A more stable semi-implicit scheme for MC2

Claude Girard & Michel Desgagné

Introduction

The problem: Original & New Stability Analysis

Searching for a solution

Conclusion: Having found a rather elegant solution

Introduction

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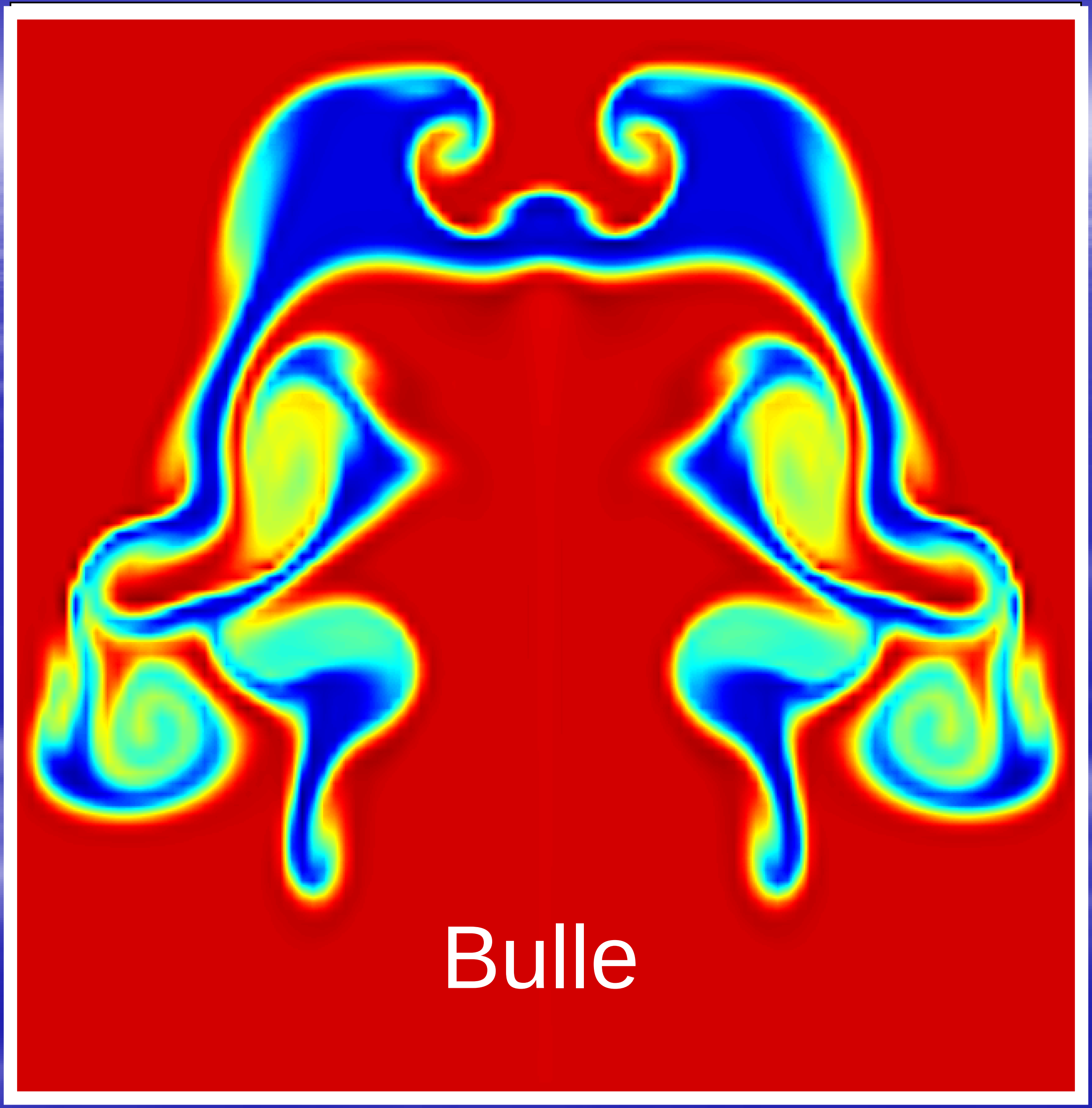
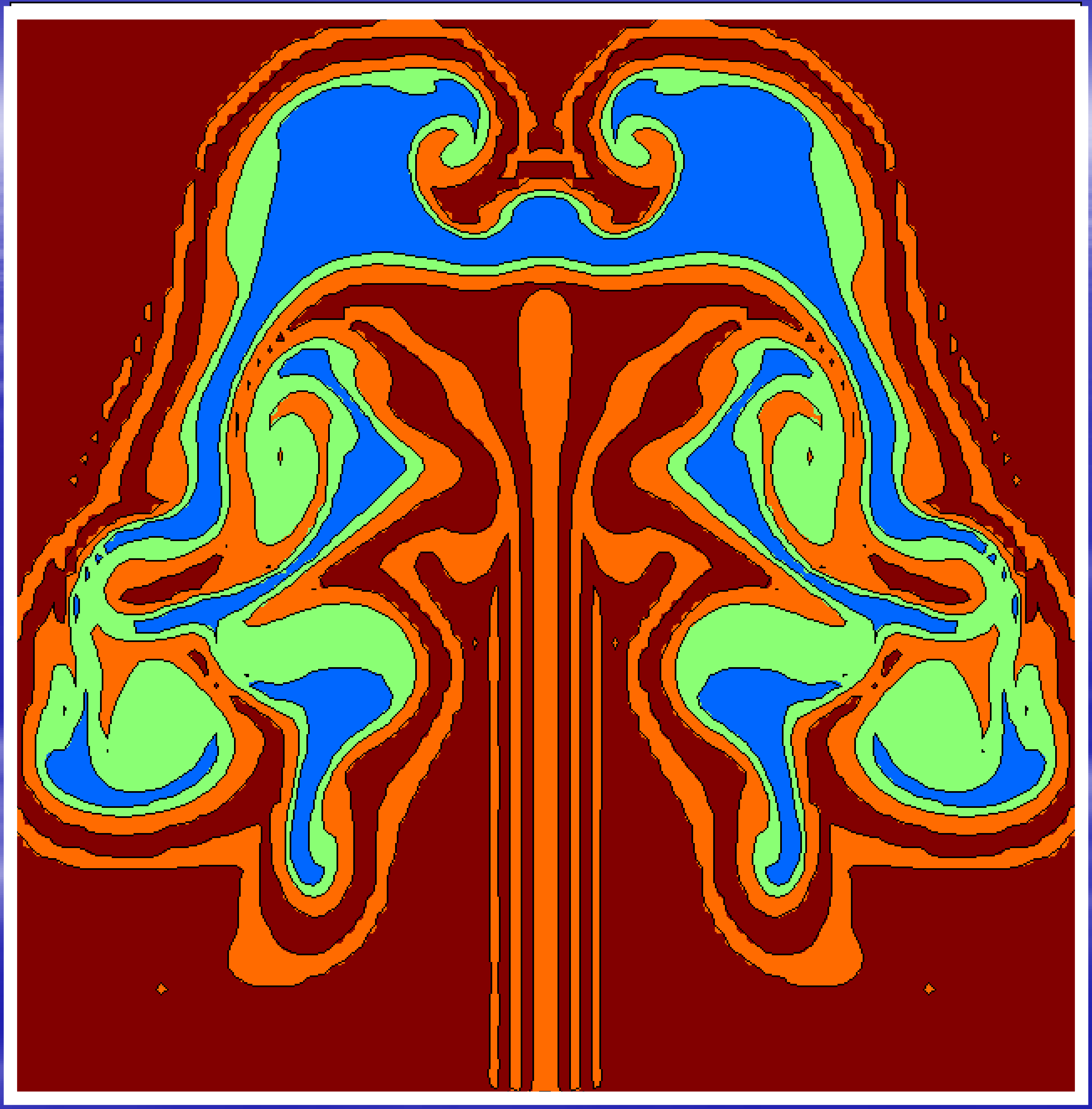
1990 TRL Tanguay, M., A. Robert, and R. Laprise, 1990: A semi-implicit semi-Lagrangian fully compressible regional forecast model. *Mon. Wea. Rev*., **118**, 1970-1980.

"The proposed semi-implicit semi-Lagrangian scheme is said to be **unconditionally stable**…" (p1979) ?


Bulle



Introduction

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> “The proposed semi-implicit semi-Lagrangian scheme is said to be **unconditionally stable**…” (p1979)

1992 Tanguay, M., E. Yakimiw, H. Ritchie and A. Robert, 1992: Advantages of spatial averaging in semi-implicit semi-Lagrangian schemes. *Mon. Wea. Rev.*, **120**, 113-123.

> “The **uncentered ε first-order accuracy version of the time and spatial average operators** has been taken to eliminate high-frequency oscillations…Those oscillations appear to be induced by imbalances in the initial fields as a result of an imperfect initialization produced by the currently used dynamic initialization procedure.” (p116)

?

"To remove small-scale noise, an **implicit spatial filter … has been applied to the mountain field**. **The topography is therefore smoother compared to the one in the spectral model**." (p117)

Result "…indicates that **the *εΔt* first-order accuracy is not negligible**." (p118)

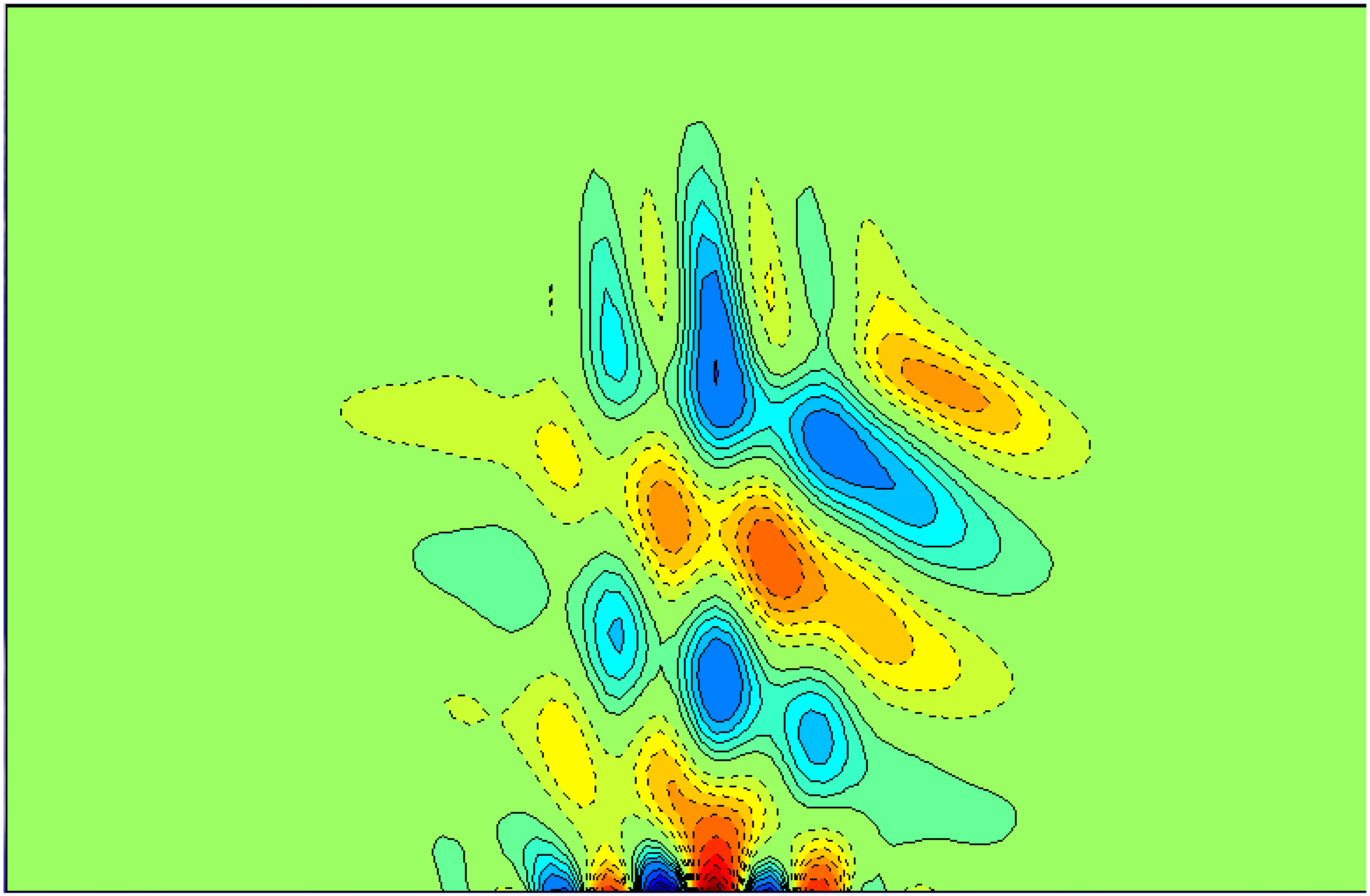
1996 Héreil, P. and R. Laprise, 1996: Sensitivity of **internal gravity waves** solutions to the time step of a semi-implicit semi-Lagrangian nonhydrostatic model, *Mon. Wea. Rev*., **124**, 972-999.

1998 Thomas, S. J., C. Girard, R. Benoit, M. Desgagné, and P. Pellerin, 1998: A **new adiabatic kernel** for the MC2 model. *Atmosphere-Ocean*, **36**, 241-270.

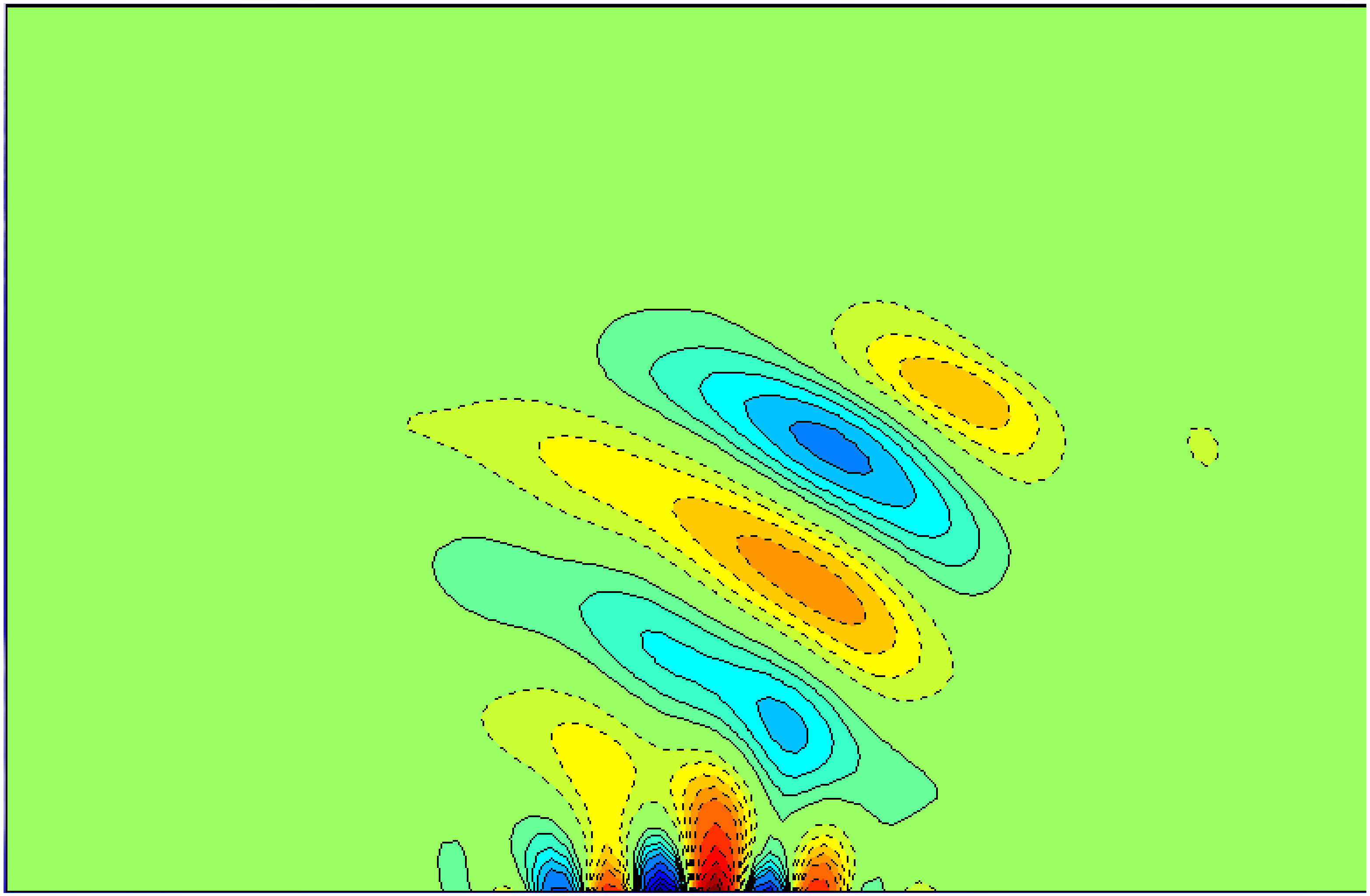
2002 Schär, C., D. Leuenberger, O. Fuhrer, D. Lüthi and C. Girard, 2002: A **new terrain-following vertical coordinate** formulation for high-resolution numerical weather prediction models, *Mon. Wea. Rev*., **130**, 2459-2480.

2002 Girard, C., M. Desgagné, R. Benoit, 2002 & **2004**: **Finescale topography** and the MC2 dynamics kernel, Seminaire RPN & MWR ready for submission.

Schaer



Girard



2003 Bénard, P., 2003: **Stability of semi-implicit** and iterative centered-implicit time discretizations for various equation systems used in NWP. *Mon. Wea. Rev.*, **131**, 2479-2491.

"For the 3-TL SI scheme, the external structure $\nu = 0$ is unconditionally stable for -.25 < α <1, but **slightly shorter structures as described earlier are found unstable at large time steps as soon as** $\alpha \neq 0$ (very short modes are stable however). Figure 1 depicts the asymptotic growth-rates for two structures: the external structure $\nu = 0$, and a long structure $\nu = 0.0001\ m^{-1}$. The growth rate of the long structure for a moderate time step Δt = 30 s with a time-decentering ε=0.1 … is also depicted: the practical instability becomes small under these conditions, and **the 3-TL scheme cannot be positively rejected**…" (p2489)

$$\alpha = \frac{T - T_*}{T_*}$$

This can be unstable

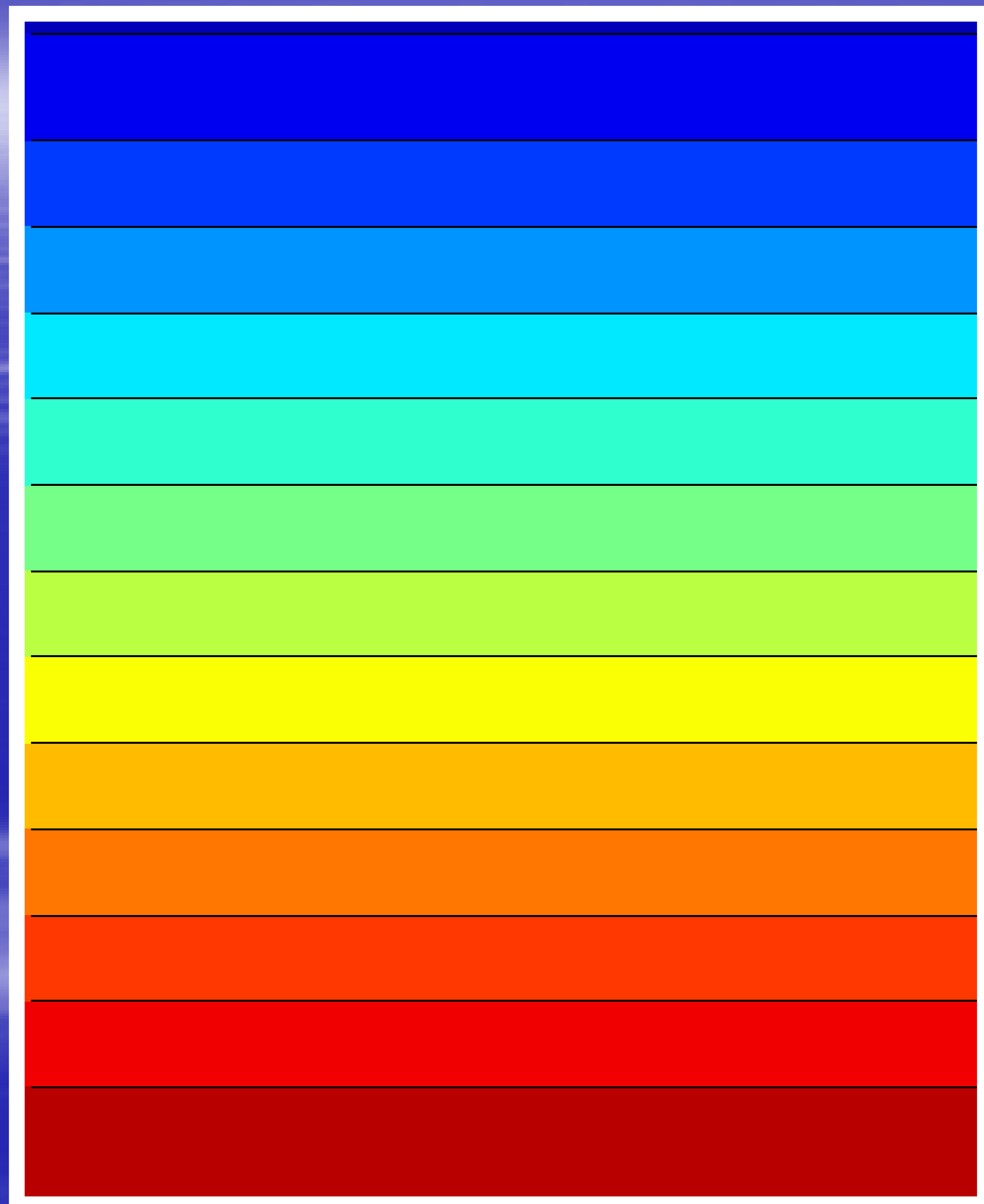
$\delta=0$

$\varepsilon=0$

$\alpha=.05$

$\Delta t=120$

$$\alpha = \frac{T - T_*}{T_*}$$



This can be unstable

$\delta=.05$

$\varepsilon=0$

$\alpha=.5$

$\Delta t=60$

The problem: Original & New Stability Analysis

A) The fundamental SISL (semi-implicit semi-Lagrangian) scheme

1) The Basic Model Equations (2D isentropic version):

2) Change thermodynamic variables *T, q* to deviations *T', q'*

$$\frac{du}{dt} + RT\frac{\partial q}{\partial x} = 0$$

$$\frac{dw}{dt} + RT\frac{\partial q}{\partial z} + g = 0$$

$$\frac{dT}{dt} - \frac{RT}{c_p}\frac{dq}{dt} = 0$$

$$\frac{c_v}{c_p}\frac{dq}{dt} + D = 0$$

$$q = \ln p$$

$$D = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

$$T' = T - T_*$$

$$q' = q - q_*$$

$$T_* = const.$$

$$\frac{\partial q_*}{\partial z} = -\frac{g}{RT_*}$$

A) The fundamental SISL (semi-implicit semi-Lagrangian) scheme

1) The Basic Model Equations (2D isentropic version):

$$\frac{du}{dt} + RT\frac{\partial q}{\partial x} = 0$$

$$\frac{dw}{dt} + RT\frac{\partial q}{\partial z} + g = 0$$

$$\frac{dT}{dt} - \frac{RT}{c_p}\frac{dq}{dt} = 0$$

$$\frac{c_v}{c_p}\frac{dq}{dt} + D = 0$$

2) Change thermodynamic variables *T, q* to deviations *T', q'*

$$P = RT_* q'$$

$$b = g\frac{T'}{T_*}$$

$$N_*^2 = \frac{g^2}{c_p T_*}$$

$$c_*^2 = \frac{c_p}{c_v} RT_*$$

$$N_*^2 = g\gamma_*$$

3) Change again to generalized pressure *P* and buoyancy *b*

4) Apply the semi-implicit semi-Lagrangian (SISL) scheme:

$$\frac{du}{dt} + \frac{\partial \overline{P}}{\partial x} = -\frac{T'}{T_*}\frac{\partial P}{\partial x}$$

$$\frac{dw}{dt} + \frac{\partial \overline{P}}{\partial z} - \overline{b} = -\frac{T'}{T_*}\frac{\partial P}{\partial z}$$

$$\frac{d}{dt}\left(b - \gamma_* P\right) + N_*^2 \overline{w} = -g\frac{R}{c_v}\frac{T'}{T_*}D$$

$$\frac{1}{c_*^2}\left(\frac{dP}{dt} - g\overline{w}\right) + \overline{D} = 0$$

$$\frac{dX}{dt} \equiv \frac{X^+ - X^-}{2\Delta t}$$

$$\overline{X} \equiv \frac{X^+(1+\varepsilon) + X^-(1-\varepsilon)}{2}$$

B) Original Stability Analysis

1) **Linearize around basic state *T****

2) **Consider eigenmodes**

$$X(x,z,t)=e^{ikx+nz}X(t)$$

$$n=i\nu+1/2H_*$$

$$\frac{du}{dt}+\frac{\partial \overline{P}}{\partial x}=0$$

$$\frac{dw}{dt}+\frac{\partial \overline{P}}{\partial z}-\overline{b}=0$$

$$\frac{db}{dt}-\gamma_*\frac{dP}{dt}+N_*^2\overline{w}=0$$

$$\frac{1}{c_*^2}\left(\frac{dP}{dt}-g\overline{w}\right)+\overline{D}=0$$

$$\frac{du}{dt}+\frac{\partial \overline{P}}{\partial x}=-\frac{T'}{T_*}\frac{\partial P}{\partial x}$$

$$\frac{dw}{dt}+\frac{\partial \overline{P}}{\partial z}-\overline{b}=-\frac{T'}{T_*}\frac{\partial P}{\partial z}$$

$$\frac{d}{dt}\left(b-\gamma_*P\right)+N_*^2\overline{w}=-g\frac{R}{c_v}\frac{T'}{T_*}D$$

$$\frac{1}{c_*^2}\left(\frac{dP}{dt}-g\overline{w}\right)+\overline{D}=0$$

B) Original Stability Analysis

1) **Linearize around basic state T^***

2) Consider eigenmodes

3) Use trigonometry

$$X(x,z,t) = e^{ikx+nz} X(t)$$

$$n = i\nu + 1/2H_*$$

$$\frac{du}{dt} + \frac{\partial \overline{P}}{\partial x} = 0$$

$$\frac{dw}{dt} + \frac{\partial \overline{P}}{\partial z} - \overline{b} = 0$$

$$\frac{db}{dt} - \gamma_* \frac{dP}{dt} + N_*^2 \overline{w} = 0$$

$$\frac{1}{c_*^2}\left(\frac{dP}{dt} - g\overline{w}\right) + \overline{D} = 0$$

$$\frac{\Lambda^{-4}}{c_*^2} + \Lambda^{-2}\Lambda^{+2}\left(k^2 + nn^*\right) + N_*^2 k^2 \Lambda^{+4} = 0$$

$$\frac{1}{\Delta t^4 c_*^2}\tan^4\gamma - \frac{1}{\Delta t^2}\tan^2\gamma\left(k^2 + nn^*\right) + N_*^2 k^2 = 0$$

$$\Lambda^- = \left(\lambda^2 - 1\right)/2\Delta t$$

$$\Lambda^+ = \left(\lambda^2 + 1\right)/2$$

$$\lambda = X^+ / X = X / X^-$$

B) Original Stability Analysis

1) **Linearize around basic state *T****

2) Consider eigenmodes

3) Use trigonometry

4) Analyze: There is in fact no restriction on tanγ

$$X(x,z,t) = e^{ikx+nz} X(t)$$

$$n = i\nu + 1/2H_*$$

$$\frac{\Lambda^{-4}}{c_*^2} + \Lambda^{-2}\Lambda^{+2}\left(k^2 + nn^*\right) + N_*^2 k^2 \Lambda^{+4} = 0$$

$$\lambda = \pm \frac{(1 + i\tan\gamma)}{\sqrt{1+\tan^2\gamma}} = \pm e^{i\omega\Delta t}$$

$$\lambda\lambda^* = 1$$

$$\frac{1}{\Delta t^4 c_*^2}\tan^4\gamma - \frac{1}{\Delta t^2}\tan^2\gamma\left(k^2 + nn^*\right) + N_*^2 k^2 = 0$$

$$\Lambda^- = \left(\lambda^2 - 1\right)/2\Delta t$$

$$\Lambda^+ = \left(\lambda^2 + 1\right)/2$$

$$\lambda = X^+/X = X/X^-$$

"The proposed semi-implicit semi-Lagrangian scheme is said to be **unconditionally stable**…" (TRL p1979)

C) New stability analysis

1a) Consider true perturbation variables around mean state T_o, q_o : T'', q''

$$T'' = T - T_o; \qquad T' = T'' + \alpha T_*$$

$$q'' = q - q_o; \qquad q' = q'' + \frac{z\alpha}{H_o}$$

$$H_o = \frac{g}{RT_o}$$

$$b_1 = g\frac{T''}{T_*}$$

$$\Rightarrow b = b_1 + g\alpha$$

$$\Rightarrow db = db_1$$

$$P_1 = RT_* q''$$

$$\Rightarrow P = P_1 + gz\frac{\alpha}{1+\alpha}$$

$$\alpha = \frac{T_o}{T_*} - 1$$

$$\Rightarrow dP = dP_1 + g\frac{\alpha}{1+\alpha}dz$$

$$\frac{du}{dt} + \frac{\partial \overline{P}_1}{\partial x} = -\frac{1}{g}(b_1 + g\alpha)\frac{\partial P_1}{\partial x}$$

$$\frac{dw}{dt} + \frac{\partial \overline{P}_1}{\partial z} + \frac{g\alpha}{1+\alpha} - \left(\overline{b}_1 + g\alpha\right) = -\frac{1}{g}(b_1 + g\alpha)\left(\frac{\partial P_1}{\partial z} + \frac{g\alpha}{1+\alpha}\right)$$

$$\frac{d}{dt}(b_1 - \gamma_* P_1) - \gamma_* w\frac{g\alpha}{1+\alpha} + N_*^2\overline{w} = -\frac{R}{c_v}(b_1 + g\alpha)D$$

$$\frac{1}{c_*^2}\left(\frac{dP_1}{dt} + w\frac{g\alpha}{1+\alpha} - g\overline{w}\right) + \overline{D} = 0$$

$$\frac{\partial z}{\partial z} = 1$$

$$\frac{dz}{dt} = w$$

C) New stability analysis

1a) Consider true perturbation variables around mean state T_o, q_o : T'', q''

1b) Linearize around mean state T_o

2) Consider eigenmodes

$$\frac{du}{dt}+\frac{\partial \overline{P}_1}{\partial x}=-\alpha\frac{\partial P_1}{\partial x}$$

$$\frac{dw}{dt}+\frac{\partial \overline{P}_1}{\partial z}-\overline{b}_1=-\alpha\frac{\partial P_1}{\partial z}\qquad -b_1\frac{\alpha}{1+\alpha}$$

$$\frac{d}{dt}\left(b_1-\gamma_* P_1\right)+N_*^2\overline{w}=-\frac{R}{c_v}g\alpha D+N_*^2 w\frac{\alpha}{1+\alpha}$$

$$\frac{1}{c_*^2}\left(\frac{dP_1}{dt}-g\overline{w}\right)+\overline{D}=\qquad -\frac{g}{c_*^2}w\frac{\alpha}{1+\alpha}$$

$$\frac{du}{dt}+\frac{\partial \overline{P}_1}{\partial x}=-\frac{1}{g}\left(b_1+\cdots\right)\cdots\frac{\cdots}{\partial x}$$

$$\frac{dw}{dt}+\frac{\partial \overline{P}_1}{\partial z}+\frac{g\alpha}{1+\alpha}-\left(\overline{b}_1+g\alpha\right)=-\frac{1}{g}\left(b_1+g\alpha\right)\left(\frac{\partial P_1}{\partial z}+\frac{g\alpha}{1+\alpha}\right)$$

$$\frac{d}{dt}\left(b_1-\gamma_* P_1\right)-\gamma_* w\frac{g\alpha}{1+\alpha}+N_*^2\overline{w}=-\frac{R}{c_v}\left(b_1+g\alpha\right)D$$

$$\frac{1}{c_*^2}\left(\frac{dP_1}{dt}+w\frac{g\alpha}{1+\alpha}-g\overline{w}\right)+\overline{D}=0$$

C) New stability analysis

1a) Consider true perturbation variables around mean state T_o, q_o : T'', q''

1b) Linearize around mean state T_o

2) Consider eigenmodes

3) Use trigonometry

$$\Lambda_1^+ = \Lambda^+ + \alpha\lambda$$

$$\Lambda_2^+ = \Lambda^+ - \alpha\lambda / (1+\alpha)$$

$$\frac{du}{dt} + \frac{\partial \overline{P_1}}{\partial x} = -\alpha \frac{\partial P_1}{\partial x}$$

$$\frac{dw}{dt} + \frac{\partial \overline{P_1}}{\partial z} - \overline{b_1} = -\alpha \frac{\partial P_1}{\partial z} \qquad - b_1 \frac{\alpha}{1+\alpha}$$

$$\frac{d}{dt}\left(b_1 - \gamma_* P_1\right) + N_*^2 \overline{w} = -\frac{R}{c_v} g \alpha D + N_*^2 w \frac{\alpha}{1+\alpha}$$

$$\frac{1}{c_*^2}\left(\frac{dP_1}{dt} - g\overline{w}\right) + \overline{D} = \qquad -\frac{g}{c_*^2} w \frac{\alpha}{1+\alpha}$$

$$\frac{\Lambda^{-4}}{c_*^2} + \Lambda^{-2}\Lambda_1^+ \left\{ \left(k^2 + nn^*\right)\Lambda^+ + \frac{n\alpha}{H_o}\left(\Lambda^+ - \lambda\right) \right\} + N_*^2 k^2 \Lambda_1^{+2} \Lambda_2^{+2} = 0$$

$$\frac{\sin^4\gamma}{\Delta t^4 c_*^2} - \frac{\sin^2\gamma}{\Delta t^2}\left(\cos\gamma + \alpha\right)\left\{ \left(k^2 + nn^*\right)\cos\gamma + \frac{n\alpha}{H_o}\left(\cos\gamma - 1\right) \right\}$$

$$+ N_*^2 k^2 \left(\cos\gamma + \alpha\right)^2 \left(\cos\gamma - \frac{\alpha}{1+\alpha}\right)^2 = 0$$

$$\frac{\sin^4\gamma}{\Delta t^4 c_*^2} - \frac{\sin^2\gamma}{\Delta t^2}(\cos\gamma+\alpha)\left\{\left(k^2+nn^*\right)\cos\gamma + \frac{n\alpha}{H_o}(\cos\gamma-1)\right\}$$

$$+N_*^2 k^2(\cos\gamma+\alpha)^2\left(\cos\gamma-\frac{\alpha}{1+\alpha}\right)^2=0$$

$$n = i\nu + 1/2H_*$$

4) Analyze (asymptotic behavior) $\Delta t \to \infty$

n.b. α=0

$$\frac{1}{\Delta t^4 c_*^2}\tan^4\gamma - \frac{1}{\Delta t^2}\tan^2\gamma\left(k^2+nn^*\right)+N_*^2 k^2=0$$

$$\frac{\sin^4\gamma}{\Delta t^4 c_*^2} - \frac{\sin^2\gamma}{\Delta t^2}(\cos\gamma+\alpha)\left\{(k^2+nn^*)\cos\gamma + \frac{n\alpha}{H_o}(\cos\gamma-1)\right\}$$

$$+N_*^2 k^2 (\cos\gamma+\alpha)^2\left(\cos\gamma - \frac{\alpha}{1+\alpha}\right)^2 = 0$$

$n = i\nu + 1/2H_*$

5) Analysis (asymptotic behavior) $\Delta t \to \infty$

a) ν=0 (external mode $\Leftrightarrow$ shallow water model)

i) $k \neq 0$

$$(\cos\gamma+\alpha)^2\left(\cos\gamma - \frac{\alpha}{1+\alpha}\right)^2 = 0$$

$|\alpha| \le 1$ $\quad$ $\left|\frac{\alpha}{1+\alpha}\right| \le 1;\ \alpha \ge -1/2$

$-1/2 \le \alpha \le 1$

$$\frac{\sin^4\gamma}{\Delta t^4 c_*^2} - \frac{\sin^2\gamma}{\Delta t^2}(\cos\gamma+\alpha)\left\{\left(k^2+nn^*\right)\cos\gamma + \frac{n\alpha}{H_o}(\cos\gamma - 1)\right\}$$

$$+ N_*^2 k^2 (\cos\gamma+\alpha)^2\left(\cos\gamma - \frac{\alpha}{1+\alpha}\right)^2 = 0$$

$n = i\nu + 1/2H_*$

5) Analysis (asymptotic behavior) $\Delta t \to \infty$

a) ν=0 (external mode $\Leftrightarrow$ shallow water model)

ii) $k = 0$ (1D version in vertical)

$$\left(1-\cos^2\gamma\right)(\cos\gamma+\alpha)\left\{\cos\gamma - \frac{2\alpha}{1+2\alpha}\right\} = 0$$

$$\left|\frac{2\alpha}{1+2\alpha}\right| \le 1 \to \alpha \ge -1/4$$

$T_* = 273 \implies 205 \le T \le 546$

iii) all *k*

$-1/4 \le \alpha \le 1$

$$\frac{\sin^4\gamma}{\Delta t^4 c_*^2} - \frac{\sin^2\gamma}{\Delta t^2}(\cos\gamma+\alpha)\left\{\left(k^2+nn^*\right)\cos\gamma + \frac{n\alpha}{H_o}(\cos\gamma-1)\right\}$$

$$+N_*^2 k^2(\cos\gamma+\alpha)^2\left(\cos\gamma-\frac{\alpha}{1+\alpha}\right)^2=0$$

$$n = i\nu + 1/2H_*$$

5) Analysis (asymptotic behavior) $\Delta t \rightarrow \infty$

a) ν=0 (external mode $\Leftrightarrow$ shallow water model)

b) $\nu \neq 0$ $k=0$ (1D version in vertical)

$$(\cos\gamma+\alpha)\left\{\cos\gamma - \frac{2\alpha}{1+2\alpha-2i\nu H_o}\right\}=0$$

the scheme is always unstable as soon as

$\alpha \neq 0$!!!

2003
Bénard

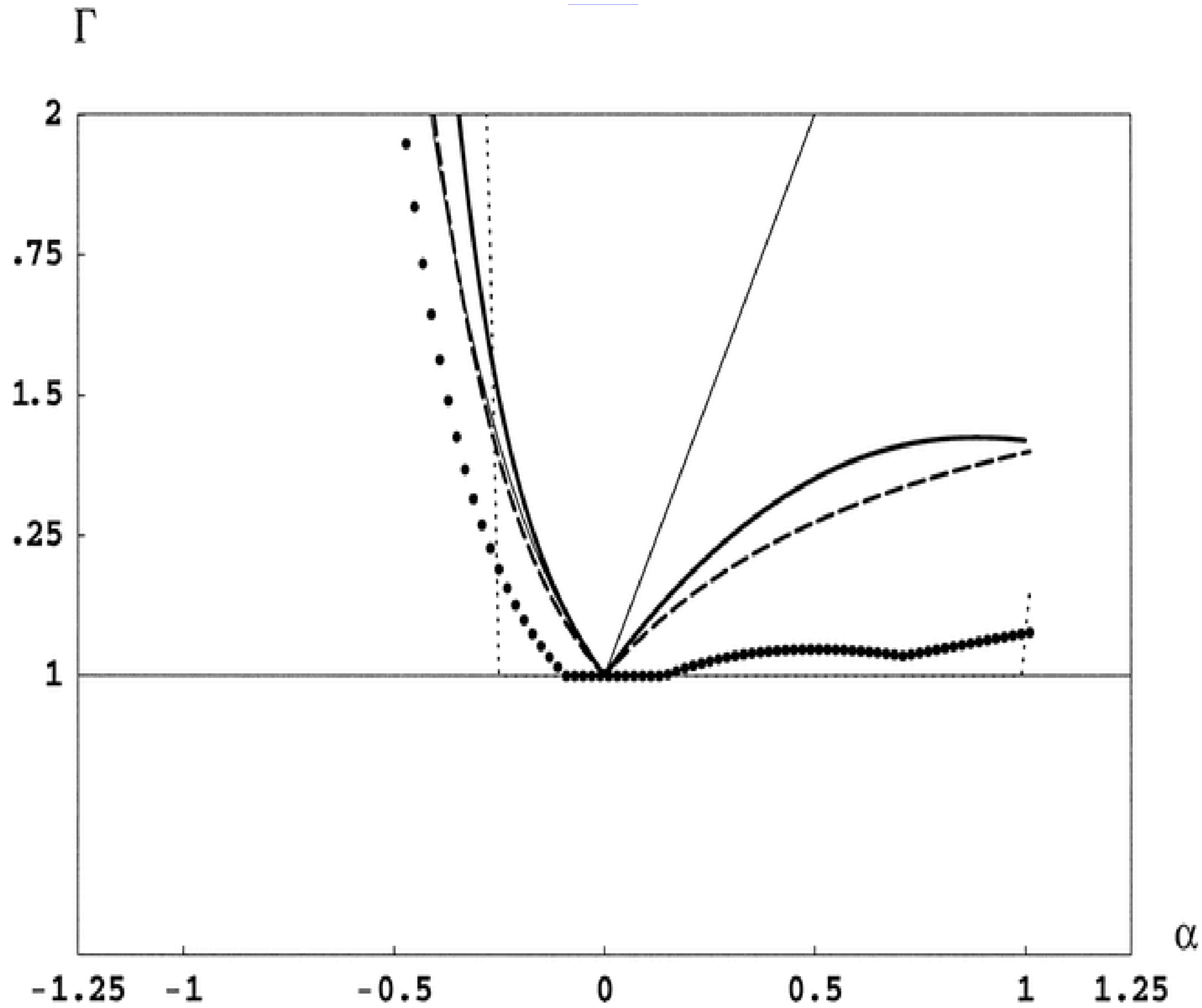


FIG. 1. Asymptotic growth-rates Γ for a 1D vertical system in *z* coordinates as a function of the nonlinearity parameter α. Long mode ($\nu = 0.0001\ m^{-1}$) with 2-TL SI scheme (thin line); long mode with 2-TL ICI scheme $N_{iter} = 2$ (thick line); external mode ($\nu = 0$) with 3-TL SI scheme (dotted line); long mode with 3-TL SI scheme (dashed line); practical growth-rate of 3-TL SI scheme for the long mode with $\Delta t = 30$ s and $\epsilon = 0.1$ (circles).

Summary: Dispersion Relations

$$\frac{\sin^4\gamma}{\Delta t^4 c_*^2} - \frac{\sin^2\gamma}{\Delta t^2}(\cos\gamma+\alpha)\left\{\left(k^2+nn^*\right)\cos\gamma + \frac{n\alpha}{H_o}(\cos\gamma-1)\right\} + N_*^2 k^2(\cos\gamma+\alpha)^2\left(\cos\gamma - \frac{\alpha}{1+\alpha}\right)^2 = 0$$

$$\frac{\Lambda^{-4}}{c_*^2} + \Lambda^{-2}\Lambda_1^{+}\left\{\left(k^2+nn*\right)\Lambda^{+} + \frac{n\alpha}{H_o}\left(\Lambda^{+}-\lambda\right)\right\} + N_*^2 k^2 \Lambda_1^{+2}\Lambda_2^{+2} = 0$$

$$\frac{\sin^2\gamma}{\Delta t^2 c_*^2} - (\cos\gamma+\alpha)\ \left\{nn^*\cos\gamma + \frac{n\alpha}{H_o}(\cos\gamma-1)\right\} = 0$$

$k = 0$

$$\frac{\Lambda^{-2}}{c_*^2} + \Lambda_1^{+}\left\{nn*\Lambda^{+} + \frac{n\alpha}{H_o}\left(\Lambda^{+}-\lambda\right)\right\} = 0$$

Searching for a solution

Searching for a solution

Scheme 1: A naïve fully implicit scheme

$$\frac{dw}{dt}+\frac{\partial\overline{P}}{\partial z}-\overline{b}=-\alpha_o\frac{\partial\overline{P}}{\partial z}$$

$$\frac{d}{dt}\left(b-\gamma_* P\right)+N_*^2\overline{w}=-g\frac{R}{c_v}\alpha_o\frac{\partial w}{\partial z}$$

$$\frac{1}{c_*^2}\left(\frac{dP}{dt}-g\overline{w}\right)+\frac{\partial\overline{w}}{\partial z}=0$$

$$\frac{dw}{dt}+\frac{\partial\overline{P}}{\partial z}-\overline{b}=-\alpha_o\frac{\partial\overline{P}}{\partial z}$$

$$\frac{d}{dt}\left(b-\gamma_* P\right)+N_*^2\overline{w}=-g\frac{R}{c_v}\alpha_o\frac{\partial\overline{w}}{\partial z}$$

$$\frac{1}{c_*^2}\left(\frac{dP}{dt}-g\overline{w}\right)+\frac{\partial\overline{w}}{\partial z}=0$$

$$\alpha_o=\frac{b^o}{g}=\frac{(T')^o}{T_*}$$

$$\frac{\Lambda^{-2}}{c_o^2}+\Lambda^+\left\{nn*\Lambda^+ + \frac{n\alpha_o}{H_o}\left(\Lambda^+-\lambda\frac{c_v}{c_p}\right)\right\}=0$$

Searching for a solution

Scheme 1: A naïve fully implicit scheme

Scheme 2: The full thing (dropping subindex o de α**)**

$$\frac{dw}{dt}+\frac{\partial \overline{P_1}}{\partial z}-\overline{b_1}=-\alpha\frac{\partial \overline{P_1}}{\partial z} \qquad -\overline{b_1}\frac{\alpha}{1+\alpha}$$

$$\frac{d}{dt}\left(b_1-\gamma_* P_1\right)+N_*^2\overline{w}=-\frac{R}{c_v}g\alpha\frac{\partial \overline{w}}{\partial z}+N_*^2\overline{w}\frac{\alpha}{1+\alpha}$$

$$\frac{1}{c_*^2}\left(\frac{dP_1}{dt}-g\overline{w}\right)+\frac{\partial \overline{w}}{\partial z}= \qquad -\frac{g}{c_*^2}\overline{w}\frac{\alpha}{1+\alpha}$$

$$\frac{dw}{dt}+\left(1+\alpha\right)\frac{\partial \overline{P_1}}{\partial z}-\frac{1}{1+\alpha}\overline{b_1}=0$$

$$\frac{d}{dt}\left(b_1-\gamma_* P_1\right)+N_o^2\overline{w}+\frac{R}{c_v}g\alpha\frac{\partial \overline{w}}{\partial z}=0$$

$$\frac{1}{c_*^2}\frac{dP_1}{dt}+\frac{\partial \overline{w}}{\partial z}-\frac{g}{c_o^2}\overline{w}=0$$

$$\frac{\Lambda^{-2}}{c_o^2}+\Lambda^{+2}nn^*=0$$

$$N_o^2=\frac{N_*^2}{1+\alpha}=\frac{g^2}{c_pT_*\left(1+\alpha\right)}$$

$$c_o^2=c_*^2\left(1+\alpha\right)=\frac{c_p}{c_v}RT_*\left(1+\alpha\right)$$

What it means in terms of original variables

is

$$\frac{dw}{dt} + (1+\alpha)\frac{\partial \overline{P}}{\partial z} - \frac{1}{1+\alpha}\overline{b} = +b\frac{\alpha}{1+\alpha}$$

$$\frac{d}{dt}(b - \gamma_* P) + N_o^2 \overline{w} + \frac{R}{c_v} g\alpha \frac{\partial \overline{w}}{\partial z} = -N_o^2 w\alpha$$

$$\frac{1}{c_*^2}\frac{dP}{dt} + \frac{\partial \overline{w}}{\partial z} - \frac{g}{c_o^2}\overline{w} = +\frac{g}{c_o^2} w\alpha$$

Searching for a solution

Scheme 1: A naïve fully implicit scheme

Scheme 2: The full thing (dropping subindex o de α**)**

Scheme 3: The linear part only

$$\frac{dw}{dt}+\frac{\partial \overline{P}}{\partial z}-\frac{1}{1+\alpha}\overline{b}=-\frac{b}{g}\frac{\partial P}{\partial z}+b\frac{\alpha}{1+\alpha}$$

$$\frac{d}{dt}\left(b-\gamma_* P\right)+N_o^2\overline{w}=-\frac{R}{c_v}g\alpha\frac{\partial w}{\partial z}-N_o^2 w\alpha$$

$$\frac{1}{c_*^2}\frac{dP}{dt}+\frac{\partial \overline{w}}{\partial z}-\frac{g}{c_o^2}\overline{w}=+\frac{g}{c_o^2}w\alpha$$

1D analysis

$$\frac{\Lambda^{-2}}{c_*^2}+\Lambda^+\left(\Lambda^+ +\alpha\lambda\right)nn^*=0$$

$$\cos^2\gamma+\frac{\alpha}{1+K^2}\cos\gamma-\frac{K^2}{1+K^2}=0$$

$$K^2=\frac{1}{nn^* c_*^2\Delta t^2}$$

2D analysis

$$\frac{\Lambda^{-4}}{c_o^2}+\Lambda^{-2}\frac{\Lambda_1^+}{1+\alpha}\Lambda^+\left(k^2+nn^*\right)+\Lambda^{+2}\left(\frac{\Lambda_1^+}{1+\alpha}\right)^2 N_o^2 k^2=0$$

$$-1\le\alpha\le 1$$

Conclusion

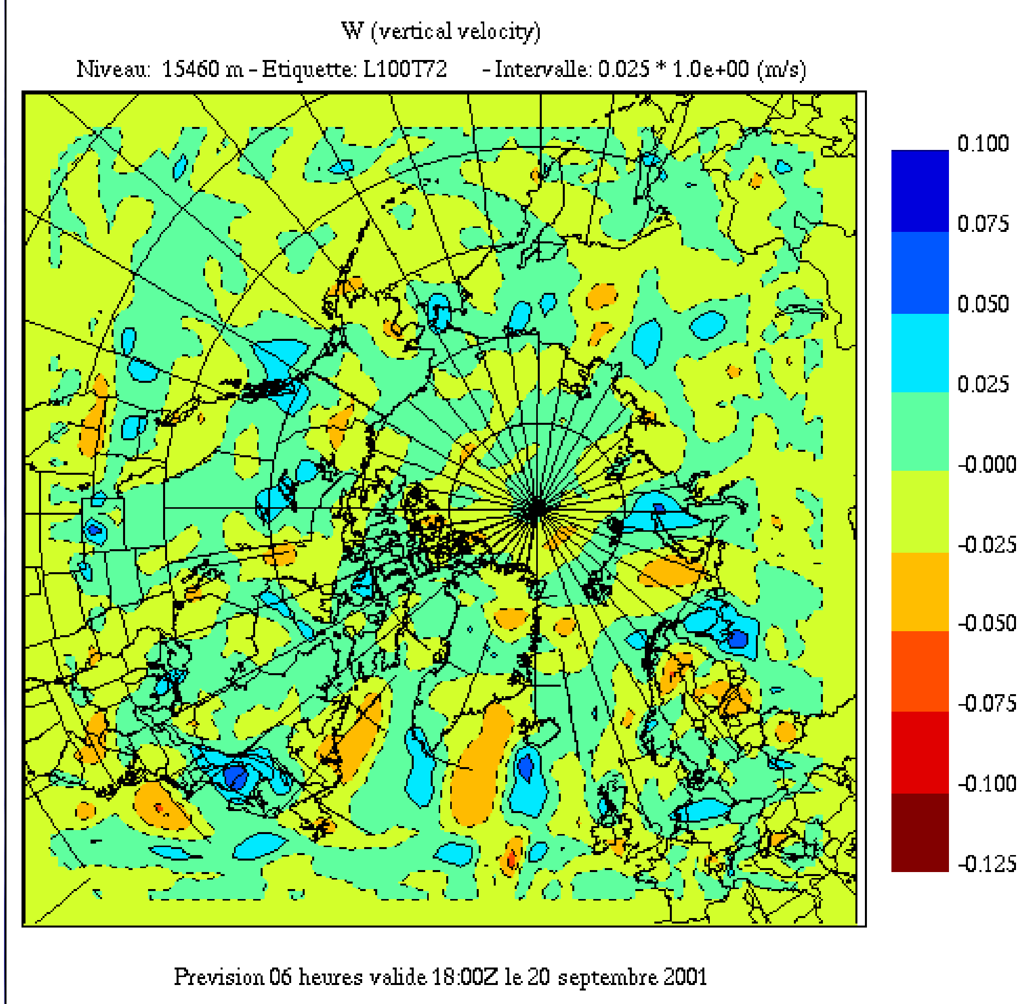
Having found a rather elegant solution

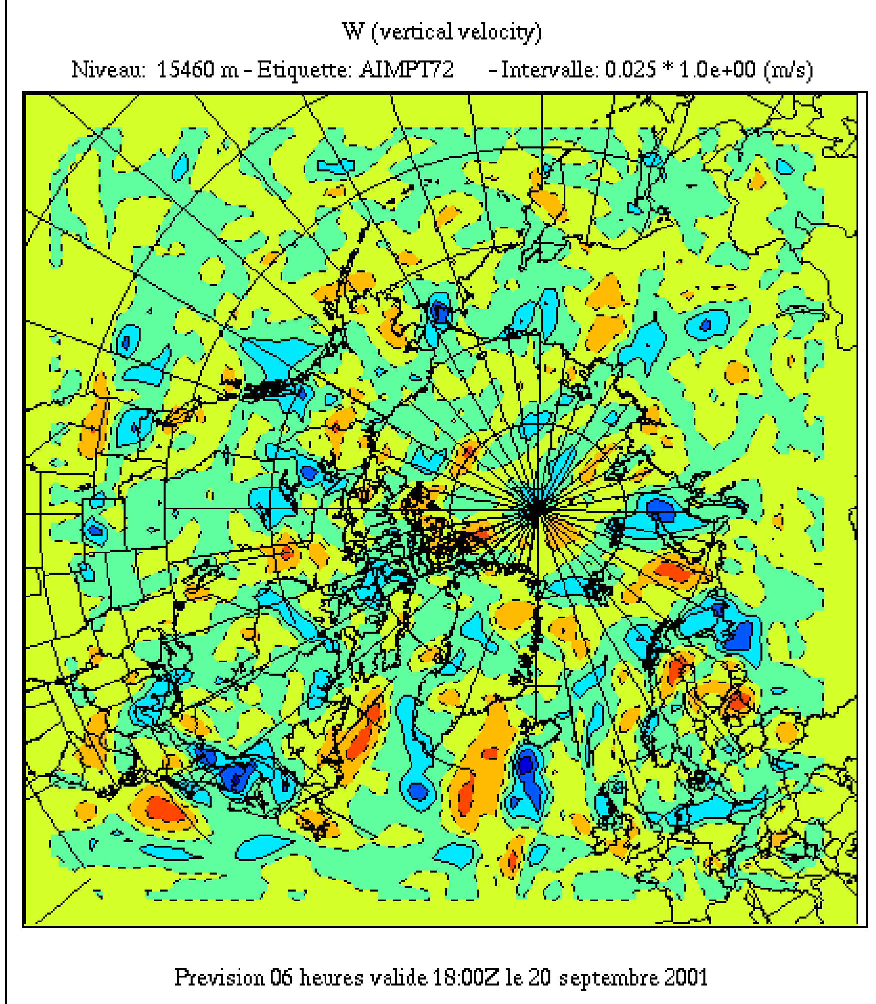
$$\frac{du}{dt} + \frac{\partial \overline{P}}{\partial x} = -\frac{b}{g}\frac{\partial P}{\partial x}$$

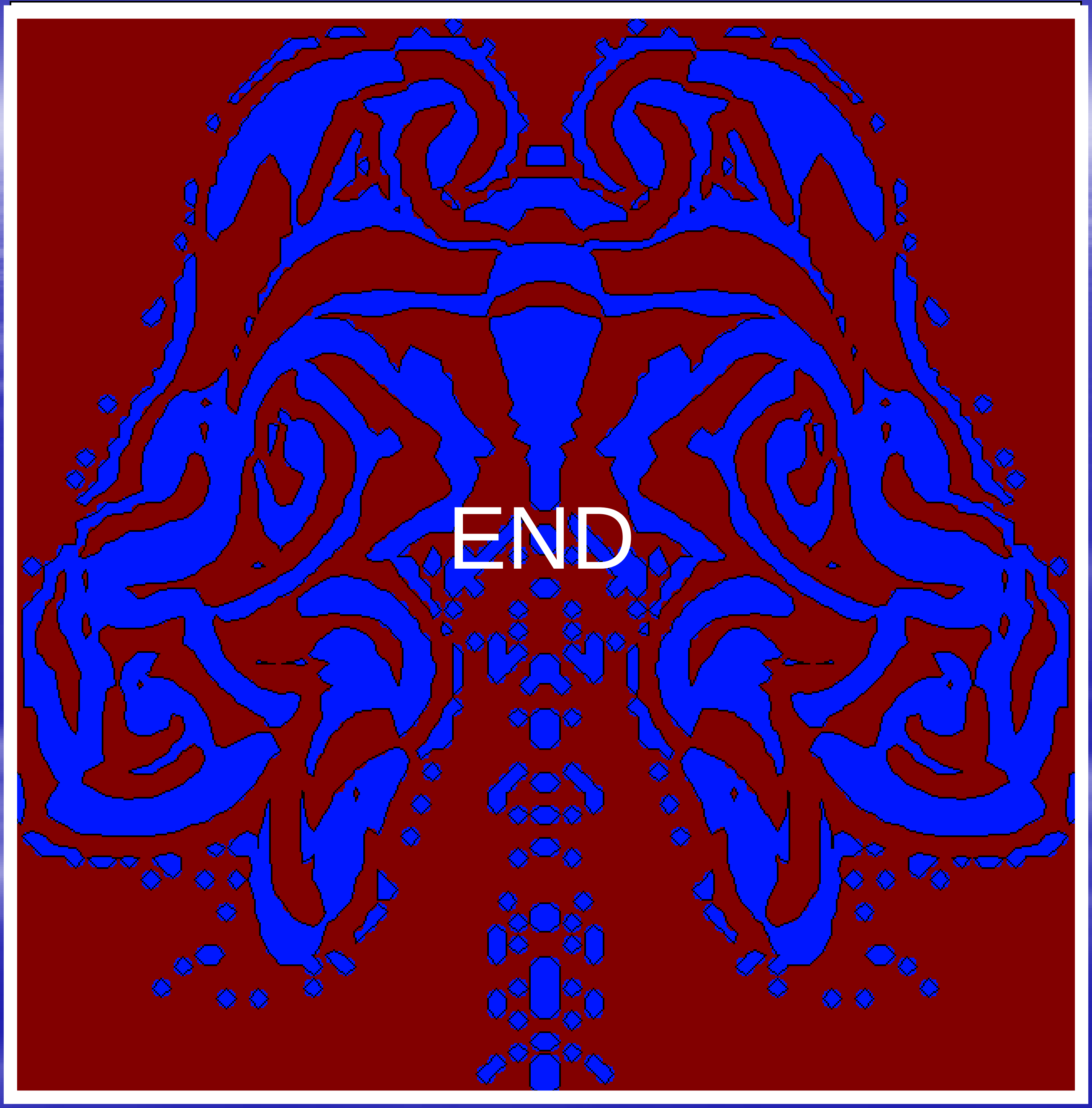
$$\frac{dw}{dt} + \frac{\partial \overline{P}}{\partial z} - \frac{\overline{b}}{1+\alpha} = -\frac{b}{g}\frac{\partial P}{\partial z} \qquad + b\frac{\alpha}{1+\alpha}$$

$$\frac{d}{dt}\left(b - \gamma_* P\right) + \frac{N_*^2}{1+\alpha}\overline{w} = -\frac{R}{c_v} g\alpha D \qquad - N_*^2 w \frac{\alpha}{1+\alpha}$$

$$\frac{1}{c_*^2}\frac{dP}{dt} + \overline{D} - \frac{g}{c_*^2(1+\alpha)}\overline{w} = \qquad + \frac{g}{c_*^2} w \frac{\alpha}{1+\alpha}$$


W (vertical velocity)
Niveau: 15460 m - Etiquette: L100T72 - Intervalle: 0.025 * 1.0e+00 (m/s)
0.100
0.075
0.050
0.025
-0.000
-0.025
-0.050
-0.075
-0.100
-0.125
Prevision 06 heures valide 18:00Z le 20 septembre 2001


W (vertical velocity)
Niveau: 15460 m - Etiquette: AIMPT72 - Intervalle: 0.025 * 1.0e+00 (m/s)
Prevision 06 heures valide 18:00Z le 20 septembre 2001


END

Atmosphere

Unification

Oceans

Pierre Pellerin

Claude Girard

André Robert

Francois Roy

Francois Saucier

Michel Desgagné

Unified Equations

Quasi-unified semi-discrete equations

Introduction of a free surface

Solid object

Unified Equations

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p + \rho \nabla \Phi = \rho \mathbf{f}$$

$$\rho c_p \frac{dT}{dt} - \alpha T \frac{dp}{dt} = \rho Q$$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$$

$$\gamma = c_p / c_v$$

$$c^2 = \frac{c_p(\gamma - 1)}{\alpha^2 T}$$

$$d\rho = -\rho \alpha dT + \frac{\gamma}{c^2} dp$$

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Quasi-unified semi-discrete equations

$$\frac{d\mathbf{v}}{dt} + \left[\nabla - \frac{N_*^2}{g}\mathbf{k}\right]P - B\mathbf{k} = R_V$$

$$\frac{dB}{dt} + N_*^2 w = R_B$$

$$\frac{d}{dt}\left(\frac{P}{c_*^2}\right) + \nabla \cdot \mathbf{v} - \frac{g}{c_*^2} w = R_P$$

generalized pressure

$$P = \frac{p'}{\rho_*}$$

$$P = RT_* q'$$

buoyancy

$$b = -g\frac{\rho'}{\rho_*}$$

$$b = g\frac{T'}{T_*}$$

generalized buoyancy

$$B = b - \gamma_W P$$

$$B = b - \gamma_A P$$

Introduction of a free surface

We used to assume that $p_*=p_*(z)$ and $\rho_*=\rho_*(z)$ only, i.e. the domain was assumed closed by a rigid top surface

$$\nabla p_* = \frac{\partial p_*}{\partial z}\mathbf{k} = -g\rho_*\mathbf{k}$$

$$\frac{dp_*}{dt} = \mathbf{v}\cdot\nabla p_* = w\frac{\partial p_*}{\partial z} = -g\rho_* w$$

We now want allow for $p_*=p_*(x, y, z, t)$ and $\rho_*=\rho_*(x, y, z, t)$ as generated by a free top surface

$$\rho_* = \rho_T e^{\beta_W (z_T - z)}$$

$$\beta_W = \partial \ln\rho_* / \partial z = const.$$

$$p_* = p_T + \frac{g\rho_T}{\beta_W}\left(e^{\beta_W (z_T - z)} - 1\right)$$

$$\frac{1}{\rho_*}\nabla_H p_* = \nabla_H g z_T$$

$$\frac{1}{\rho_*}\frac{dp_*}{dt} = g\left(\frac{dz_T}{dt} - w\right)$$

$$\frac{d\mathbf{v}}{dt} + \nabla g z_T + \left[\nabla - \frac{N_*^2}{g}\mathbf{k}\right]P - B\mathbf{k} = \mathbf{R}_V$$

$$\frac{dB}{dt} + \left\{N_*^2 - \beta_W B\right\}\left(w - \frac{dz_T}{dt}\right) = R_B$$

$$\frac{d}{dt}\left(\frac{P}{c_*^2}\right) + \nabla\cdot\mathbf{v} - \left\{\frac{g}{c_*^2} + \beta_W\frac{P}{c_*^2}\right\}\left(w - \frac{dz_T}{dt}\right) = R_P$$

Introduction of a free surface

$$\frac{d\mathbf{v}}{dt} + \left\{ \left[\nabla - \frac{N_*^2}{g}\mathbf{k} \right] P^\varsigma - B^\varsigma \mathbf{k} \right\} = \mathbf{R}_\mathbf{V}$$

$$\frac{dB^\varsigma}{dt} + N_*^2 w = R_B^\varsigma$$

$$\frac{d}{dt}\left(\frac{P^\varsigma}{c_*^2}\right) + \nabla \cdot \mathbf{v} - \frac{g}{c_*^2} w = R_P^\varsigma$$

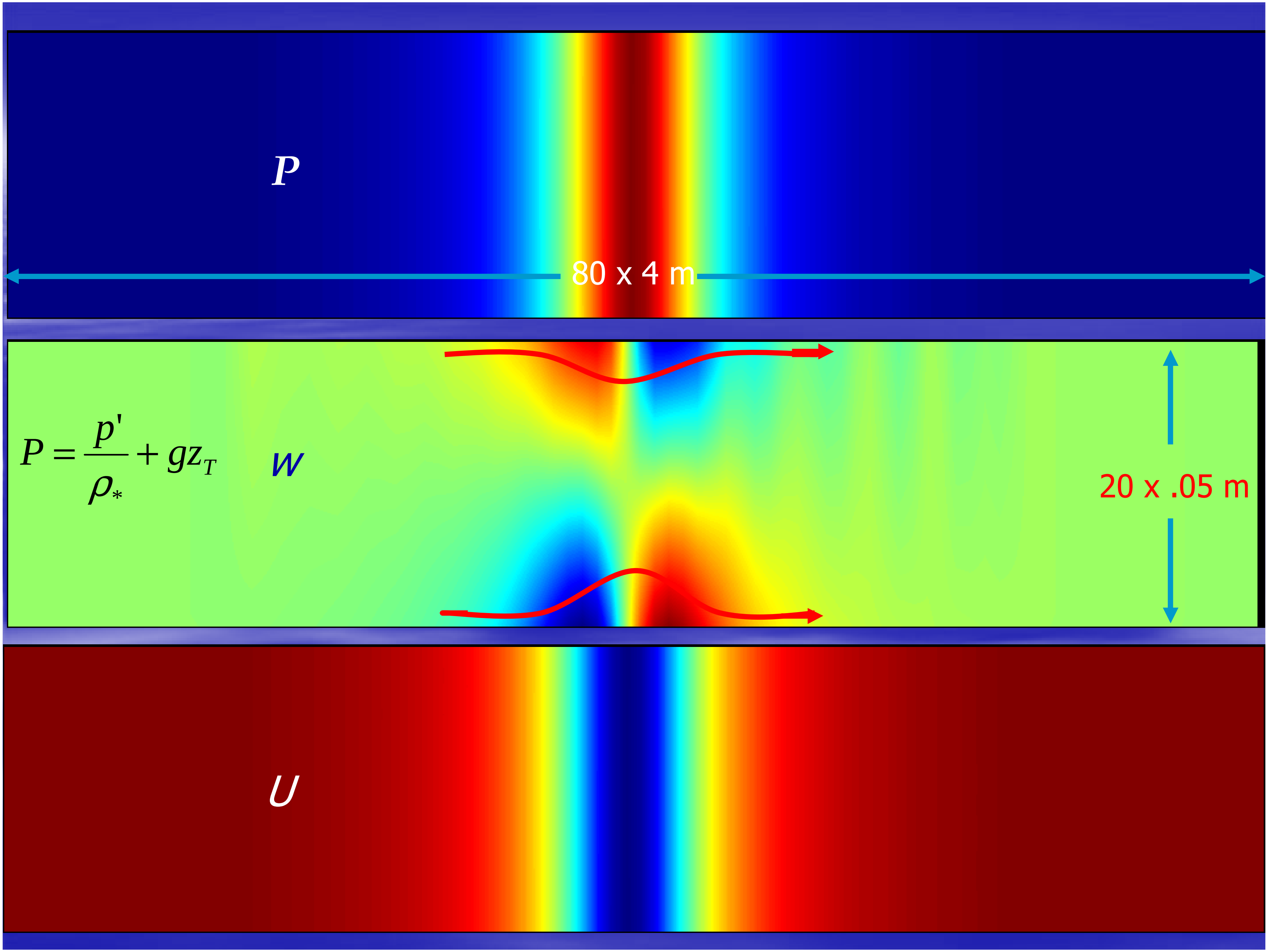
$$B^\varsigma = B - N_*^2 z_T$$

$$P^\varsigma = P + g z_T$$

$$\frac{d\mathbf{v}}{dt} + \nabla g z_T + \left[\nabla - \frac{N_*^2}{g}\mathbf{k} \right] P - B\mathbf{k} = \mathbf{R}_\mathbf{V}$$

$$\frac{dB}{dt} + \left\{ N_*^2 - \beta_W B \right\} \left(w - \frac{dz_T}{dt} \right) = R_B$$

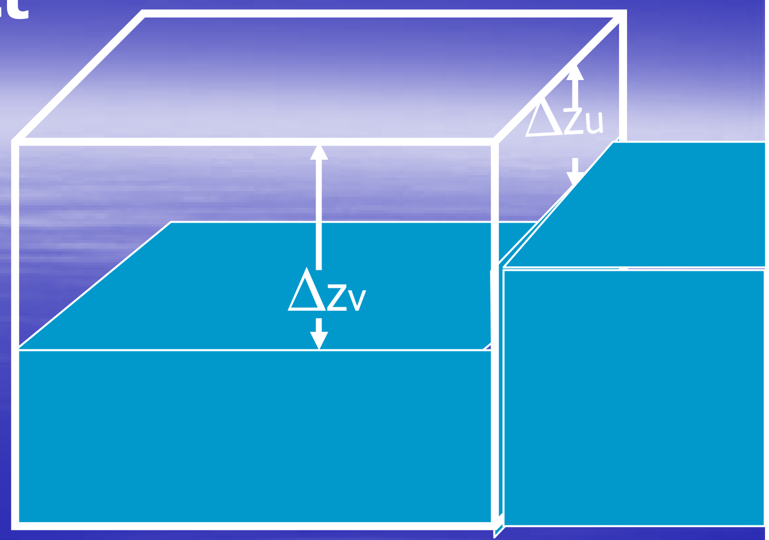
$$\frac{d}{dt}\left(\frac{P}{c_*^2}\right) + \nabla \cdot \mathbf{v} - \left\{ \frac{g}{c_*^2} + \beta_W \frac{P}{c_*^2} \right\} \left(w - \frac{dz_T}{dt} \right) = R_P$$



P
80 x 4 m
$P = \frac{p'}{\rho_*} + g z_T$
W
20 x .05 m
U

Solid object

to block flow



Space discretization (finite volume approach)

$$\iiint (\nabla \cdot \mathbf{A}) dV = \iint \mathbf{A} \cdot d\mathbf{S}$$

$$D = \frac{\delta_x a_x^P U + \delta_y a_y^P V + \delta_z a_z^P W}{\Delta V}$$

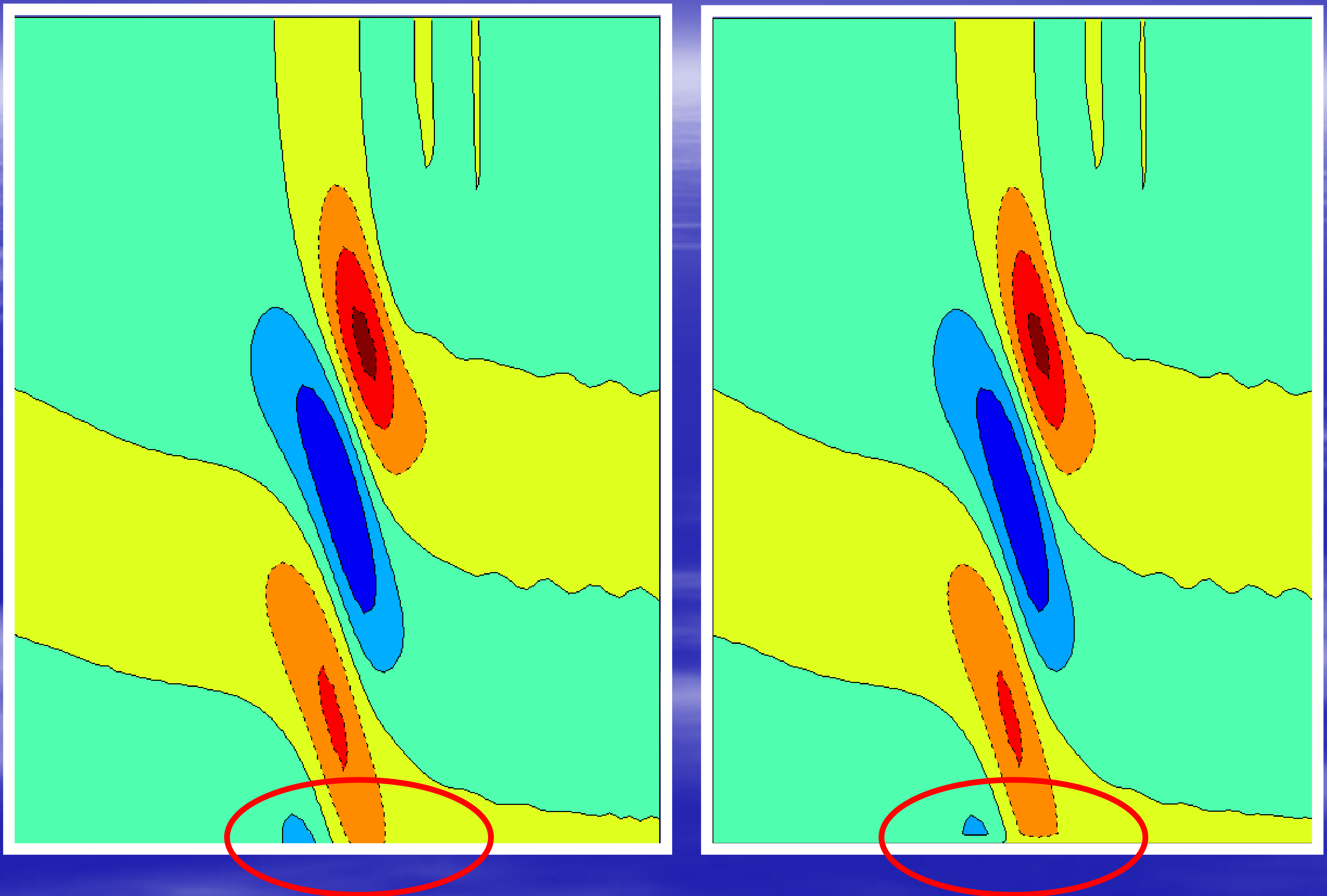
1. *Full or partial step/cell topography/bathygraphy*: the *a_x, a_y* are functions of Δz:

$$a_x = \Delta y \Delta z_U;\ a_y = \Delta x \Delta z_V;\ a_z = M_W \Delta x \Delta y;\ \Delta V = \Delta x \Delta y \Delta z$$

2. The *solid body* is enforced using *masks* (integer values 0 or 1)

3. *Boundary conditions* are applied as usual on walls

Pinty



Merci

Energy and energy-like invariants for deep non-hydrostatic atmospheres

QJ 2003

A. Staniforth, N. Wood, C. Girard

$$\frac{D}{Dt}\iiint_{i_t c i_m} \rho E dV = 0$$

$i_t c i_m$ *thermally isolated*

& closed

& mechanically isolated

$$E=K+\Phi+I$$

$$K=\frac{\mathbf{v}\cdot\mathbf{v}}{2}=\frac{\mathbf{v}_H\cdot\mathbf{v}_H}{2}+\delta_h\frac{w^2}{2}$$

$$\Phi=\begin{cases} gz & \partial\Phi/\partial z=g=const \\ \Phi(r) & \partial\Phi/\partial r=g(r) \end{cases}$$

$$I=c_vT$$

closed rigid material volume

finite
volume

z=const

$$\iiint \rho\left(K+\Phi+I\right)dV=const$$

Laprise & Girard (1990): closed elastic material volume
shallow atmosphere approx.
hydrostatic approx.

p=const.

finite
volume

$$\iiint \rho\left(K+\alpha p+I\right)dV=const$$

$$H=c_pT=RT+I=\alpha p+I$$

aterial volume
here approx.
non-hydrostatic

inite
lume

π=const.

$$\iiint \rho\left(K+\alpha\pi+I\right)dV=const$$

Margules' formula

$$\int_0^{\infty} p\,dz = \int_0^{\infty} \rho\Phi\,dz = \int_0^{p_S} z\,dp$$

Generalized Margules' formula

$$\int_0^{z_T} \pi\,dz = \pi z\Big|_0^T - \int_{p_S}^{p_T} z\,d\pi$$

$$= p_T z_T - \int_0^{z_T} z \frac{\partial \pi}{\partial z} dz$$

$$= p_T \int_0^{z_T} dz + \int_0^{z_T} zg\rho\,dz$$

$$\int_0^{z_T} \pi\,dz = \int_0^{z_T} \left(p_T + \rho\Phi\right) dz$$

closed rigid material volume

$$\iiint \rho (K + \Phi + I) dV = const$$

Laprise & Girard (1990): closed elastic material volume
shallow atmosphere approx.
hydrostatic approx.

$$\iiint [\rho (K + \Phi + I) + p_T] dV = const$$

Laprise (1992): closed elastic material volume
shallow atmosphere approx.
non-hydrostatic

$$\iiint [\rho (K + \Phi + I) + p_T] dV = const$$

closed rigid material volume

$$\iiint \rho E \, dV = const$$

closed elastic material volume
shallow atmosphere approximation

$$\iiint \left[\rho E + p_T \right] dV = const$$

closed elastic material volume
deep atmosphere

$$\iiint \left[\rho E + p_T \right] dV = const$$

?

Gauss theorem

$$\iiint (\nabla \cdot \mathbf{A}) dV = \iint \mathbf{A} \cdot d\mathbf{S}$$

Transport theorem

$$\frac{D}{Dt}\iiint F dV = \iiint \left(\frac{\partial F}{\partial t} + \nabla \cdot F\mathbf{v} \right) dV$$

F=1

transport *Gauss*

$$\frac{D}{Dt}\iiint dV = \iiint (\nabla \cdot \mathbf{v}) dV = \iint \mathbf{v} \cdot d\mathbf{S}$$

F=ρ

transport *continuity*

$$\frac{D}{Dt}\iiint \rho dV = \iiint \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{v} \right) dV = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{v} = 0$$

F=ρG

transport *continuity*

$$\frac{D}{Dt}\iiint \rho G dV = \iiint \left(\frac{\partial \rho G}{\partial t} + \nabla \cdot \rho G\mathbf{v} \right) dV = \iiint \left(\rho \frac{dG}{dt} \right) dV$$

$$= \iiint \left(\rho \left[\frac{\partial G}{\partial t} + \mathbf{v} \cdot \nabla G \right] + G \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{v} \right] \right) dV$$

Gauss theorem

$$\iiint (\nabla \cdot \mathbf{A})\, dV = \iint \mathbf{A} \cdot d\mathbf{S}$$

Transport theorem

$$\frac{D}{Dt}\iiint F\, dV = \iiint \left(\frac{\partial F}{\partial t} + \nabla \cdot F\mathbf{v} \right) dV$$

F=1

transport + Gauss *transport* *Gauss*

$$\frac{D}{Dt}\iiint dV = \iiint (\nabla \cdot \mathbf{v})\, dV = \iint \mathbf{v} \cdot d\mathbf{S}$$

F=ρ

transport + continuity *transport* *continuity*

$$\frac{D}{Dt}\iiint \rho\, dV = 0 = \iiint \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{v} \right) dV = 0 \;\longrightarrow\; \frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{v} = 0$$

F=ρG

transport + continuity *transport* *continuity*

$$\frac{D}{Dt}\iiint \rho G\, dV = \iiint \left(\frac{\partial \rho G}{\partial t} + \nabla \cdot \rho G\mathbf{v} \right) dV = \iiint \left(\rho \frac{dG}{dt} \right) dV$$

Atmospheric Equations of Motion:

Momentum:

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = \mathbf{f}$$

Thermodynamic:

$$\frac{DT}{Dt} - \frac{1}{\rho c_p}\frac{Dp}{Dt} = \frac{Q}{c_p}$$

Continuity:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$$

Kinetic Energy, $K = \frac{\mathbf{v}\cdot\mathbf{v}}{2}$, Equation:

$$\rho\frac{DK}{Dt} + \mathbf{v}\cdot\nabla p + \rho\mathbf{v}\cdot\nabla\Phi = \rho\mathbf{v}\cdot\mathbf{f}$$

Mechanical Energy, $K+\Phi$, Equation:

$$\rho\frac{D(K+\Phi)}{Dt} + \mathbf{v}\cdot\nabla p = \rho\left\{\frac{\partial\Phi}{\partial t} + \mathbf{v}\cdot\mathbf{f}\right\}$$

$$\rho\frac{Dc_pT}{Dt} - \rho\frac{DRT}{Dt} - RT\frac{D\rho}{Dt} = \rho Q$$

Internal Energy, $I = c_v T$, Equation:

$$\rho\frac{DI}{Dt} + p\nabla\cdot\mathbf{v} = \rho Q$$

Total Energy, $E=K+\Phi+I$, Equation:

$$\rho\frac{DE}{Dt} + \nabla\cdot p\mathbf{v} = \rho\left\{\frac{\partial\Phi}{\partial t} + \mathbf{v}\cdot\mathbf{f} + Q\right\}$$

Gauss ***Transport+Gauss*** ***Transport+Continuity***

$$\iiint (\nabla \cdot \mathbf{A})dV = \iint \mathbf{A} \cdot d\mathbf{S} \qquad \frac{D}{Dt}\iiint dV = \iint \mathbf{v} \cdot d\mathbf{S} \qquad \frac{D}{Dt}\iiint \rho G dV = \iiint \rho \frac{dG}{dt} dV$$

Total Energy, $E=K+\Phi+I$, Equation:

$$\rho \frac{DE}{Dt} + \nabla \cdot p\mathbf{v} = \rho\left\{\frac{\partial \Phi}{\partial t} + \mathbf{v} \cdot \mathbf{f} + Q\right\}$$

↓	↓	↓
M	**E**	**S**
o	a	u
o	r	n
n	t	
	h	

$$\boxed{\rho \frac{DE}{Dt} + \nabla \cdot p\mathbf{v} = 0}$$

Integrated Total Energy Equation:

$$\iiint \rho \frac{DE}{Dt} dV + \iiint \nabla \cdot p\mathbf{v} dV = 0$$

↓ *Transport* ↓   ↓ *Gauss* ↓

[illegible]

$$\frac{D}{Dt}\iiint \rho E dV + \iint p\mathbf{v}\cdot d\mathbf{S} = 0$$

Boundary Conditions:

A) *Rigid Closed Volume*:

$$\mathbf{v}\cdot d\mathbf{S} = 0$$

$$\frac{D}{Dt}\iiint \rho E dV = 0$$

B) *Elastic Closed Volume with* uniform pressure p_T=const. exerted on boundary:

$$\frac{D}{Dt}\iiint \rho E dV + p_T \iint \mathbf{v}\cdot d\mathbf{S} = 0$$

$$\frac{D}{Dt}\iiint \rho E dV + p_T \frac{D}{Dt}\iiint dV = 0$$

$$\frac{D}{Dt}\iiint (\rho E + p_T) dV = 0$$

QED