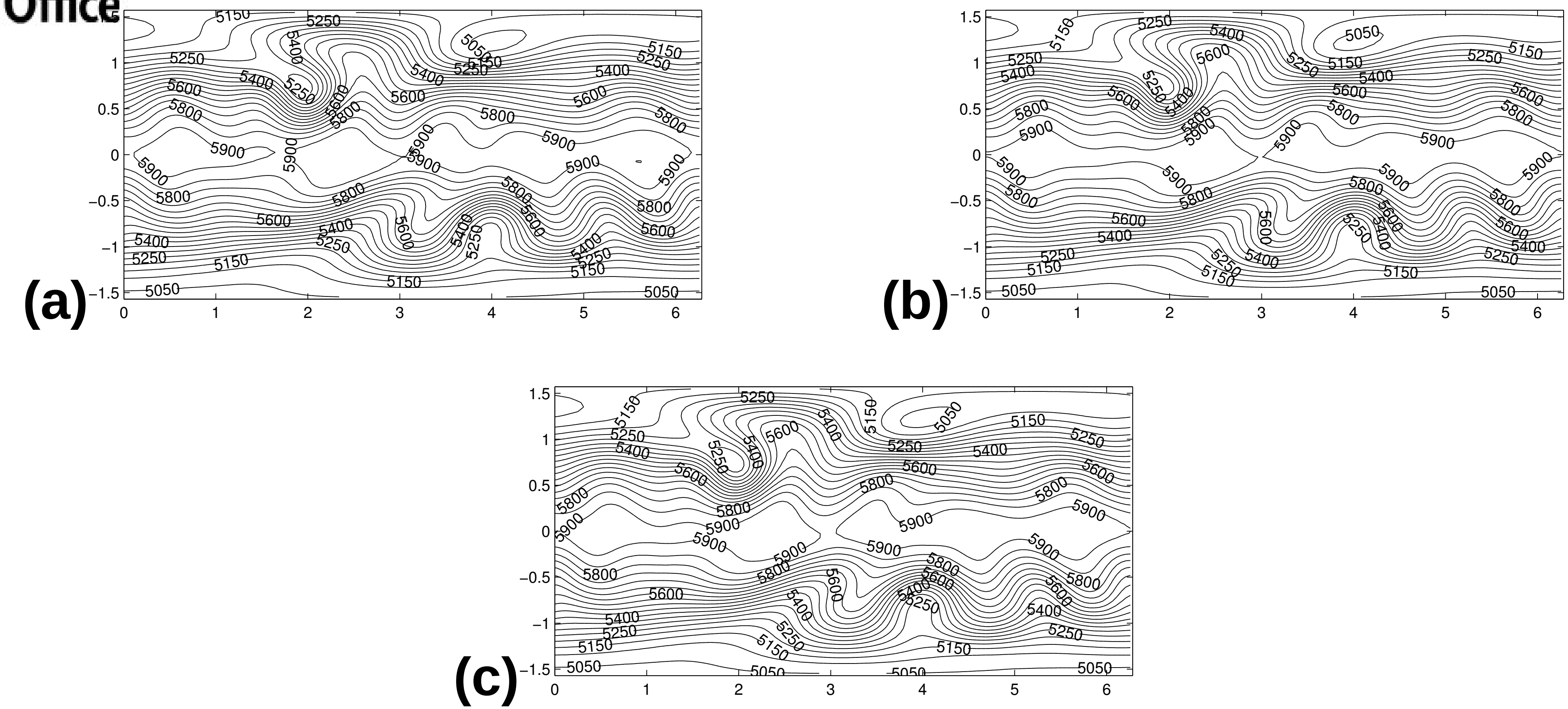
Williamson Test Case 5 - Height Fields 15 Days



a) Mass-conserving $\Delta t = 600$ s; b) $\Delta t = 6000$ s;

c) Non-conserving model $\Delta t = 6000$ s.

Comparison with SISL - L2 Error Norms

Stationary jets over zonal orography

$I \times J$	Δt(mn)	SLICE	SL	SLICE	SL
		$L_2(h)$	$L_2(h)$	$L_2(\mathbf{v})$	$L_2(\mathbf{v})$
64×32	**12**	**0.165E-02**	**0.178E-02**	**0.333E-01**	**0.363E-01**
128×64	**6**	**0.406E-03**	**0.464E-03**	**0.839E-02**	**0.946E-02**
256×128	**3**	**0.989E-04**	**0.116E-03**	**0.208E-02**	**0.238E-02**

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Comparison with SISL - L2 Error Norms

Exact Unsteady Flow

$I \times J$	Δt (mn)	SLICE	SL	SLICE	SL
		$L_2(h)$	$L_2(h)$	$L_2(\mathbf{v})$	$L_2(\mathbf{v})$
64×32	**12**	**0.176E-02**	**0.179E-02**	**0.428E-01**	**0.429E-01**
128×64	**6**	**0.447E-03**	**0.447E-03**	**0.114E-01**	**0.114E-01**
256×128	**3**	**0.113E-03**	**0.111E-03**	**0.289E-02**	**0.285E-02**

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Reducing the dependency on the reference state

- **The generalised approach of using a reference profile to couple SLICE to a SISL treatment of the momentum equations has a weak dependency on the reference state.**
- **There is an alternative approach that reduces this dependency.**
- **A concomitant benefit of the alternative approach is the improved consistency between mass field and conserved scalar fields.**

Coupling SLICE to SISL: Reference-State Scheme (RSS)

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The shallow water continuity equation is written in conservative Lagrangian form as

$$\frac{D}{Dt}\left(\int_{\delta\mathcal{A}} \Phi \, d\mathcal{A}\right) = 0, \tag{4}$$

Consider Φ^{ref} as constant reference value for Φ. Substitution of $\Phi = \Phi - \Phi^{ref} + \Phi^{ref}$ in (4) gives:

$$\frac{D}{Dt}\left[\int_{\delta\mathcal{A}} \left(\Phi - \Phi^{ref}\right) d\mathcal{A}\right] = -\Phi^{ref} \int_{\delta\mathcal{A}} \nabla \cdot \mathbf{u} \, d\mathcal{A}. \tag{5}$$

Eq. (5) also implies the evolution of the material area element $d\mathcal{A}$,

$$\frac{D\left(\int_{\delta\mathcal{A}} d\mathcal{A}\right)}{Dt} = \int_{\delta\mathcal{A}} \nabla \cdot \mathbf{u} \, d\mathcal{A}, \tag{6}$$

A CSISL discretization of (5) is:

$$\left\{\int_{\delta\mathcal{A}}\left(\Phi-\Phi^{ref}\right)d\mathcal{A}\right\}_{A}^{n+1}-\left\{\int_{\delta\mathcal{A}}\left(\Phi-\Phi^{ref}\right)d\mathcal{A}\right\}_{D}^{n}=$$
$$-\alpha\Delta t\Phi^{ref}\left(\int_{\delta\mathcal{A}}\nabla\cdot\mathbf{u}\,d\mathcal{A}\right)_{A}^{n+1}-\beta\Delta t\Phi^{ref}\left(\int_{\delta\mathcal{A}}\nabla\cdot\mathbf{u}\,d\mathcal{A}\right)_{D}^{n}. \quad (7)$$

A discrete form of (6) is implied by (7) and can be obtained by setting $\Phi\equiv 0$ therein as:

$$\left(\int_{\delta\mathcal{A}}d\mathcal{A}\right)_{A}^{n+1}-\left(\int_{\delta\mathcal{A}}d\mathcal{A}\right)_{D}^{n}=$$
$$\alpha\Delta t\left(\int_{\delta\mathcal{A}}\nabla\cdot\mathbf{u}\,d\mathcal{A}\right)_{A}^{n+1}+\beta\Delta t\left(\int_{\delta\mathcal{A}}\nabla\cdot\mathbf{u}\,d\mathcal{A}\right)_{D}^{n}.$$

This equation is not forced to hold and therefore in general will not do so $\Rightarrow$ Therefore the numerical solution is dependent on Φ^{ref}.

Dependence on the reference value, test 1 results

L_2 norms after 5 days; The mid-latitude jets problem; Resolution 128x64; $\Delta t = 60$ mins;

RSS (Reference-State Scheme)

Φ^{ref}	$L_2(h)$	$L_2(\mathbf{v})$
Φ_1^{ref}	**0.405E-03**	**0.859E-02**
Φ_2^{ref}	**0.406E-03**	**0.839E-02**
Φ_3^{ref}	**0.445E-03**	**0.934E-02**

Reference-State-Free Scheme (RSFS) (1)

Specifically, the discrete SISL form of (4) **is**

$$\left(\int_{\delta \mathcal{A}} \Phi d\, \mathcal{A} \right)_A^{n+1} = \left(\int_{\delta \mathcal{A}} \Phi \, d\mathcal{A} \right)_D^n , \qquad (8)$$

i.e., (7) **with** $\Phi^{ref} \equiv 0$**. This can be rewritten as**

$$\Phi_A^{n+1} = \frac{1}{\Delta \mathcal{A}} \left(\int_{\delta \mathcal{A}} \Phi \, d\mathcal{A} \right)_D^n . \qquad (9)$$

Reference-State-Free Scheme (RSFS) (2)

Then, to maintain the same implicit form as (1)**, add the term** $-\Phi^{ref} + \alpha \Delta t \Phi^{ref} (\nabla \cdot \mathbf{u})_A^{n+1}$ **on both sides of** (9) **to obtain:**

$$\left[\left(\Phi - \Phi^{ref} \right) + \alpha \Delta t \Phi^{ref} \nabla \cdot \mathbf{u} \right]_A^{n+1} = R_{\Phi 2}^{n}, \qquad (10)$$

where

$$R_{\Phi 2}^{n} \equiv \frac{1}{\Delta \mathcal{A}} \left(\int_{\delta \mathcal{A}} \Phi \, d\mathcal{A} \right)_D^n - \Phi^{ref} + \alpha \Delta t \Phi^{ref} (\nabla \cdot \mathbf{u})_A^{n+1}. \qquad (11)$$

Unified approach non/or-conservative scheme

For all the schemes, the discrete equations are:

$$\left[\mathbf{u} - \alpha \Delta t \mathbf{\Psi}\right]^{n+1} = R_u$$
$$\left[\left(\Phi - \Phi^{\mathrm{ref}}\right) + \alpha \Delta t \Phi^{\mathrm{ref}} \nabla \cdot \mathbf{u}\right]^{n+1} = R_{\Phi}^{m}$$

where,

$$m = SL,\ RSS,\ RSFS1,\ RSFS2$$

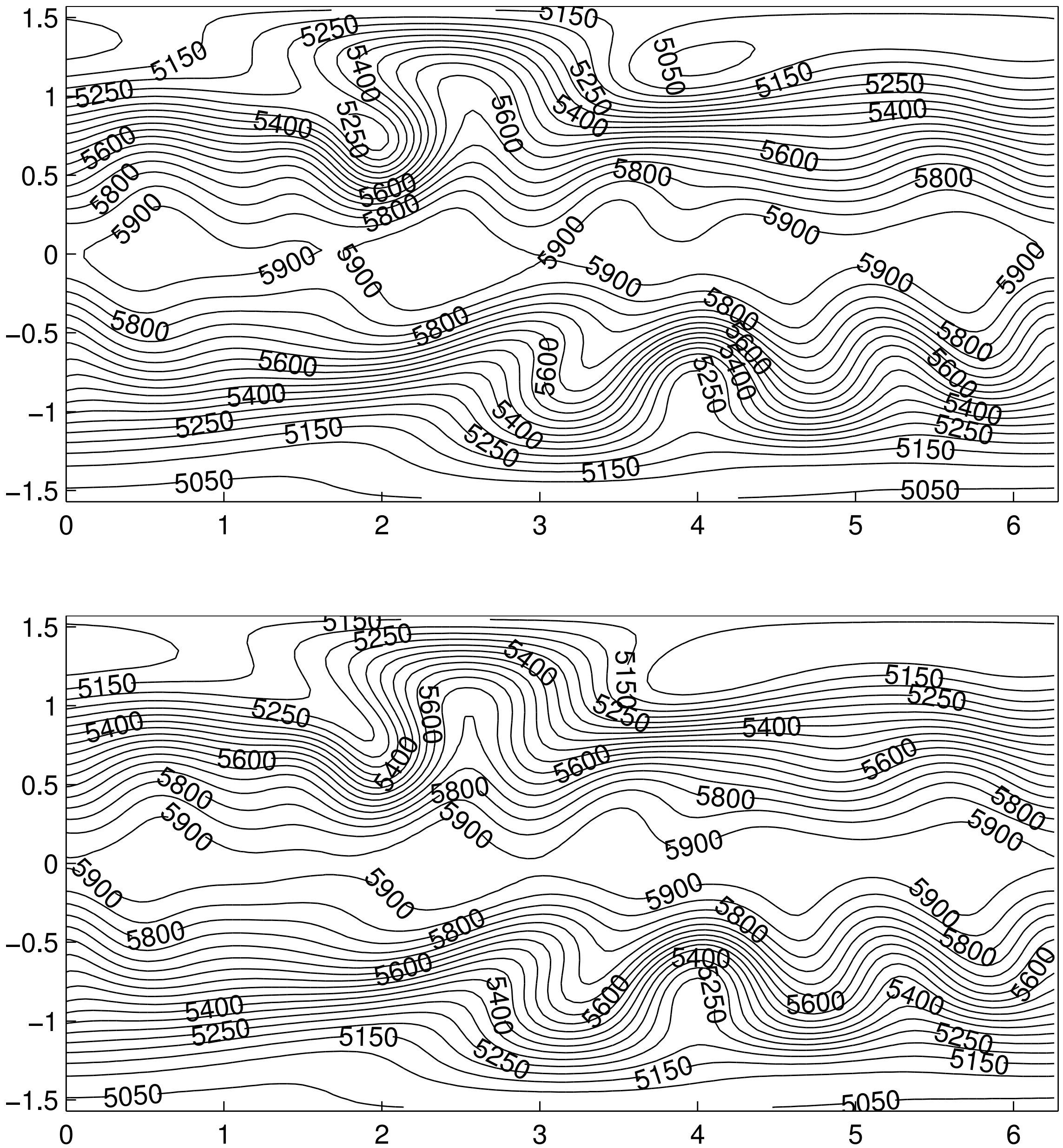
Reference-State-Free Scheme (RSFS), test 1 results

L_2 norms after 5 days; The mid-latitude jets problem; Resolution 128x64; $\Delta t = 60$ mins.

RSFS1 (Reference-State-Free Scheme with (λ, ϕ)-remapping)

Φ^{ref}	$L_2(h)$	$L_2(\mathbf{v})$
Φ_1^{ref}	**0.842E-03**	**0.207E-01**
Φ_2^{ref}	**0.845E-03**	**0.207E-01**
Φ_3^{ref}	**0.845E-03**	**0.207E-01**

Test 5 (Williamson et al.); Solution after 15 days; Resolution 128x64; (a) $\Delta t = 600$s ; (b)$\Delta t = 6000$s



An alternative remapping coordinate (FFS2 scheme)

The implied divergence averaged along the trajectory for SLICE, $\overline{D}^{SLICE}$ is:

$$\overline{D}^{SLICE} \equiv \frac{2}{\Delta t} \left(\frac{\delta \mathcal{A}_A - \delta \mathcal{A}_D^{SLICE}}{\delta \mathcal{A}_A + \delta \mathcal{A}_D^{SLICE}} \right), \qquad (12)$$

$\delta \mathcal{A}_D^{SLICE}$ **can be written as:**

$$\delta \mathcal{A}_D^{SLICE} = \delta \mathcal{A}_A \left[1 + \Delta t \overline{D}^{exact} + O\left(\Delta t^2\right) \right]$$

Substituting this into (12) **shows that the implied trajectory-average divergence is only first-order accurate in time.**

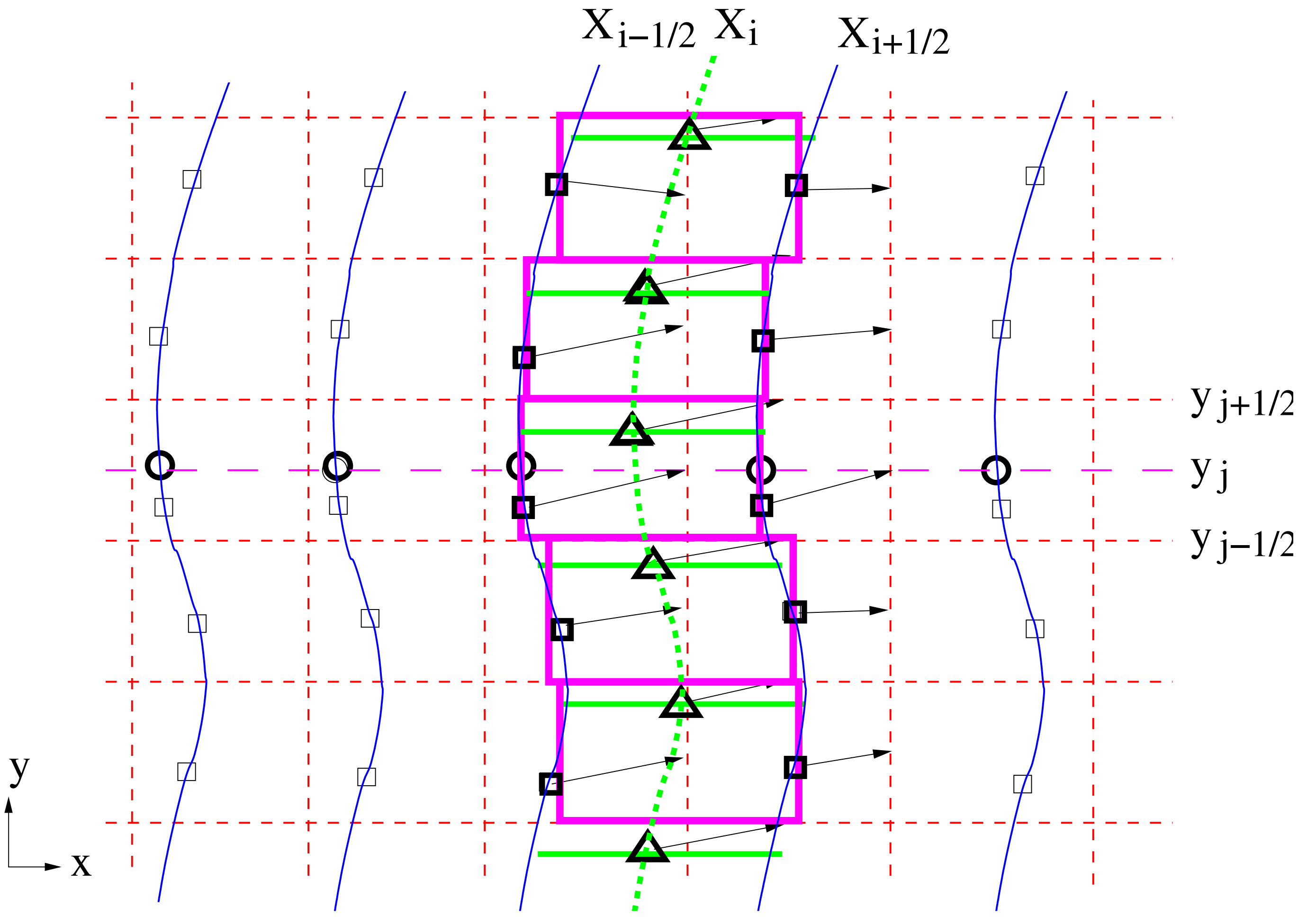
It is possible to restore the second-order accuracy in time, by:

- **computing an estimate for the departure areas whose implied trajectory-average divergence is second-order accurate, and**
- **modifying SLICE to allow the required departure cell areas to be imposed.**
  - **A satisfactory solution is to make an iterative improvement to the implied SLICE departure areas:**

$$\delta A_D^{SLICE} = \delta A_A - \frac{\Delta t}{2} \left(D_A \delta A_A + \{ D_D \delta A_D \}^{SLICE} \right) . \quad (13)$$

  - **A modification is required to SLICE to force it to see the accurate estimate of δA_D^{SLICE}.**

This achieved by changing the **second remapping coordinate, as the area integrated**.



Schematic for C-SLICE

FFS2 results

L_2 norms after 5 days; The mid-latitude jets problem.
Resolution 128x64; $\Delta t = 60$ mins.

RSFS2 [(λ, Area)-remapping]

Φ^{ref}	$L_2(h)$	$L_2(\mathbf{v})$
Φ_1^{ref}	**0.437E-03**	**0.103E-01**
Φ_2^{ref}	**0.437E-03**	**0.103E-01**
Φ_3^{ref}	**0.437E-03**	**0.103E-01**

RSFS1 [(λ, ϕ)-remapping]

Φ^{ref}	$L_2(h)$	$L_2(\mathbf{v})$
Φ_1^{ref}	**0.842E-03**	**0.207E-01**
Φ_2^{ref}	**0.845E-03**	**0.207E-01**
Φ_3^{ref}	**0.845E-03**	**0.207E-01**

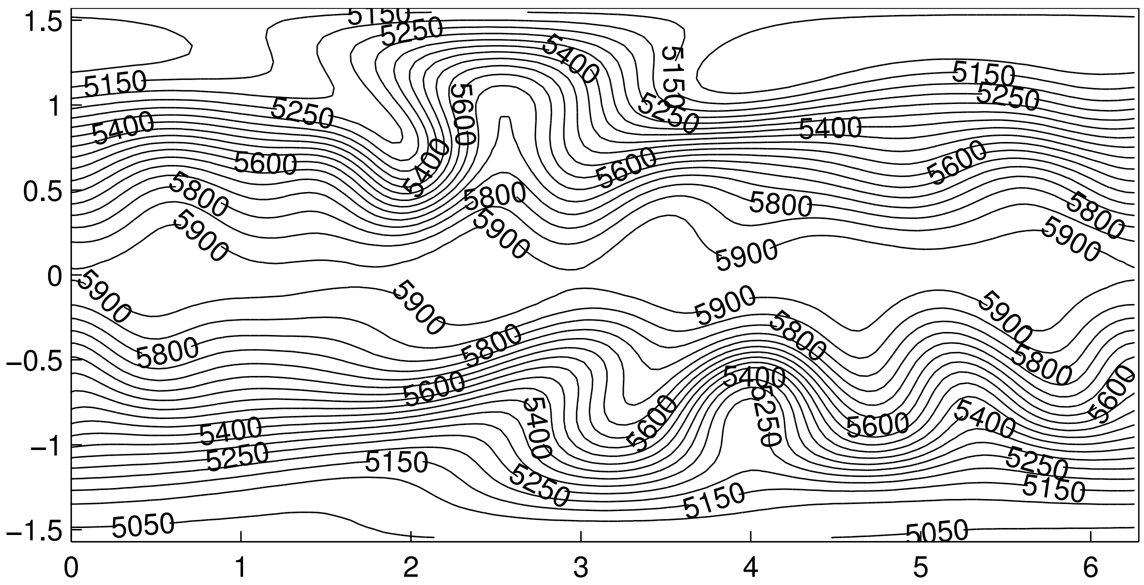


Figure 1: Test 5, 15 days, scheme SFS1,$\Delta t = 6000$s

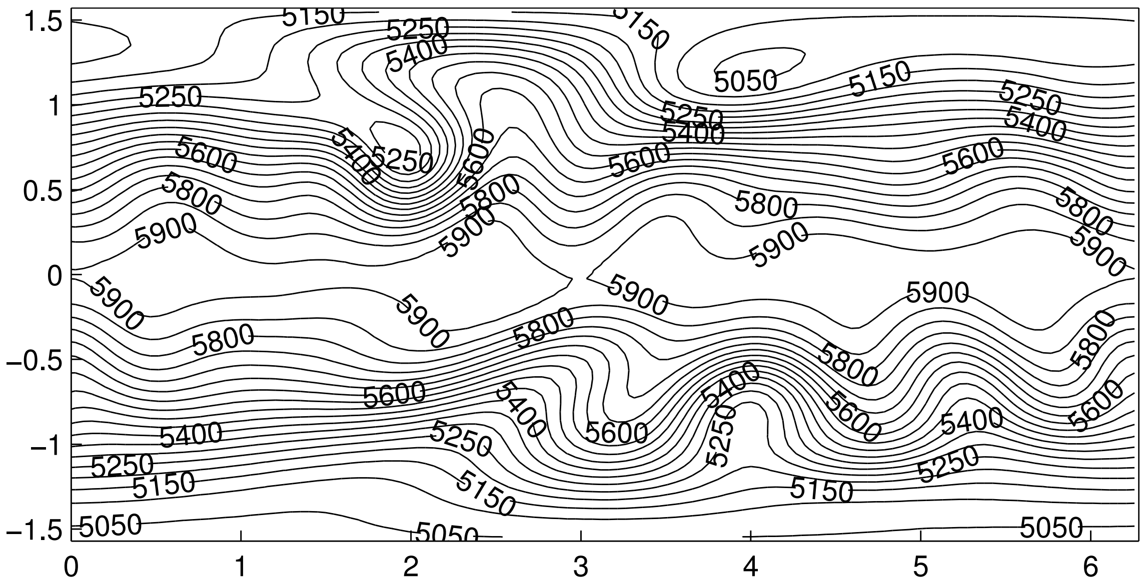
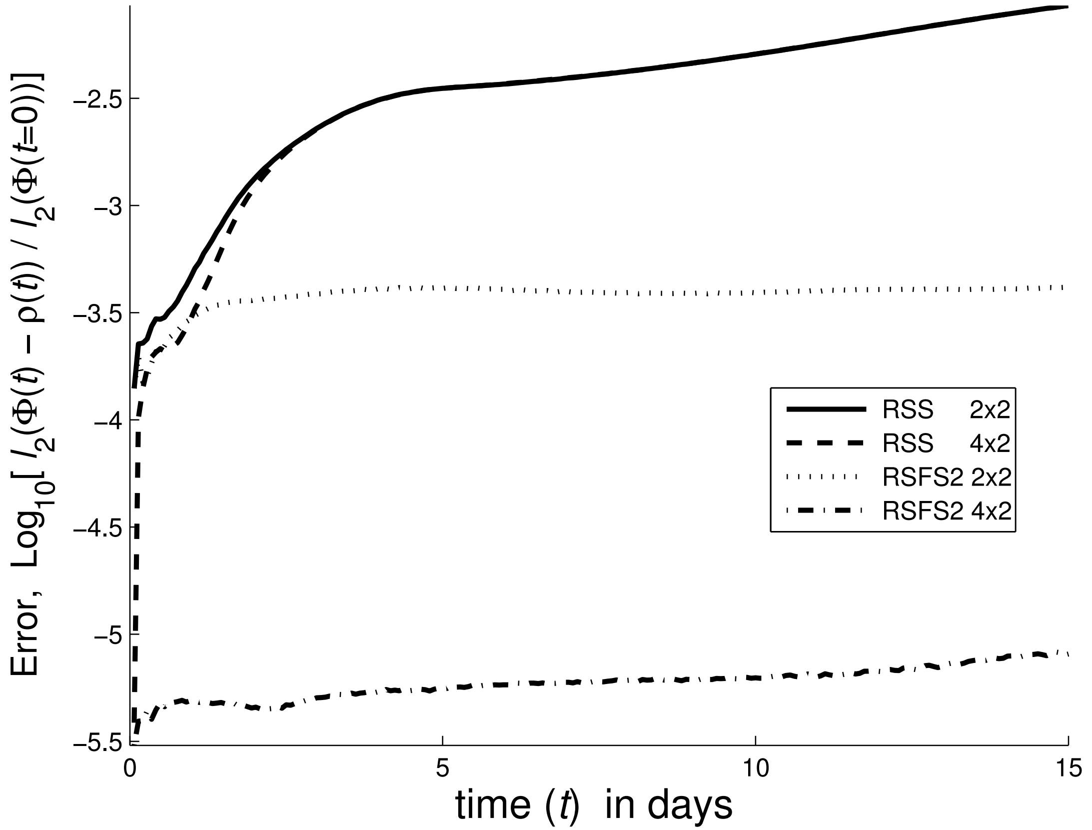


Figure 2: Test 5, 15 days, scheme SFS2,$\Delta t = 6000$s

Consistency of scalar transport: Test 5 of Williamson etal.



Conclusions

- Simple framework for implicit coupling of SLICE (or any Conservative scheme) to momentum equations
- Improved handling of Coriolis terms on C-grid.
- Performs at least as well as standard SL continuity scheme but with mass exactly conserved
- Semi-implicit semi-Lagrangian (No restriction on CFL number ==> Large time steps)
- Inherently conserves mass using a high-order remapping scheme (SLICE)
- Option to reduce the dependence on the reference profile
- Option to preserve a constant for a non-divergent flow
- Optimal generalization to three dimensions
- Option to eliminate the mass-wind inconsistency problem

Some Related Publications

M. Zerroukat, N. Wood, and A. Staniforth, An improved version of SLICE for conservative monotonic remapping on C-grid, Q. J. R. Meteorol. Soc., vol. 135, 541-546 (2009).

M. Zerroukat, N. Wood, A. Staniforth, A. A. White and J. Thuburn, An inherently mass conserving semi-implicit semi-Lagrangian discretisation of the shallow water equations on the sphere, Q. J. R. Meteorol. Soc., (in press).

J. Thuburn, M. Zerroukat, N. Wood and A. Staniforth, Coupling a mass conserving semi-Lagrangian scheme (SLICE) to a semi-implicit discretization of the shallow water equations: minimizing the dependence on a reference atmosphere; Q. J. R. Meteorol. Soc., (submitted).