# An Equivalent and Fast Alternative to Semi-Lagrangian.

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# *Outline*

- Spectral Elements
- Fast: Semi-Implicit
- OIFS theory
- OIFS $=$ semi-Lagrangian
- Second order splitting
- Faster: OIFS versus semi-Lagrangian
- Numerical results
- Localized high resolution: AMR
- Conclusion

# *Spectral Elements*

Horizontal discretisation is $\mathbb{P}_N - \mathbb{P}_N$ spectral element. Fields expanded in terms of the Lagrangian interpolants $h_i$

$$\mathbf{v}_h^k(r_1, r_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} \mathbf{v}_{ij}\, h_i(r_1)\, h_j(r_2)$$

Evaluate inner products using Gauss–Lobatto quadrature

$$\langle\, f,\ g\,\rangle_{GL} \ \ = \ \ \sum_{k=1}^{K} \sum_{i=0}^{N} \sum_{j=0}^{N} f^k(\xi_i, \xi_j)\, g^k(\xi_i, \xi_j)\, \rho_i\, \rho_j$$

# *Time-stepping: Semi-Implicit*

State $X$, explicit $\mathcal{M}$ and linear $\mathcal{L}$ parts of model

$$\frac{dX}{dt} = \mathcal{M}X$$

Define the operator

$$\Delta_{tt}X = X^{n+1} - 2X^{n} + X^{n-1}$$

Semi-implicit applied as a correction to the explicit step

$$\frac{X^{n+1} - X^{n-1}}{2\Delta t} = \mathcal{M}\,X^{n} - \frac{1}{2}\Delta_{tt}\mathcal{L}X$$

Robert–Asselin (1972) time-filter

# *Results dry-primitive equations*

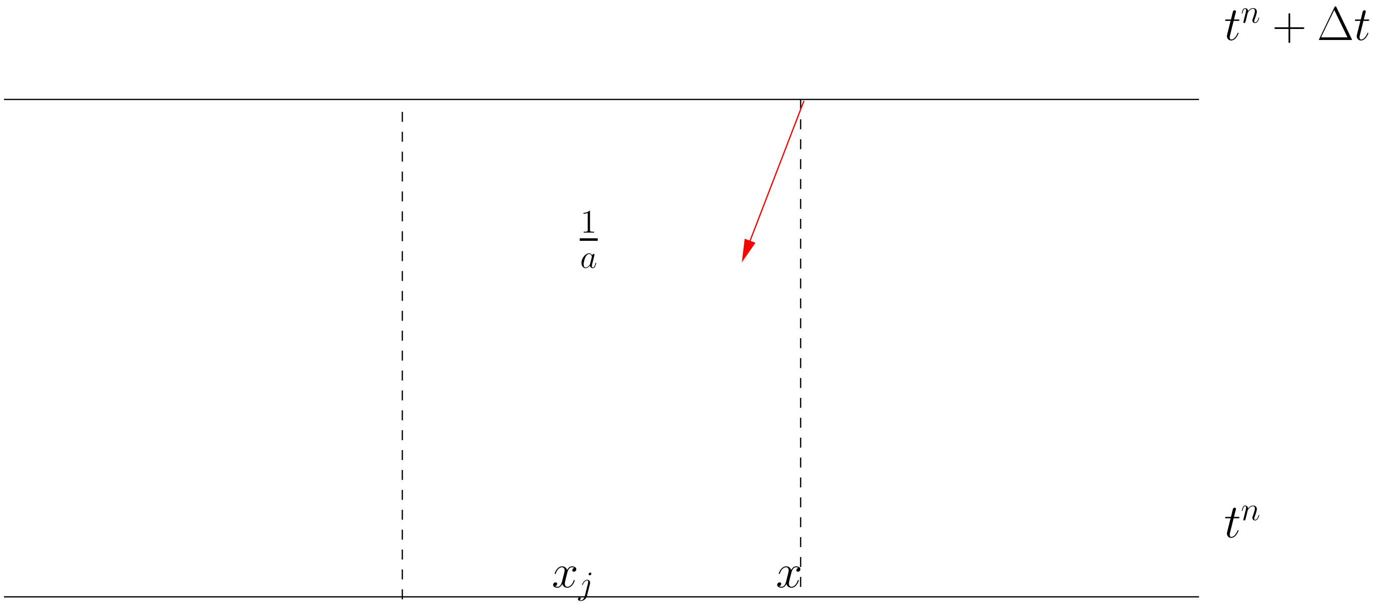
Thomas and Loft 2004.

- 3 times faster than explicit integration.
- Modified Helmholtz equation
- Vertical eigen-mode decomposition.
- Iterative conjugate gradient solver.
- Improved preconditioner on the way.

# *Operator Integration Factor Splitting*

- A. Robert 1981 (SI+SL).
- Maday, Patera, Ronquist 1990: OIFS.
- $K$ elements of order $N$, $KN^d$ grid points
- Interpolation $KN^{2d}$
- Scalar advection requires $dKN^{d+1}$
- OIFS more efficient if sub-step $< N^{d-1}$ "times"
- Purely Eulerian: regular communication patterns
- Nonlinear OIFS: St-Cyr and Thomas (2004)

# Semi-Lagrangian



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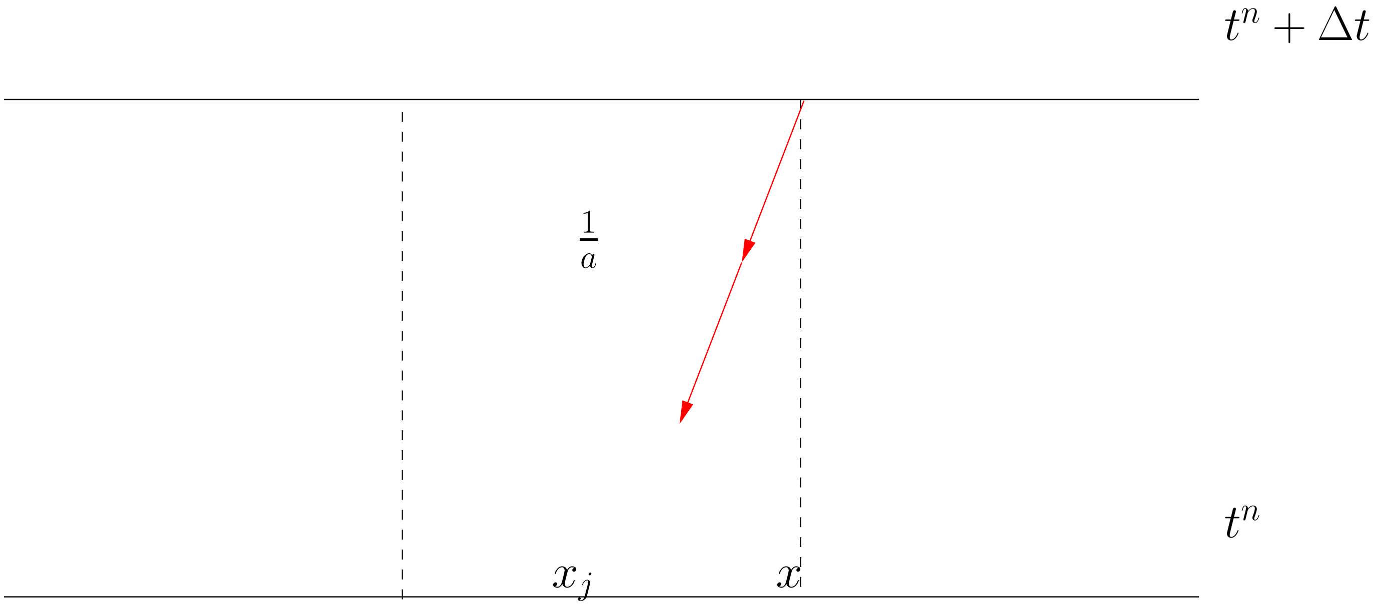
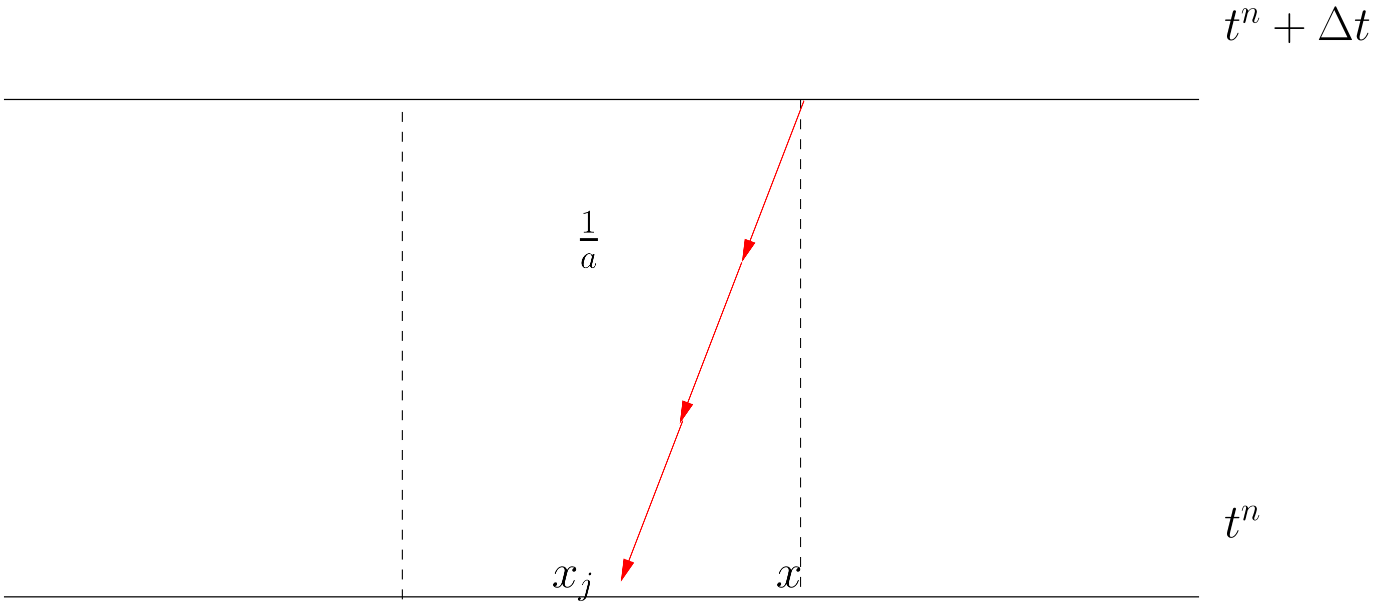
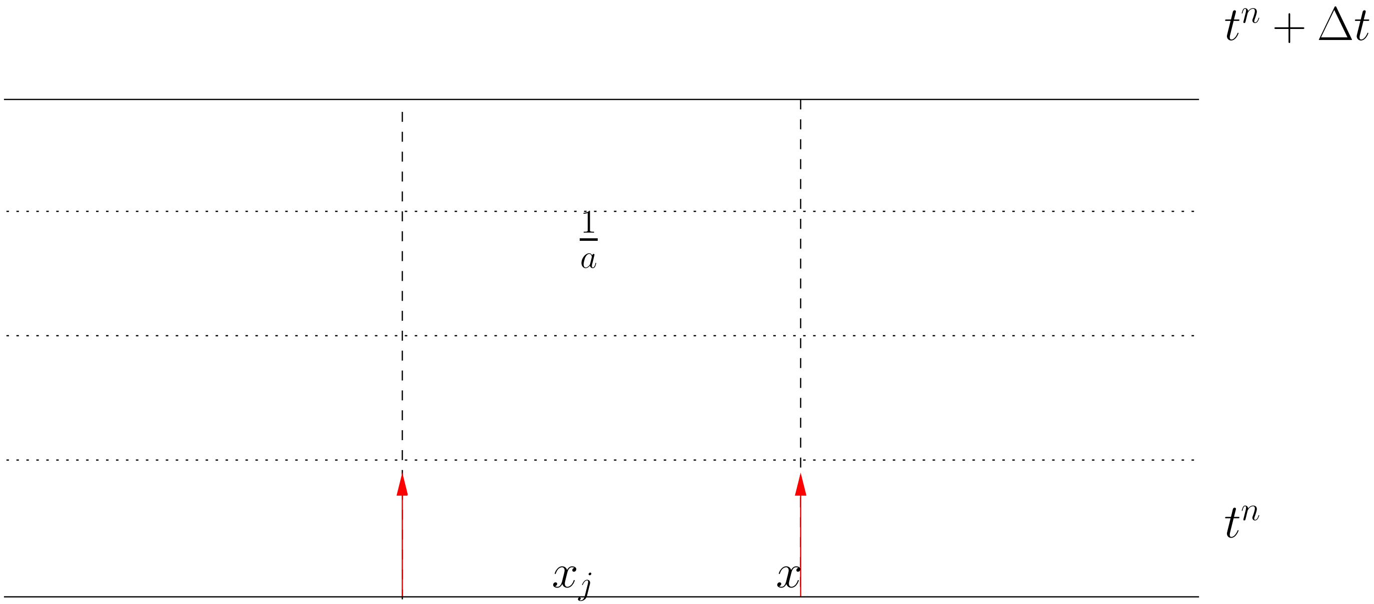


# Semi-Lagrangian



# OIFS Sub-cycling



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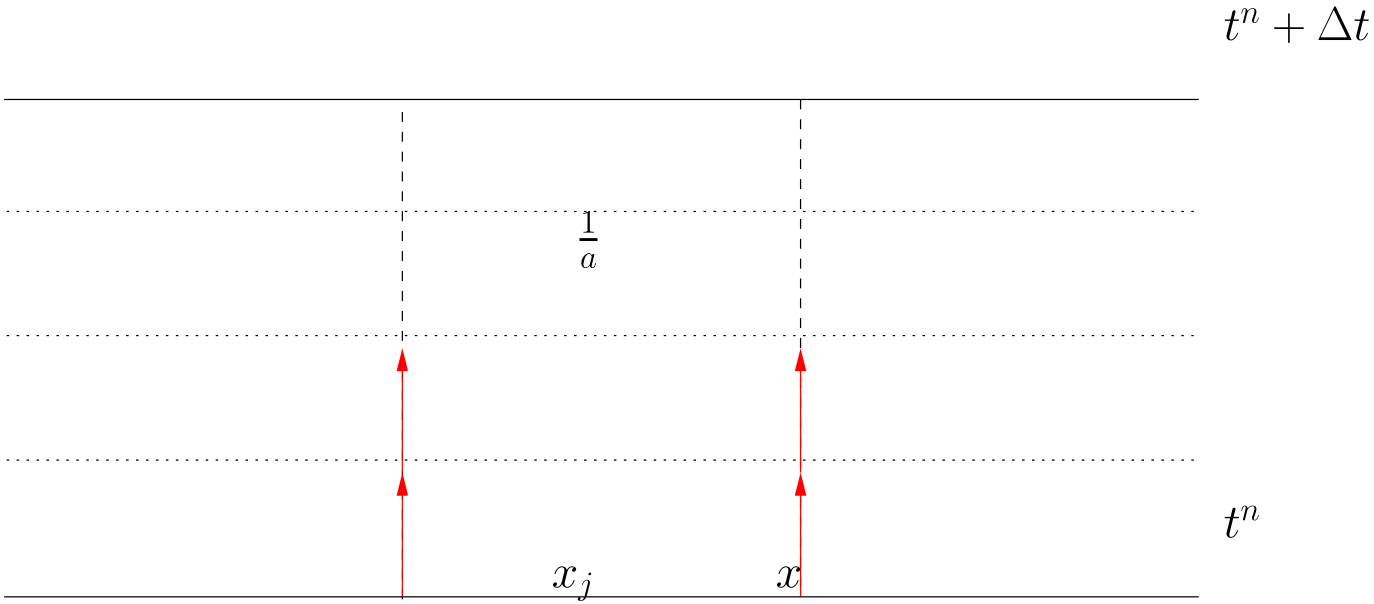

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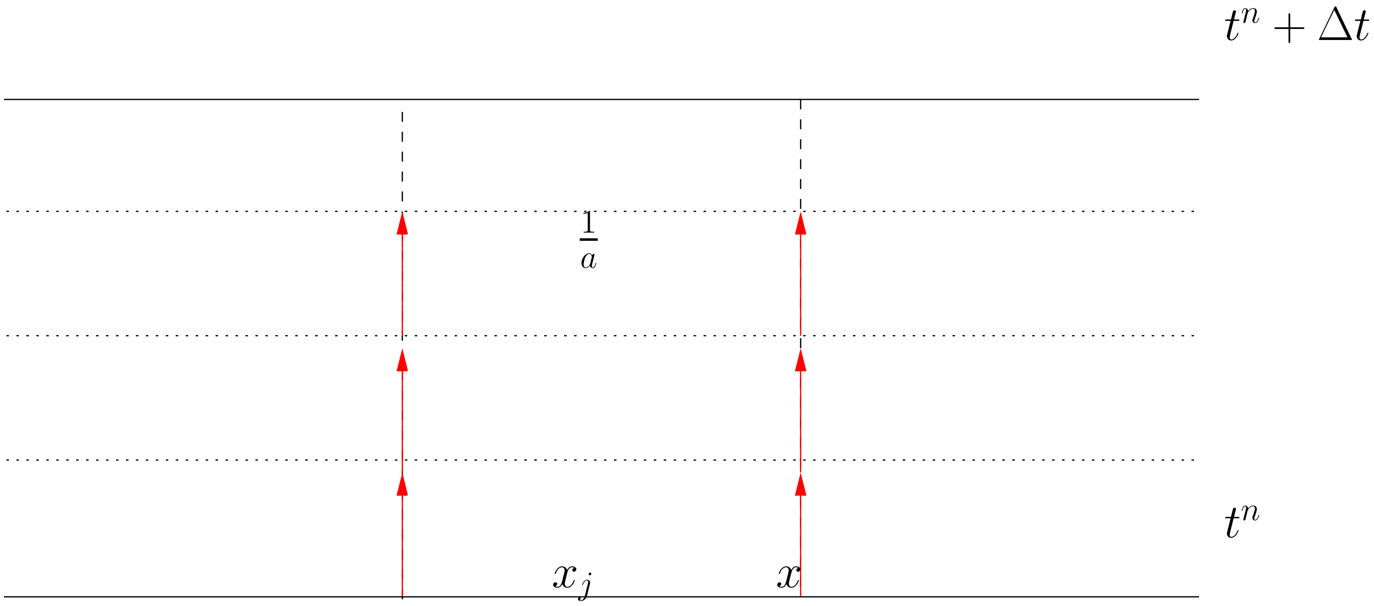

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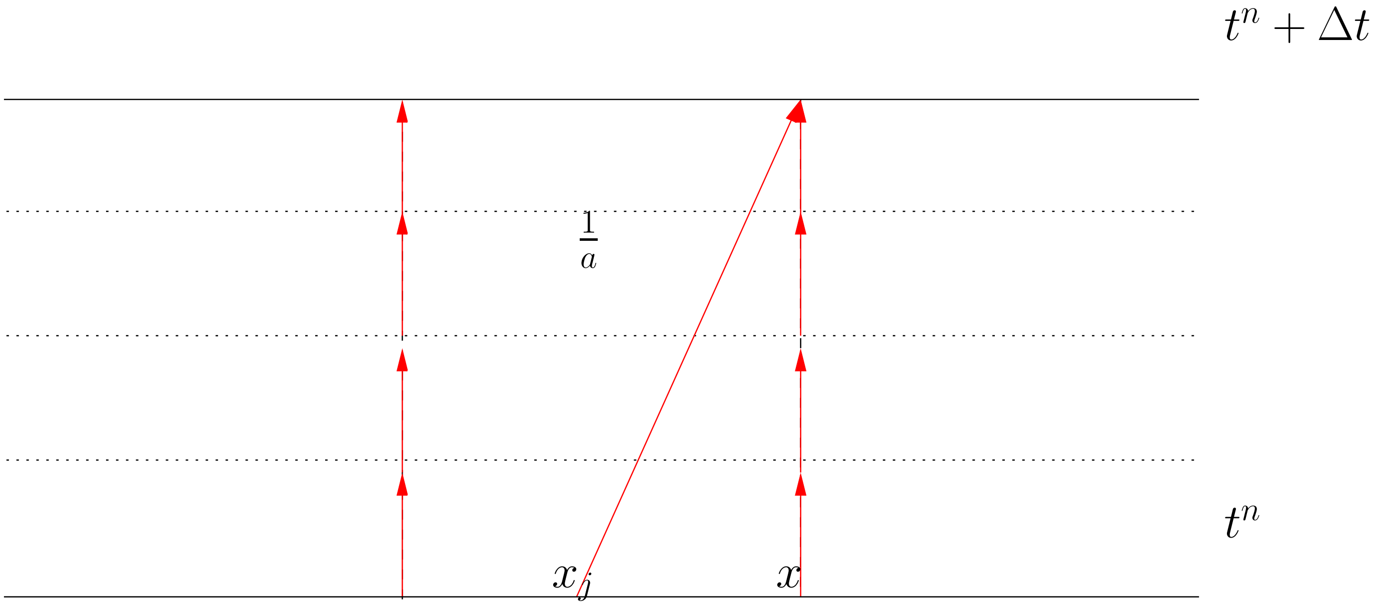

# Operator Integrating Factor Splitting

$$\frac{du(t)}{dt} = S(u(t)) + F(u(t)), \quad t \in [0, T]$$

with initial condition $u(0) = u_0$.
Find integrating factor $Q_S^{t^*}(t)$, such that $Q_S^{t^*}(t^*) = I$,

$$\frac{d}{dt} Q_S^{t^*}(t) \cdot u = Q_S^{t^*}(t) \cdot F(u).$$

To find the action of $Q_S^{t^*}(t)$ solve

$$\frac{dv^{(t^*,t)}(s)}{ds} = S(v^{(t^*,t)}), \quad 0 \le s \le t - t^*$$

with initial condition $v^{(t^*,t)}(0) = u(t)$

# OIFS theory: linear case

- If $S(u(t)) = A(t)u(t)$
- $A(t)$ is an $\mathcal{N} \times \mathcal{N}$ matrix
- $A(t_1)A(t_2) = A(t_2)A(t_1)$

$$Q_S^{t^*}(t) = \exp\left[\int_t^{t^*} A(s)\,ds\right].$$

Variation of the constant

$$u(t^*) = e^{A(t^*-t)}u(t) + \int_t^{t^*} e^{A(t^*-s)}F(u(s))\,ds.$$

# *OIFS theory: theorem*

**Theorem**: If $v^{(t^*,t)}(s)$ is the solution of

$$\frac{d}{ds}v^{(t^*,t)}(s) = S(v^{(t^*,t)}(s)), \quad 0 \le s \le t^* - t$$

with initial condition $v^{(t^*,t)}(0) = u(t)$, then

$$Q_S^{t^*}(t) \cdot u(t) = v^{(t^*,t)}(t^* - t).$$

# OIFS theory: proof

$$\frac{d}{dt}\left[Q_S^{t^*}(t)\cdot u(t)\right] = Q_S^{t^*}(t)\cdot F(u(t)),$$

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$$\frac{d}{dt}\left[Q_S^{t^*}(t)\right]\cdot u(t) + Q_S^{t^*}(t)\cdot\frac{d}{dt}\left[u(t)\right] = Q_S^{t^*}(t)\cdot F(u(t)),$$

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$$\frac{d}{dt}\left[Q_S^{t^*}(t)\right]\cdot u(t) + Q_S^{t^*}(t)\cdot\left[S(u(t)) + F(u(t))\right] = Q_S^{t^*}(t)\cdot F(u(t)),$$

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$$\frac{d}{dt}\left[Q_S^{t^*}(t)\right]\cdot u(t) + Q_S^{t^*}(t)\cdot\left[S(u(t)) + F(u(t))\right] = Q_S^{t^*}(t)\cdot F(u(t)),$$

$$\frac{d}{dt}\left[Q_S^{t^*}(t)\right]\cdot u(t) = -Q_S^{t^*}(t)\cdot S(u(t)).$$

# OIFS theory: proof

$$\frac{d}{ds}v^{(t^*,t)}(s) = \qquad S(v^{(t^*,t)}(s)).$$

# OIFS theory: proof

$$Q_S^{t^*}(t+s) \cdot \frac{d}{ds} v^{(t^*,t)}(s) = Q_S^{t^*}(t+s) \cdot S(v^{(t^*,t)}(s)).$$

# OIFS theory: proof

$$Q_S^{t^*}(t+s)\cdot\frac{d}{ds}v^{(t^*,t)}(s)=Q_S^{t^*}(t+s)\cdot S(v^{(t^*,t)}(s)).$$

(last slide): using $t\to t+s$ and $v^{(t^*,t)}(s):=u(t+s)$ (fix $t$)

$$\frac{d}{ds}\left[Q_S^{t^*}(t+s)\right]\cdot u(t+s)=-Q_S^{t^*}(t+s)\cdot S(u(t+s)).$$

# OIFS theory: proof

$$Q_S^{t^*}(t+s)\cdot\frac{d}{ds}v^{(t^*,t)}(s) = Q_S^{t^*}(t+s)\cdot S(v^{(t^*,t)}(s)).$$

(last slide): using $t \to t+s$ and $v^{(t^*,t)}(s) := u(t+s)$ (fix $t$)

$$\frac{d}{ds}\left[Q_S^{t^*}(t+s)\right]\cdot u(t+s) = -Q_S^{t^*}(t+s)\cdot S(u(t+s)).$$

$$\frac{d}{ds}\left[Q_S^{t^*}(t+s)\right]\frac{ds}{dt}\cdot v^{(t^*,t)}(s) = -Q_S^{t^*}(t+s)\cdot S(v^{(t^*,t)}(s)).$$

# OIFS theory: proof

Summing both contributions:

$$\frac{d}{dt}\left[Q_S^{t^*}(t+s)\cdot v^{(t^*,t)}(s)\right]=0.$$

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$$cste = Q_S^{t^*}(t+s)\cdot v^{(t^*,t)}(s)$$

# OIFS theory: proof

Summing both contributions:

$$\frac{d}{dt}\left[Q_S^{t^*}(t+s)\cdot v^{(t^*,t)}(s)\right]=0.$$

$$cste = Q_S^{t^*}(t+s)\cdot v^{(t^*,t)}(s)$$

with $s=0$ and $v^{(t^*,t)}(0)=u(t)$

$$Q_S^{t^*}(t)\cdot u(t) = Q_S^{t^*}(t+s)\cdot v^{(t^*,t)}(s)$$

# OIFS theory: proof

Using $t^* - t = s$ and applying the relation

$$Q_S^{t^*}(t^*) = I$$

$$Q_S^{t^*}(t) \cdot u(t) = Q_S^{t^*}(t^*) \cdot v^{(t^*,t)}(t^* - t)$$

$$Q_S^{t^*}(t) \cdot u(t) = v^{(t^*,t)}(t^* - t).$$

# OIFS = semi-Lagrangian

$-S(u) =$ semi-discrete advection
Material derivative is

$$\frac{d}{dt}u(X(x,t),t) = \frac{\partial u(x,t)}{\partial t} - S(u(x,t))$$

$X(x,t)$ satisfies the ordinary differential equation

$$\frac{d}{dt}X(x,t) = u(X(x,t),t).$$

# OIFS = semi-Lagrangian

Integration on $[t^{n-q}, t^n]$

$$\int_{t^{n-q}}^{t^n} \frac{d}{ds} u(X(x,s),s)ds = \int_{t^{n-q}}^{t^n} \left\{ \frac{\partial u(x,s)}{\partial s} - S(u(x,s)) \right\} ds$$

$$u(X(x,t^n),t^n) - u(X(x,t^{n-q}),t^{n-q}) =$$

$$u(x,t^n) - u(x,t^{n-q}) - \int_{t^{n-q}}^{t^n} S(u(x,s))\, ds$$

# OIFS = semi-Lagrangian

$u(X(x,t^n),t^n) = u(x,t^n)$ because $X(x,t^n) = x$

$$u(X(x,t^{n-q}),t^{n-q}) = u(x,t^{n-q}) + \int_{t^{n-q}}^{t^n} S(u(x,s))\,ds.$$

# OIFS = semi-Lagrangian

Integrating on the unshifted interval

$$\frac{dv}{ds}(s) = S(v(s)), \quad 0 \le s \le t^n - t^{n-q}$$

with $v(0) = u(x, t^{n-q})$ ($x$ fixed).

$$\begin{aligned} v(t^n) &= v(t^{n-q}) + \int_0^{t^n - t^{n-q}} S(v(s))\, ds \\ &= u(x, t^{n-q}) + \int_0^{t^n - t^{n-q}} S(u(x, s+t))\, ds \\ &= u(x, t^{n-q}) + \int_{t^{n-q}}^{t^n} S(u(x, s))\, ds. \end{aligned}$$

## Second order splitting

If initial condition specified at $t^n$, then $t^{n+k}$ is

$$u(t^{n+k}) = e^{k\Delta t(S+F)}u(t^n).$$

Solving first for $S$ and then $F$, the solution is given by

$$u(t^{n+k}) = e^{k\Delta tF}e^{k\Delta tS}u(t^n).$$

Error is the difference.

# Nonlinear OIFS

St-Cyr and Thomas (2004) sub-step

$$\begin{aligned} \frac{\partial\,\tilde{\mathbf{v}}}{\partial s} + \tilde{\zeta}\,\mathbf{k}\times\tilde{\mathbf{v}} + \frac{1}{2}\nabla\left(\tilde{\mathbf{v}}\cdot\tilde{\mathbf{v}}\right) &= 0 \\ \frac{\partial\,\tilde{\Phi}}{\partial s} + \nabla\cdot(\tilde{\Phi}\,\tilde{\mathbf{v}}) &= 0 \end{aligned}$$

with initial conditions $\tilde{\mathbf{v}}(\mathbf{x}, t^{n-q}) = \mathbf{v}(\mathbf{x}, t^{n-q})$, $\tilde{\Phi}(\mathbf{x}, t^{n-q}) = \Phi(\mathbf{x}, t^{n-q})$.

# *Time Discretization*

Integration factor applied to the SWE's

$$\frac{d}{dt} Q_S^{t^*}(t) \left[ \begin{array}{c} \mathbf{v} \\ \Phi \end{array} \right] = -Q_S^{t^*}(t) \left[ \begin{array}{c} f\,\mathbf{k} \times \mathbf{v} + \nabla\Phi \\ \Phi_0\,\nabla \cdot \mathbf{v} \end{array} \right].$$

Backward Differentiation Formula (BDF-2):

$$\begin{aligned} \frac{3\mathbf{v}^n - 4\tilde{\mathbf{v}}^{n-1} + \tilde{\mathbf{v}}^{n-2}}{2\Delta t} &= -\mathbf{M} f \mathbf{v}^n - \nabla\Phi^n \\ \frac{3\Phi^n - 4\tilde{\Phi}^{n-1} + \tilde{\Phi}^{n-2}}{2\Delta t} &= -\Phi_0\,\nabla \cdot \mathbf{v}^n \end{aligned}$$

sNon-symmetric due to implicit Coriolis: CGS.

# OIFS: results

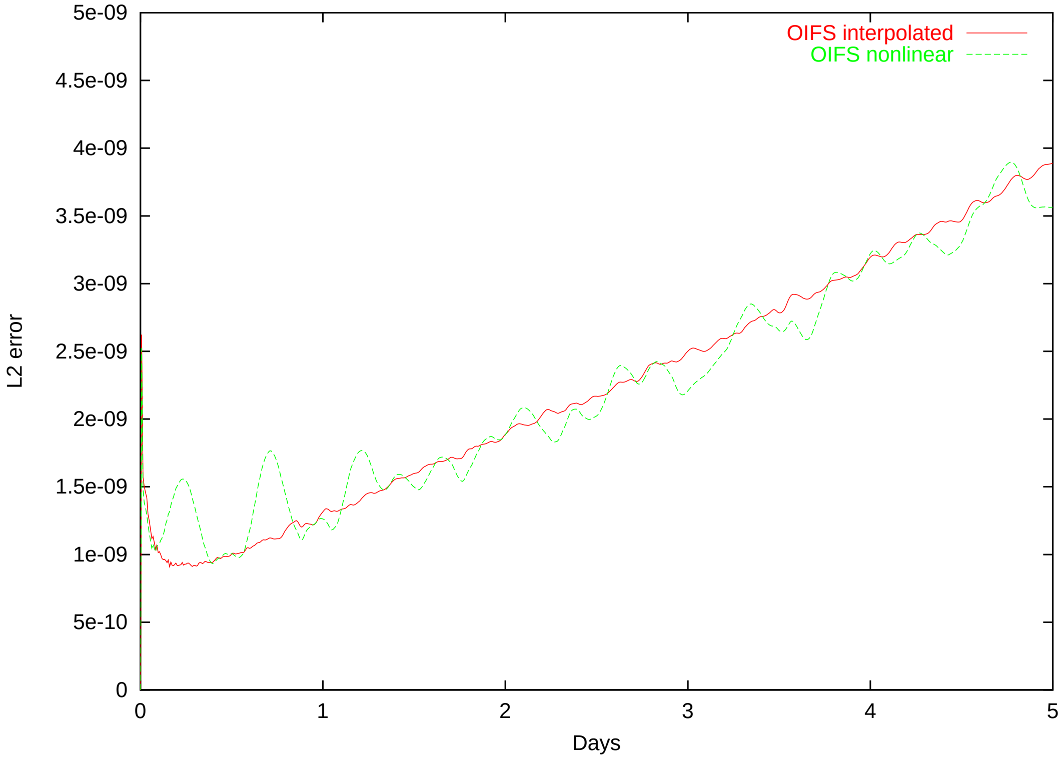


Figure 1: Shallow water test case 2

# OIFS: results

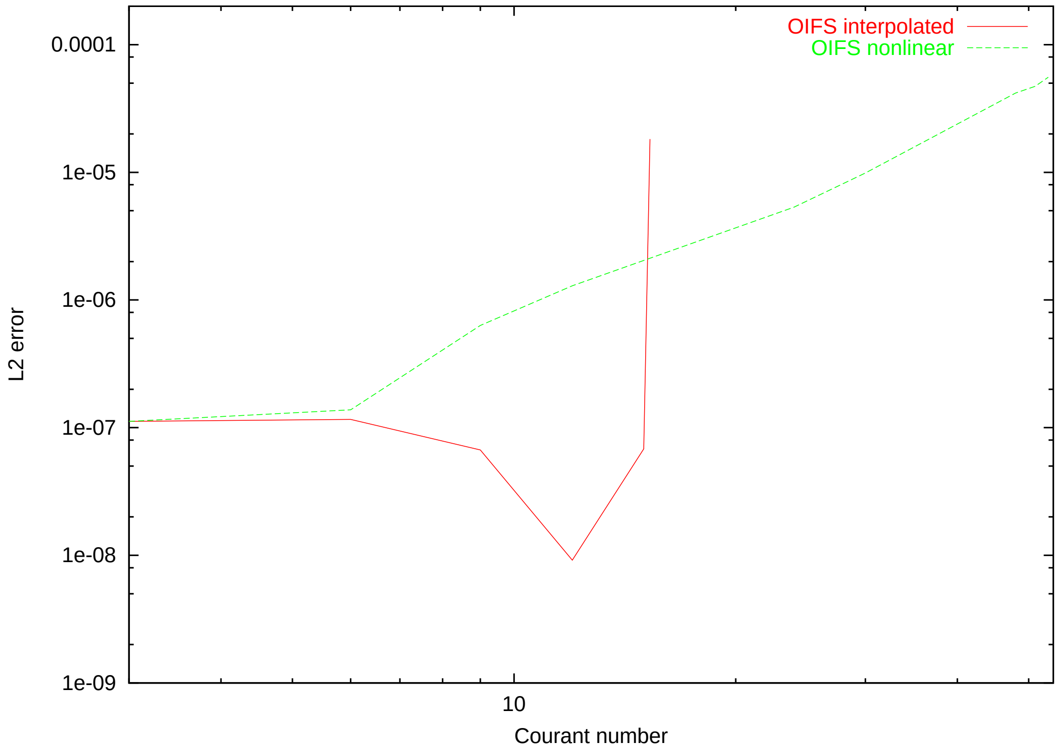


Figure 2: Shallow water test case 2

# OIFS: results

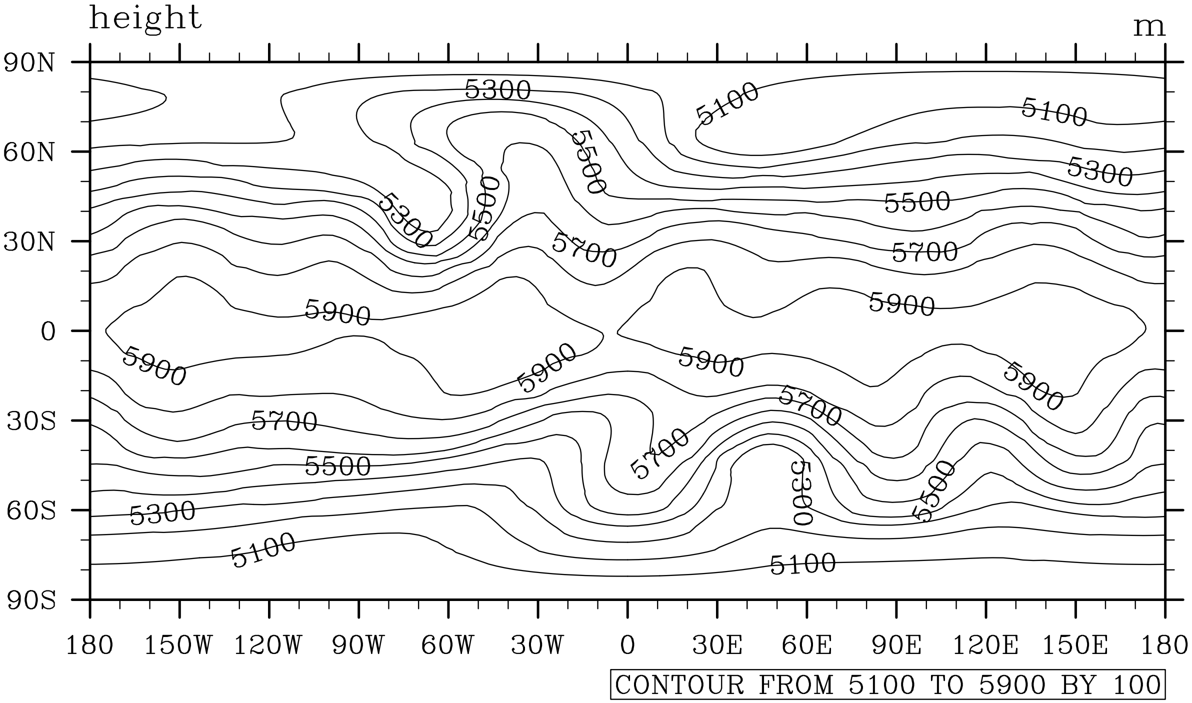


Figure 3: Nonlinear OIFS: $\Delta t = 480$ sec

# OIFS: results

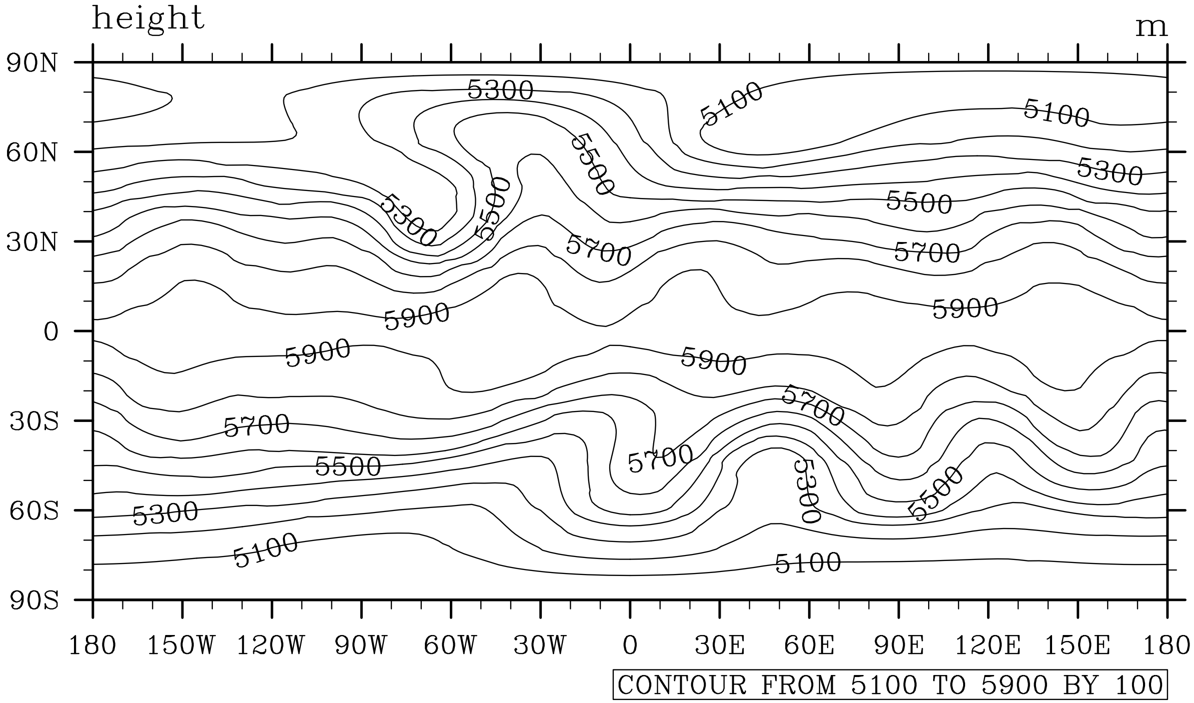


Figure 4: Nonlinear OIFS: $\Delta t = 14400$ sec

# OIFS: results

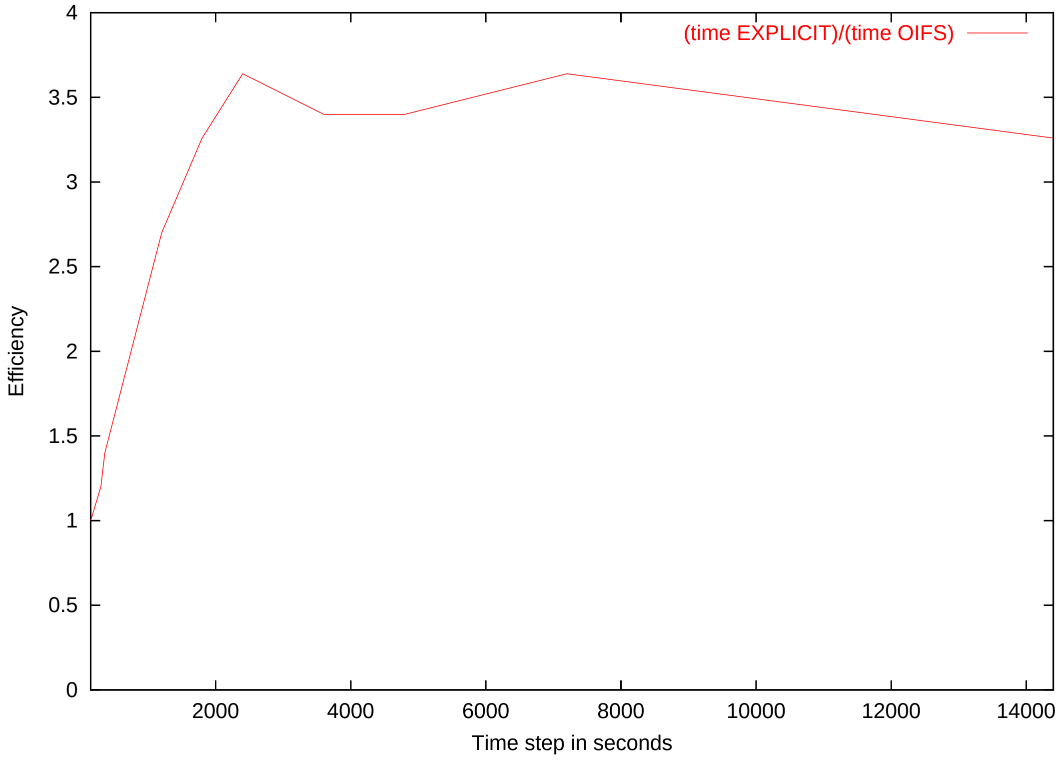


Figure 5: Efficiency NL-OIFS VS Explicit

# OIFS: results

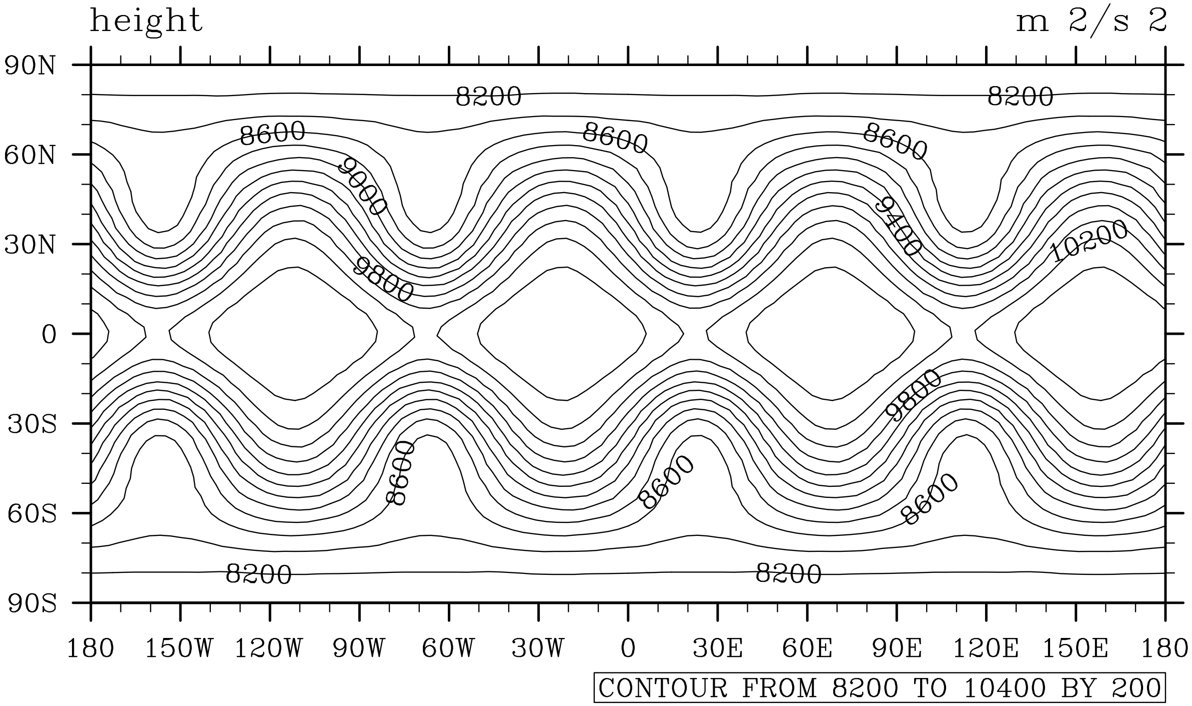


Figure 6: Swtc6: R-H : $\Delta t = 120$

# OIFS: results

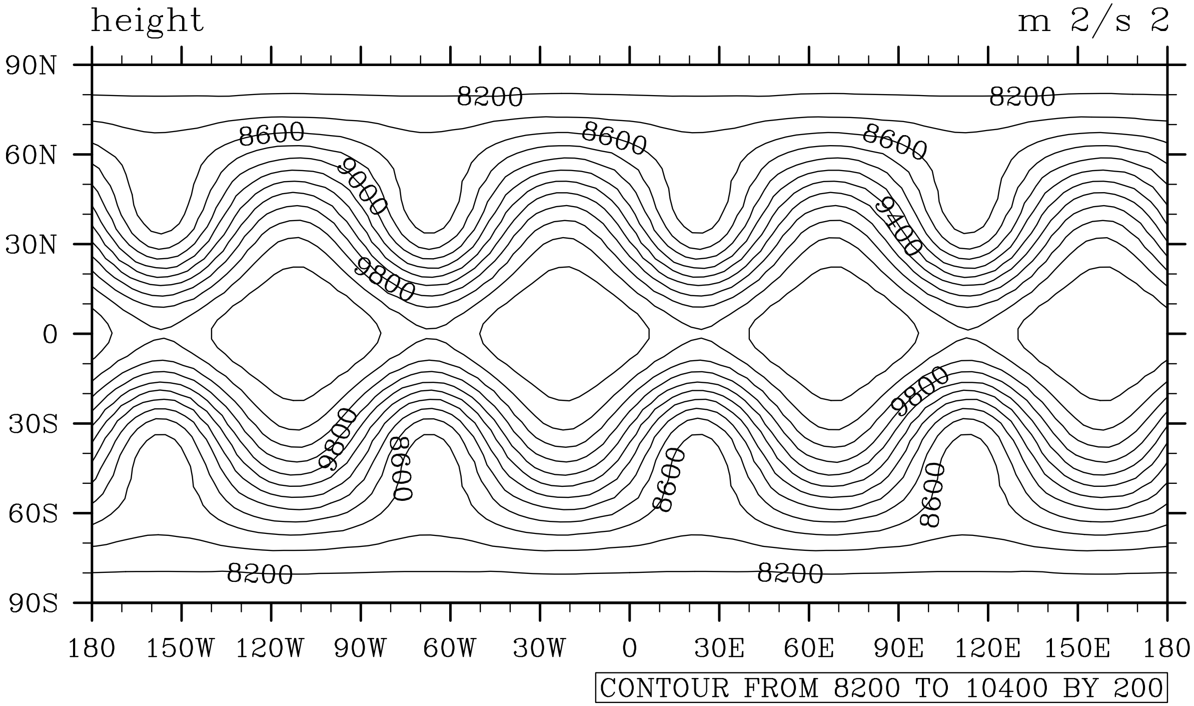
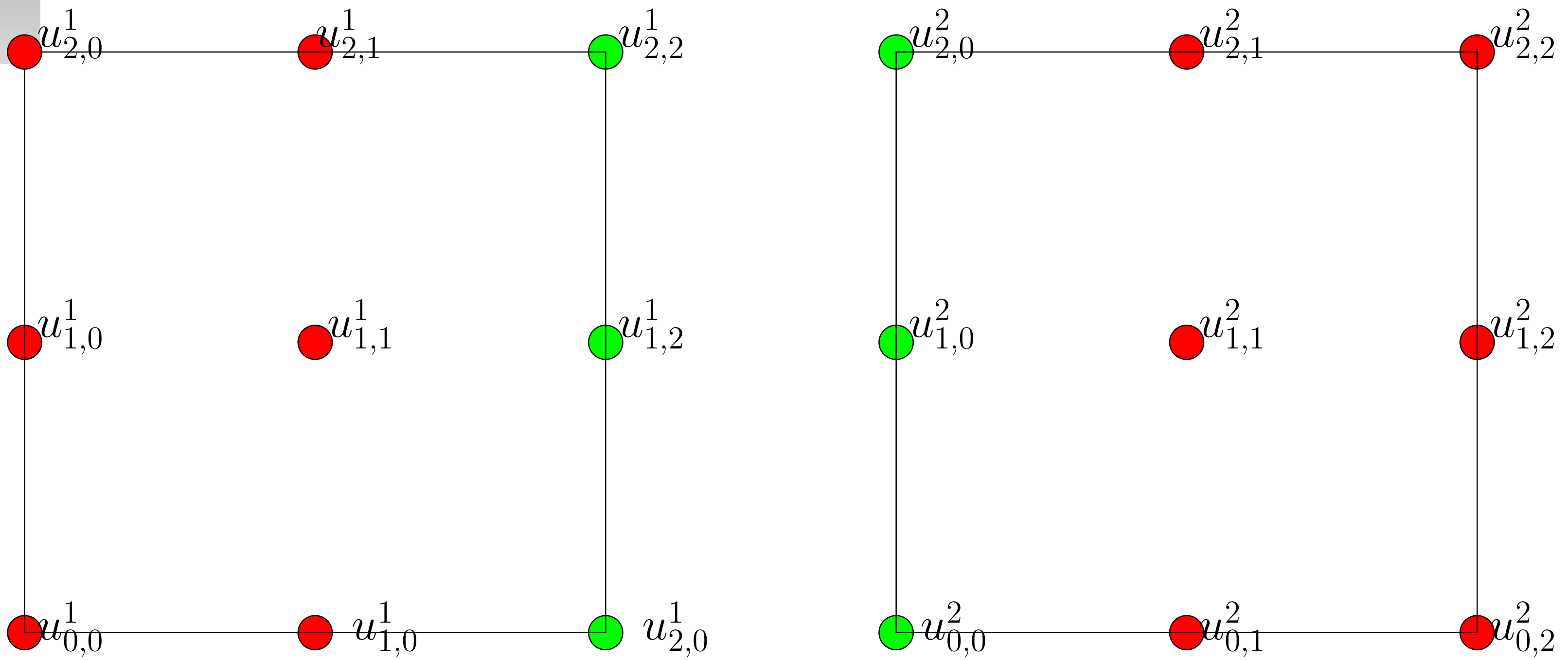


Figure 7: Swtc6: R-H : $\Delta t = 360$ sec

# ***AMR: why?***

- High-resolution at lower cost.
- Enables to work with less resources.
- Weather localized features (15 days).
- Conjecture for climate: will it work?

# AMR: Local degrees of freedom



$$v^T A u = v^T Q^T A_L Q u = v^T Q^T M_L Q f = v^T f$$

$$v^T Q^T A_L u_L = v^T Q^T M_L f_L$$

# *AMR: Interpolation based non-conforming elements*

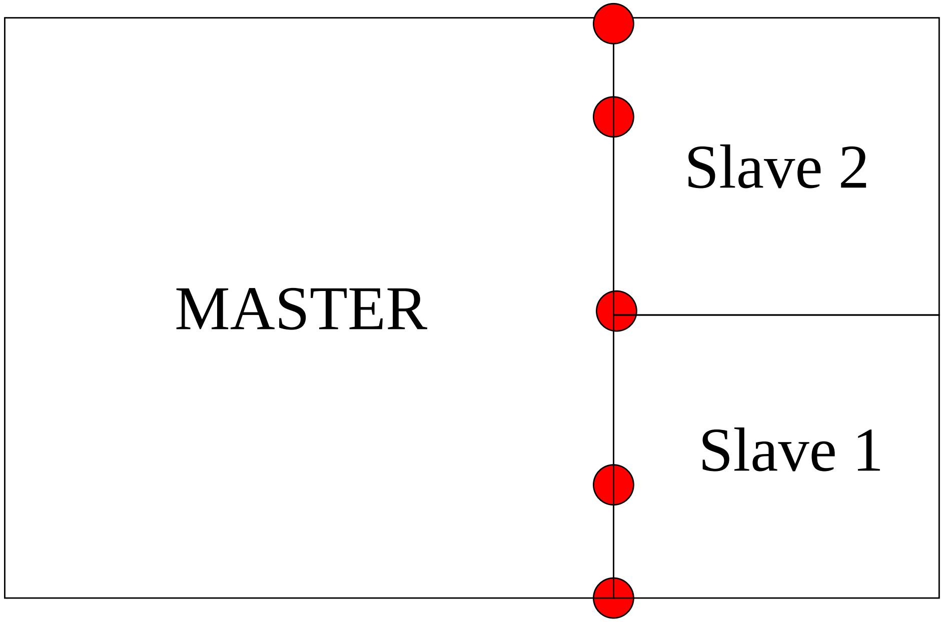


Figure 8: Non-conforming edges between parent and children

# *AMR: Trace interpolation*

Boolean matrix $Q$ is redefined as

$$Q = J_L \tilde{Q}$$

where $J_L$ is an interpolation matrix.
DSS is conceptually the same:

$$v^T A u = v^T (\tilde{Q}^T J_L^T) A_L (J_L \tilde{Q}) u = v^T Q^T A_L Q u.$$

(Fischer, Kruse and Loth 2002)

# *AMR: Validation: test description*

- Standard test suite of Williamson et al. 1992: test case 1.
- The initial condition is only $C_0$.
- Error estimator based on true solution.
- Error estimator of C. Mavriplis.
- Novel AMR algorithm uses only nearest neighbors communications. (no broadcast of the tree)
- Comparison between locally refined and uniform: same error.

# ***Conclusion***

- OIFS is equivalent to SL for advection.
- OIFS is efficient.
- Scalable alternative to semi-Lagrangian (SL).
- OIFS can be used with AMR.
- AMR: affordable high-resolution: will be attempted on primitive equations.
- Next step primitive equations.

# ***Acknowledgments***

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