A Cubed Sphere model for wave propagation on long physical time problems

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Shallow Water equation on the Sphere

- Consider the **non linear Shallow Water equation** :

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla_T \mathbf{u} + g\nabla_T h + f\mathbf{k}\times\mathbf{u} = \mathbf{0} \\ \dfrac{\partial h}{\partial t} + \nabla_T\cdot(h\mathbf{u}) = 0 \end{cases} \quad \text{for all } \mathbf{x}\in\mathbb{S}^2_a \text{ and } t>0 \tag{1}$$

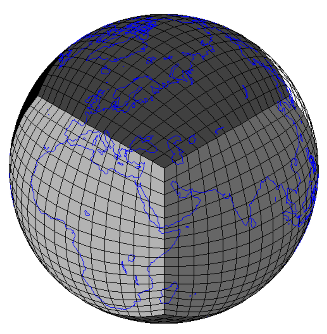
where subscript T stands for the tangential operator.
- Quasi analytic solution $q(t,\mathbf{x}) = [h(t,\mathbf{x}), \mathbf{u}(t,\mathbf{x})]^T \in \mathbb{S}^2_a \times \mathbb{TS}^2_a$ [5] of (1) :

$$q(t,\mathbf{x}) = q_0 + Q_n(\theta)\exp\left(ik(\lambda - C_{n,k}t)\right) \tag{2}$$

where (λ,θ) are the spherical coordinates.
- **Goal :** Simulation over a long physical time of (1) with a nonlinear solver of wave like problems.
- **Difficulties :** Minimal dissipation and dispersion while keeping good conservation properties.

1. Space discretization

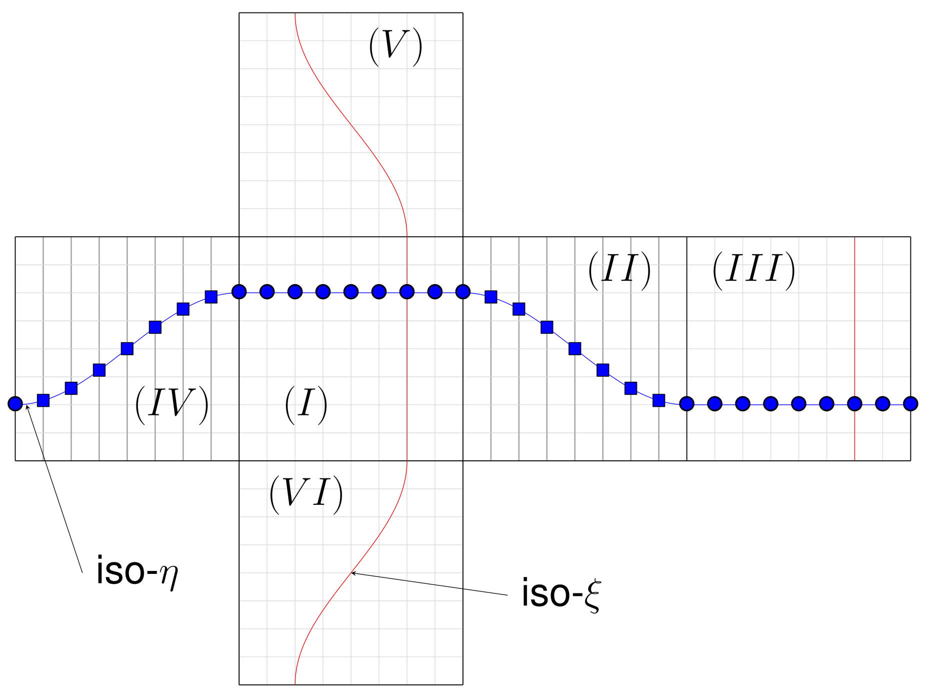
- **Cubed-Sphere grid** :



- The scheme HCCS (Hermitian Compact Cubed Sphere) is based on the **fourth order hermitian scheme** :

$$\frac{1}{6}\delta^H_\xi \mathbf{u}_{j+1} + \frac{4}{6}\delta^H_\xi \mathbf{u}_j + \frac{1}{6}\delta^H_\xi \mathbf{u}_{j-1} = \frac{\mathbf{u}_{j+1}-\mathbf{u}_{j-1}}{2\Delta\xi}. \tag{3}$$

- Essential use of the structure of the Cubed Sphere with great circles coordinates lines on each panel.



- **4th order** accuracy.

2. Time discretization

- The semi-discretizaion of (1) is

$$\frac{d\mathrm{q}}{dt}(t) = F_\Delta(\mathrm{q}(t)). \tag{4}$$

- $\mathrm{q}(t)$ satisfies

$$\mathrm{q}(t^n+\Delta t) = e^{\Delta t L_n}\mathrm{q}(t^n) + \int_0^{\Delta t} e^{(\Delta t-\tau)L_n} N_n(\mathrm{q}(t^n+\tau))d\tau. \tag{5}$$

with $L_n = \mathrm{Jac}_{\mathrm{q}^n} F_\Delta$ and $N_n(\mathrm{q}) = F_\Delta(\mathrm{q}) - L_n\mathrm{q}$ for the **Rosenbrock approach**.
- The time scheme consists of the **second order** exponential procedure (ERK2). Considering the approximation $N_n(q(t^n+\tau)) \approx N_{\Delta,n}(q(t^n))$ (5) is approximated by :

1. $\hat{q}^{n+1} = e^{\Delta t L_{\Delta,n}} q^n + \Delta t\varphi_1(\Delta t L_n)\, N_n(q^n)$,
2. $q^{n+1} = \mathcal{F}(\hat{q}^{n+1})$.

with $\varphi_1(z) = z^{-1}(\exp(z)-1))$ and $\mathcal{F}$ a high order spatial filter.
- A **third order** variant is (ERK3) :

1. $a^n = q^n + \Delta t\varphi_1(\Delta t L_n)\, F_\Delta(q^n)$,
2. $\hat{q}^{n+1} = a^n + 2\Delta t\varphi_3(\Delta t L_n)\,(N_n(a^n) - N_n(q^n))$
3. $q^{n+1} = \mathcal{F}(\hat{q}^{n+1})$.

with $\varphi_3(z) = z^{-3}(\exp(z) - 1 - z - z^2/2))$ and $\mathcal{F}$ a high order spatial filter.

- **Properties** of exponential Rosenbrock integrators for linear problems :
  - **Exact** for linear problems.
  - **A-stability**.
  - **No dissipation** in the time scheme : the extrema are preserved after many iterations.
  - **No dispersion** : no phase error if the space scheme is exact.

- **Implementation with Krylov methods :**

Computational principle : all products matrix-vector $f(A)b$ are performed using a **Krylov method**.

Define the Krylov subspace $\mathcal{K}_m = \mathrm{Span}(b, Ab, ...A^{m-1}b)$.
The matrices V_m and H_m are defined such that
  - V_m is an orthogonal matrix representing the projection from $\mathbb{R}^n$ onto "small" $\mathcal{K}_m$.
  - H_m is projection of action of A on $\mathcal{K}_m$, $H_m = V_m^T A V_m$.

Krylov approximation : $A \approx V_m H_m V_m^T$.
Approximation used :

$$f(A)b \approx \|b\|_2 V_m f(H_m)\mathbf{e}_1. \tag{6}$$

$f(H_m)$ is computed using scaling and squaring with a Padé approximant in the right hand side of (6).

3. Comparaison between ERK and RK4

- **Barotropic Instability [2].** Initial condition: pertubation of an equilibrium state. Discriminant test case for a Cubed Sphere due to a large interpanel gradient.

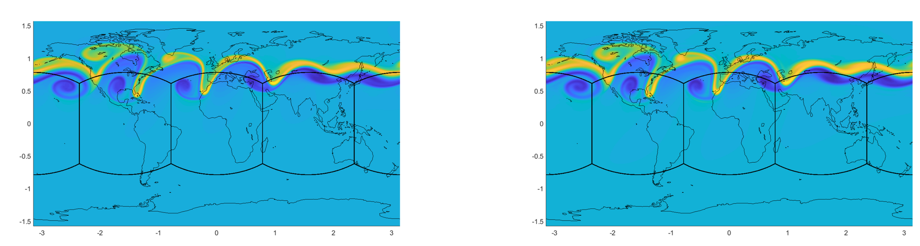


Figure 3: *Vorticity at 6 days on the grid $6\times 96\times 96$. Top : ERK2 with $\Delta t = 1$ hour. Bottom : RK4 with $\Delta t = 240$seconds.*

No loss in accuracy using the exponential scheme !

4. Spherical quasi-analytic wave like solution [5]

- Assessing the long time behavior of the HCCS scheme using a family of quasi-analytic solutions.

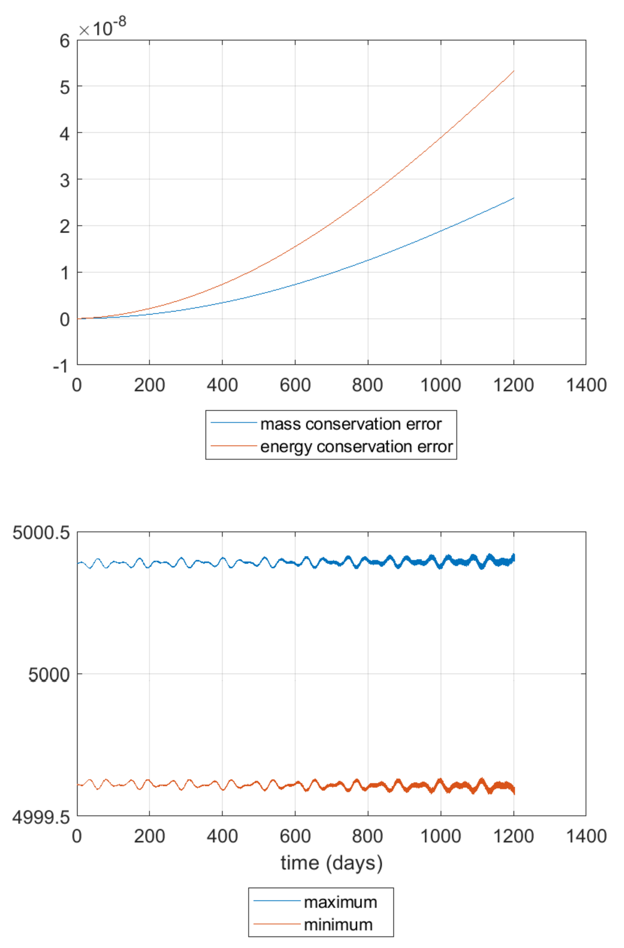


Figure 4: *Rossby wave test with $\Delta t = 2h54$min. and the grid $6\times 64\times 64$ with ERK2. Left : Conservation error. Right : extremums of height h.*

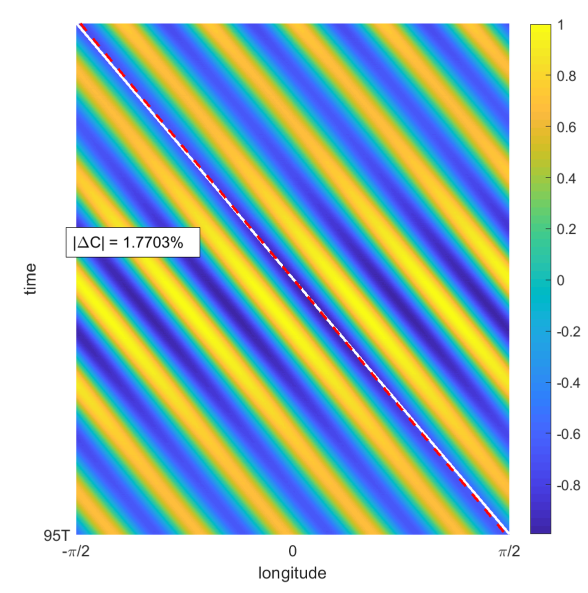


Figure 5: *Hovmöller diag. for Rossby wave test with $\Delta t = 2h54$min. and the grid $6\times 64\times 64$ with ERK2 for the last 5 periods.*

- On Fig. 5, **small phase error** (close to 1.77%).

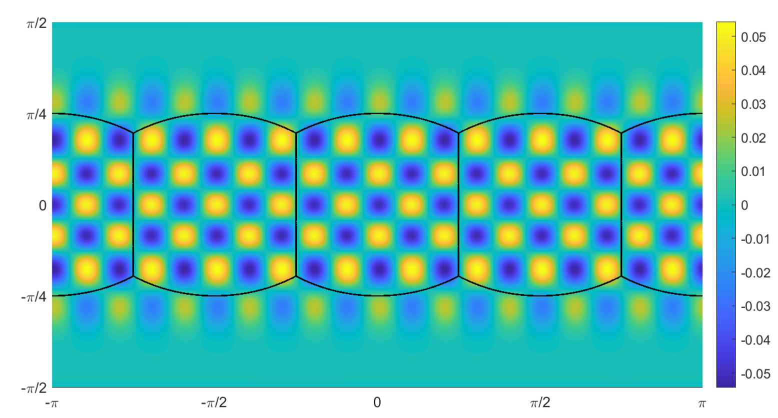


Figure 6: *zonal component u_λ of u for Rossby wave test at final time.*

Wave	$T_{\max}$	Scheme	Δt	ΔC
EIG	13 days	RK4	629s.	0.15%
		ERK2	1h10min $\approx 6.7\Delta t_{RK4}$	0.44%
		ERK3	1h10min $\approx 6.7\Delta t_{RK4}$	0.44%
Rossby	1200 days	RK4	629s	1.77%
		ERK2	2h54min $\approx 16.6\Delta t_{RK4}$	1.77%

Table 1: *Phase error ΔC for the wave propagation with different scheme on the Cubed-Sphere $6\times 64\times 64$.*

- **Main outcome** of the HCCS scheme:
  - Very good accuracy for long physical time propagation problems.
  - Very small dispersion and dissipation error.
  - Good conservation properties.

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