Topography based local refinement in spherical Voronoi grids

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Abstract
This work explores global quasi-uniform Voronoi grids that allow locally refined regions with smooth transitions to the coarser regions. The method adopted to build these grids is based on Centroidal Voronoi Tessellation algorithms. Interpolation techniques and topography smoothing methods were investigated. Aiming for better weather forecasting in Brazil, we developed a grid that captures well the Andes region. We show numerical experiments for the shallow water equations on the sphere using a finite volume method in this locally refined grid.

SCVT and Lloyd’s method
A Spherical Centroidal Voronoi Tessellation (SCVT) is a Voronoi diagram on the sphere such that the center of mass of the Voronoi cells with respect to a given density function \( \rho \) are also the generators of the Voronoi cells. Denote the diameter and the generator of a Voronoi cell \( \gamma_i \) and \( x_i \), resp. Then, we have the following approximation (Ju et al. 2011):

\[
\frac{h_i}{\rho(x)} \approx \left(\frac{\rho(x)}{\rho(x_i)}\right)^{1/4}
\]  

(1)

This relation between the local size of the Voronoi cells and density function is very useful to build grids with local refinement. Lloyd’s method is a deterministic algorithm for constructing SCVTs that is known for being effective and simple, and was adopted for the numerical experiments of this work.

Interpolating and smoothing the topography data
Since the elevation is known only for some points on the sphere, we applied the bilinear interpolation method in the data from ETOP01, so we have an approximation of the elevation for every point on the sphere. We also applied the 2-dimensional moving average technique in elevation data to smooth the data. The elevation table is defined in a lat-lon grid with 720x1440 points.

Andes density function
We first set the parameters \( \gamma = 3 \), \( \varepsilon = \pi/12 \), \( s(x) = (4 \pi \varepsilon)^{-1/6} \) and \( x \), which is given in geographic coordinates by \( ( -\frac{\pi}{2}, -\frac{\pi}{4} ) \). A density function \( \rho \) that yields a grid with local refinement in Andes may be defined as follow:

- In the region where \( d(x, x_i) \leq \pi/6 \), \( \rho(x) \) is obtained using bilinear interpolation in the normalized elevation data.
- In the transition zone \( \pi/6 < d(x, x_i) \leq \pi/6 + \varepsilon \), we define:
  \[
  \rho(x) = \left( 1 - s(x) \right)^\alpha s(x)/\gamma_i^4, \quad \text{where} \quad \alpha = \sqrt{\frac{500}{0.020}}
  \]
- In the outside region, we define \( \rho(x) = 1/\gamma_i^4 \).

Generated grid

![Generated grid](image)

Figure: SCVT generated with Andes refinement for 2526 and 10242 generators

Measures of grid quality
From relation (1), we can deduce that the size of the Voronoi cells in the region where \( d(x, x_i) \leq \pi/6 \) is approximately \( \gamma_i \) smaller than in the region where \( d(x, x_i) > \pi/6 + \varepsilon \).

![Measures of grid quality](image)

Figure: Voronoi cells diameters and distortions distributions - 40962 generators, \( \gamma_i = 3 \)

Shallow water equations in vector invariant form
The first test problem that we analyze is the test case 2 proposed in Williamson et al. 1992.

![Shallow water equations in vector invariant form](image)

Figure: Test Case 2: Steady state zonal geostrophic flow - Error of the fluid height field \( h \) after 12 days of integration - grid with 163842 generators

In the second test problem, we adapted the test case from Galewsky et al. 2004. This adapted test consists of a barotropically unstable jet in South Hemisphere without topography.

![Shallow water equations in vector invariant form](image)

Figure: Barotropically unstable jet in South Hemisphere with the perturbation - Potential vorticity at day 7 - grid with 163842 generators

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References