A nonhydrostatic dynamical core on cubed sphere using multi-moment scheme

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Outline

1. Background
2. Numerical formulations
3. Results of benchmark tests
4. Summary
Multi-moment method

Multi-moment method adopts two or more kinds of moments

Rigorous numerical conservation, High-order accuracy, Flexibility in dealing with very different grid structures, Good scalability on massively parallel computers, ……
icosahedral grid with a resolution of m o o t e r d e c l i n e i n t e c o n s e r v a t i o n e r r o s w h e nt h e c o m p u t a t i o n a l g r i di s r e fi n e d . I t is evident that the results show improvement in high resolution simulation. The icosahedral–hexagonal grid demonstrates overall more accurate results and energy of bottom mountain due to the satisfaction of the C-property.

The models using cubed-sphere and icosahedral grids rigorously conserve the total mass. The conservation errors of total mass, momentum, and energy are also controlled to a low level. The normalized conservation errors of total energy and potential enstrophy are shown in the figure.

Since there is no analytic solution available for this test, we consider the spectral transform solution on fine T213 grid given in the reference.[11] The solution is usually considered as the reference solution for verification of the models.

The Rossby–Haurwitz wave provides a good test bed for global shallow water models to simulate the middle-range dynamic processes up to two weeks. Being case 6 in Williamson’s test set, the details of initial conditions of this test were given in Fig. 15.

The normalized conservation errors of total energy and potential enstrophy are also controlled to a low level. The normalized conservation errors of total energy and potential enstrophy are shown in Figs. 17 and 18.

The models using cubed-sphere and icosahedral grids converge to the reference solution. The convergence of normalized errors on refined grids of time dependent zonal flow is shown in Fig. 19. The convergence rate of multi-moment model on icosahedral-hexagonal grid is shown in the graph.

Convergence rate of multi-moment model on icosahedral-hexagonal grid

Nonhydrostatic model in vertical direction

Steady-state flowing over the Schär mountain

Contour plot of vertical velocity
Computational mesh

Horizontal mesh

Cubed-sphere grid

Vertical mesh

Gnomonic projection

Resolution

Coarse

Fine

Height-based terrain-following grid

PDEs on Sphere 2019
Flux-form nonhydrostatic governing equations in local curvilinear coordinates (with shallow atmosphere assumption)

(Ullrich & , Jablonowski, JCP, 2012; Clark, JCP, 1977)

\[
\frac{\partial q}{\partial t} + \frac{\partial E(q)}{\partial \xi} + \frac{\partial F(q)}{\partial \eta} + \frac{\partial H(q)}{\partial \zeta} = S(q)
\]

Model variables

\[
q = \left[J \rho', J \rho \omega , J \rho \omega, J \omega, J (\rho \theta) \right]^T
\]

\(E, F\) and \(H\) are flux functions in different directions, \(S\) are source terms including gravity force, Coriolis force, Rayleigh friction and the terms representing the grid transformations.
Definition of DOFs (prognostic variables)

Three-point MCV scheme

Two kinds of DOFs

Extension to 3D case

- Same formulations are used for spatial discretizations in all directions.
Three-point multi-moment scheme

Considering the 1D hyperbolic system

$$\frac{\partial q}{\partial t} + \frac{\partial e}{\partial x} = 0$$

1. Updating the DOF at cell interface through solving Derivative Riemann Problem (DRP)

$$\frac{\partial q_{il}}{\partial t} \approx -\frac{1}{2} \left\{ \left[ \frac{\partial}{\partial x} E_{i-1}(x) \right]_{i-\frac{1}{2}} + \left[ \frac{\partial}{\partial x} E_i(x) \right]_{i-\frac{1}{2}} \right\} - \frac{1}{2} A \left\{ \left[ \frac{\partial}{\partial x} Q_{i-1}(x) \right]_{i-\frac{1}{2}} - \left[ \frac{\partial}{\partial x} Q_i(x) \right]_{i-\frac{1}{2}} \right\}$$

- MM_FVM4 (Chen & Xiao, JCP, 2008) is used for evaluating the derivatives of flux functions and dependent variables.
- Other schemes, e.g. MCV3_WENO (Su et al. CCP, 2015), MCV3_BGS (Deng et al., JSC, 2017), GLPCC (chen et al., GMD, 2015), can be straightforwardly applied for achieving different numerical properties.
Three-point multi-moment scheme

Considering the 1D hyperbolic system

$$\frac{\partial q}{\partial t} + \frac{\partial e}{\partial x} = 0$$

1. Updating the DOF at cell interface through solving Derivative Riemann Problem (DRP)

$$\frac{\partial q_{il}}{\partial t} \approx -\frac{1}{2} \left\{ \left[ \frac{\partial}{\partial x} E_{i-1}(x) \right]_{i-\frac{1}{2}} + \left[ \frac{\partial}{\partial x} E_i(x) \right]_{i-\frac{1}{2}} \right\} - \frac{1}{2} A \left\{ \left[ \frac{\partial}{\partial x} Q_{i-1}(x) \right]_{i-\frac{1}{2}} - \left[ \frac{\partial}{\partial x} Q_i(x) \right]_{i-\frac{1}{2}} \right\}$$

- Local Lax-Friedrich (LLF) approximate Riemann solver is used in this study for its simplicity.
- A modification is adopted for more accurate representation of large-scale wave propagations in atmosphere.
Three-point multi-moment scheme

Considering the 1D hyperbolic system

\[
\frac{\partial q}{\partial t} + \frac{\partial e}{\partial x} = 0
\]

2. Updating DOF at cell center through formulation derived by using the constraint on Volume-inetgrated average (VIA)

\[
\frac{\partial q_{i2}}{\partial t} \approx \frac{E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}}}{\Delta x} + \frac{1}{4} \left( \frac{\partial q_{i1}}{\partial t} + \frac{\partial q_{i3}}{\partial t} \right)
\]

Flux-form (finite volume) formulation for VIA guarantees the numerical conservation.

\[
\frac{\partial \bar{q}_i}{\partial t} = \frac{1}{6} \frac{\partial q_{i1}}{\partial t} + \frac{2}{3} \frac{\partial q_{i2}}{\partial t} + \frac{1}{6} \frac{\partial q_{i3}}{\partial t}
\]
Horizontal-Explicit/Vertical-Implicit (HEVI) time integration

\[
\begin{align*}
q^{n+1} &= q^n + \Delta t \sum_{l=0}^{s} b_l H(q^{(l)}) + \Delta t \sum_{l=0}^{s} \tilde{b}_l V(q^{(l)}) \\
q^{(0)} &= q^n \\
q^{(s)} &= q^n + \Delta t \sum_{l=0}^{s-1} a_{sl} H(q^{(l)}) + \Delta t \sum_{l=0}^{s} \tilde{a}_{sl} V(q^{(l)})
\end{align*}
\]

- Time step is determined by stability condition of horizontal explicit integration (CFL \( \approx 0.8 \) with MM FVM4 discretization).
- \( H \): Derivatives of flux functions in horizontal directions, Coriolis force and source terms for grid transformation.
- \( V \): Directives of flux functions in vertical directions, gravity force and Rayleigh friction.
- HEVI (horizontal explicit-vertical implicit) time integration is implemented by 3-stage IMEX Runge-Kutta algorithm (of 3rd-order accuracy).
- Nonlinear equations is solved using Newton’s method and the linear equation set at each iteration is currently solved by the direct method.
Boundary conditions

• Ghost cells in horizontal directions are built through the interpolations on adjacent patches.
• A correction is applied for DOFs defined along the patch boundaries to assure the conservation.

Top: Slip wall + Reyleigh friction
\( w = 0 \)

Bottom: Slip wall
\( \tilde{w} = 0 \)

At vertical boundaries, the derivatives of flux functions are evaluated by 3-point one-sided formulations instead of solving DRP.
Details of tests are referred to

C. Jablonowski et al., Idealized test cases for the dynamical cores of atmospheric general circulation models: a proposal for the NCAR ASP 2008 summer colloquium, Tech. rep. (2008)
Test 1: Convergence test

3D balanced flow on refining grids

Normalized $l_2$ errors and convergence rates at day 5
Test 2: Baroclinic instability (resolution 1°)
Test 2: Baroclinic instability

850 hPa Temperature Day 7
850 hPa Temperature Day 9

850 hPa rel. Vor. Day 7
850 hPa rel. Vor. Day 9
Test 3: 3D Rossby–Haurwitz wave (resolution 1.5°)
Test 3: 3D Rossby–Haurwitz wave

Numerical results with linear interpolation and different time marching schemes

500hPa height

Surface pressure

3rd IMEX

Dimensional-splitting time integration
Test 4: Mountain-induced Rossby wave-train (resolution 1.5°)

MMFV results

Mcore results
Test 4: Mountain-induced Rossby wave-train

MMFV results
Test 5: Gravity waves (resolution 1°)

Mcore results

MMFV results
Summary

- The proposed dynamical core achieves fourth-order accuracy on cubed sphere with satisfying performance in the preliminary verification by the ideal dry benchmark cases.
- The initial release of the parallel multi-moment dynamical core is currently ready for coupling with the physical parametrization processes through C-coupler.

Further work

1. Improvement of dynamical core and advection solver
2. Dynamical core with global adaptive mesh refinement
3. Dynamical core on icosahedral-hexagonal grid
4. ......
THANK YOU!

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