Development of oscillation-less nonhydrostatic atmospheric model by multimoment finite volume method

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High order numerical method with local reconstruction

Accurate (high-order), Conservative, Highly scalable on massively parallel system (peta, exa, ...)

✧ Discontinuous Galerkin (DG) method
  1. Dennis et al., 2005
  2. Nair et al., 2007, 2009

✧ Spectral element (SE) method
  1. CAM-SE: Dennis et al., 2005, 2011
  2. SEE-AM, Giraldo and Rosmond, 2004
  3. SEAM, Fournier et al, 2004

✧ Multi-moment finite volume method
  1. Xiao et al, 2005; Chen and Xiao, 2008; Li and Xiao, 2009; Xiao et al., 2013

✧ Flux reconstruction

✧ Compatible finite element (FE) method
  2. McRae and Cotter, 2014; Melvin et al., 2019
High order scheme (>=2) always generate the numerical oscillations. Notable upwind schemes with limiter such as FCT, MUSCL, MPDATA, ENO, TVD, PPM, TVB, WENO can achieve the non-oscillatory properties for sharp gradients such as cold fronts, dryline, cloud boundaries and inversions.

**Requirement:** a robust and high order limiting strategy

*High order scheme (>=2) always generate the numerical oscillations*
Outlines

1 Multi-moment finite Volume (MMFV) method

2 The MMFV-BGS limiting strategy

3 Benchmark results for nonhydrostatic atmospheric model

4 Summary
## Conservative numerical solvers for computational fluid dynamics

<table>
<thead>
<tr>
<th>Method</th>
<th>DOFs (Unknowns)</th>
<th>Features</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional FVM</td>
<td>$\bar{\phi}_i(t) = \frac{1}{</td>
<td>V_i</td>
</tr>
<tr>
<td>Multi-moment FVM</td>
<td>$\bar{\phi}_i(t) = \frac{1}{</td>
<td>V_i</td>
</tr>
<tr>
<td>Spectral/hp element with flux concept, DG, SE, MFE</td>
<td>$\hat{\phi}<em>i(t) = \left\langle \left( \sum</em>{k=1}^{K} \bar{\phi}_k(t) \hat{\psi}_k(X) \right), \hat{\psi}_i(X) \right\rangle$</td>
<td>Compact stencil, High accuracy, Complexity</td>
</tr>
</tbody>
</table>
Multi-moment finite Volume method (3-points MMFV)

Consider the conservation law
\[ \frac{\partial q}{\partial t} = -\frac{\partial f(q)}{\partial x} \]

Equidistant solution points
\[ \xi_1 = -1, \xi_2 = 0, \xi_3 = 1 \]

Evolution equations to update the nodal values
\[ \frac{dq_{ik}}{dt} = -\hat{f}_{\xi i}(\xi_k) = -\hat{f}_{\xi ik}, k = 1, 2, 3 \]

Reconstruction of the reconstructed flux function of degree 3
\[ \hat{f}_i(\xi) \]

Impose Hermite constraints on the boundary

\[
\begin{aligned}
\hat{f}_{\xi i 1} &= f_{\xi i}^B(-1), \\
\hat{f}_{\xi i 2} &= \frac{1}{4}(3f_i^B(1) - 3f_i^B(-1) - f_{\xi i}^B(-1) - f_{\xi i}^B(1)), \\
\hat{f}_{\xi i 3} &= f_{\xi i}^B(1),
\end{aligned}
\]

Numerical conservation
\[
\int_{-1}^{1} \hat{f}_{\xi i}(\xi) d\xi = \sum_{k=1}^{3} (w_k \hat{f}_{\xi ik}) = \frac{1}{3} \hat{f}_{\xi i 1} + \frac{4}{3} \hat{f}_{\xi i 2} + \frac{1}{3} \hat{f}_{\xi i 3} = f_i^B(1) - f_i^B(-1)
\]
High order non-oscillatory BGS limiter

*The derivative Riemann problem*

\[
\hat{f}_{\xi i-1/2} = \frac{1}{2} \left( \hat{f}_{\xi i-1/2}^{+} + \hat{f}_{\xi i-1/2}^{-} \right) - \frac{1}{2} A_{i-1/2} \left( \frac{\partial q^{+}}{\partial \xi} - \frac{\partial q^{-}}{\partial \xi} \right) = \hat{f}_{\xi i}^{B}(-1)
\]

\[
\hat{f}_{\xi i}^{B}(\pm 1) = \frac{\partial F_{i}^{\pm}(\xi)}{\partial \xi} \approx A_{i-1/2} \frac{\partial Q_{i}^{\pm}(\xi)}{\partial \xi}
\]

\[d_{ci} : \text{Slope limiter (xiao & Yabe, JCP, 2001)}\]

\[Q_{i}^{AL}, Q_{i}^{AR}, Q_{i}^{TVD} \text{ are the Lagrangian cubic polynomials}\]

\[
\begin{cases}
Q_{i}^{TVD}(x_{i1}) = q_{i1} \\
Q_{i}^{TVD}(x_{i2}) = q_{i2} \\
Q_{i}^{TVD}(x_{i3}) = q_{i3} \\
Q_{i}^{TVD}(x_{i2}) = d_{ci}
\end{cases}
\]

\[
\begin{cases}
Q_{i}^{AL}(x_{i-1}) = q_{(i-1)2} \\
Q_{i}^{AL}(x_{i-1/2}) = q_{i1} \\
Q_{i}^{AL}(x_{i}) = q_{i2} \\
Q_{i}^{AL}(x_{i+1/2}) = q_{i3} \\
Q_{i}^{AL}(x_{i+1}) = q_{(i+1)2}
\end{cases}
\]

\[
\begin{cases}
Q_{i}^{AR}(x_{i-1/2}) = q_{i1} \\
Q_{i}^{AR}(x_{i}) = q_{i2} \\
Q_{i}^{AR}(x_{i+1/2}) = q_{i3} \\
Q_{i}^{AR}(x_{i+1}) = q_{(i+1)2}
\end{cases}
\]
High order non-oscillatory BGS limiter

Boundary Gradient Switching (BGS)

(Basic idea similar to ENO)

\[
\text{BGS} \left[ \left( \frac{\partial Q_i(x)}{\partial x} \right)_{i - \frac{1}{2}} \right] = \begin{cases} 
\text{dmin} \left( d_1, d_2 \right) & \text{if } \text{sign} \left( d_1 \right) = \text{sign} \left( d_2 \right) = \text{sign} \left( d_3 \right) \\
d_1 & \text{only if } \text{sign} \left( d_1 \right) = \text{sign} \left( d_3 \right) \\
d_2 & \text{only if } \text{sign} \left( d_2 \right) = \text{sign} \left( d_3 \right) \\
\text{absmin} \left( d_1, d_2 \right) & \text{otherwise}
\end{cases}
\]

\[
\begin{align*}
q(i - 1) & \quad q(i) & \quad q(i + 1) \\
\frac{\partial Q_i^{4L}(x)}{\partial x} & \quad \frac{\partial Q_i^{4R}(x)}{\partial x} & \quad \frac{\partial Q_i^{TVD}(x)}{\partial x}
\end{align*}
\]

\[
\begin{align*}
d_1 &= \left( \frac{\partial Q_i^{4L}(x)}{\partial x} \right)_{i - \frac{1}{2}} \\
d_2 &= \left( \frac{\partial Q_i^{4R}(x)}{\partial x} \right)_{i - \frac{1}{2}} \\
d_3 &= \left( \frac{\partial Q_i^{TVD}(x)}{\partial x} \right)_{i - \frac{1}{2}}
\end{align*}
\]

\[
\text{dmin} \left( d_1, d_2 \right) = \begin{cases} 
d_1 & \text{if } |d_1 - d_3| < |d_2 - d_3| \\
d_2 & \text{otherwise}
\end{cases}
\]

\[
\text{absmin} \left( d_1, d_2 \right) = \begin{cases} 
d_1 & \text{if } |d_1| < |d_2| \\
d_2 & \text{otherwise}
\end{cases}
\]
Some remarks:

² The compact local MCV3-BGS make full use of local information.
² No trouble cell indicators
² Do not suffer from loss of accuracy.

High order slope BGS limiter

Convergence rate test (3 point MCV with BGS limiter)

<table>
<thead>
<tr>
<th>Grid number</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
</tr>
</thead>
<tbody>
<tr>
<td>Errors/orders</td>
<td>1.66e-5</td>
<td>-</td>
<td>1.12e-6</td>
<td>3.80</td>
</tr>
</tbody>
</table>

 pedestal radius 2.5

max: 1.005
min: -1.22E-03

Max: 1.
Min: 0.
1D compressible Euler equation simulations with BGS limiter


Sod’s problem

Lax’s problem

Shock-turbulence interaction

Two interacting blast waves
The MCV nonhydrostatic atmospheric model

Splitting of thermaldynamic variables

\[ \rho(x, t) = \bar{\rho}(z) + \rho'(x, t), \]
\[ p(x, t) = \bar{p}(z) + p'(x, t), \]
\[ (\rho\theta)(x, t) = (\bar{\rho}\theta)(z) + (\rho\theta)'(x, t) \]

\[ \tilde{w} = \frac{d\zeta}{dt} = \frac{1}{\sqrt{G}}(w + \sqrt{G}G^{13}u) \]
\[ u = (u, w) \]
\[ G^{13} = \frac{\partial\zeta}{\partial x} \]
\[ \rho = (\rho_d + \rho_v + \rho_c), \quad r_v = \frac{\rho_v}{\rho_d}, \quad r_c = \frac{\rho_c}{\rho_d} \]

\[ q = [\sqrt{G}\rho', \sqrt{G}\rho u, \sqrt{G}\rho w, \sqrt{G}(\rho\theta)'] \]
\[ f = [\sqrt{G}\rho u, \sqrt{G}\rho u^2 + \sqrt{G}p', \sqrt{G}\rho w u, \sqrt{G}\rho u w u, \sqrt{G}\rho u w]^T \]
\[ g = [\sqrt{G}\rho \tilde{w}, \sqrt{G}\rho u \tilde{w} + \sqrt{G}G^{13}p', \sqrt{G}\rho w \tilde{w} + p', \sqrt{G}\rho \tilde{w} \tilde{w}]^T \]
\[ s = (0, 0, -\sqrt{G}\rho' g, s_\theta) \]

Time integration: Strong Stability-Preserving Runge-Kutta scheme (5,4)

Method of Line

\[ \frac{dq_{lm}}{dt} = \mathcal{L}(q_{lm}) \]

\[ q^{(1)} = q^n \]
\[ q^{(i)} = \sum_{k=0}^{i-1} \alpha_{ik} q^{(k)} + \Delta t \beta_{ik} \mathcal{L}(q^{(k)}), \ i = 1, ..., 5 \]
\[ q^{n+1} = q^{(5)} \]

Derivative Riemann Solver: ROE
Benchmark: Rising bubble

Riemman solver: ROE, dx=125m

Potential temperature, interval: 0.1 K

vertical velocity, interval: 1 m/s

MCV3-BGS has 4th order accuracy

Does not need the priori detector, such as the TVB criterion, to peak up the "troubled cells".
Benchmark: Rising bubble

**Vertical velocity**

Min: -9.42, Max: 15.51

Min: -8.61, Max: 14.31

$T=1000 \, s$
Benchmark: density current

(Straka et al., Int. J. Numer. Methods Fluids, 1993)

MCV3-BGS

grid spacing: 400m

grid spacing: 200m

grid spacing: 100m

grid spacing: 50m

Equivalent DOF resolutions

MCV4

Li et al., MWR, 2013

(a) 600m grid spacing

(b) 300m grid spacing

(c) 150m grid spacing

(d) 75m grid spacing
Benchmark: Internal gravity waves

\[ \Delta x = 1000 \, \text{m}, \Delta z = 100 \, \text{m} \quad \text{T} = 3000 \, \text{s} \]

(Giraldo and Restelli, JCP, 2008)
Benchmark: Schär mountain wave

(Schär et al., MWR, 2002)

Velocity fields

<table>
<thead>
<tr>
<th>Variables</th>
<th>MCV3-BGS</th>
<th>SE3</th>
<th>DG3</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>$8.71 \times 10^{-2}$</td>
<td>$2.26 \times 10^{-1}$</td>
<td>$1.94 \times 10^{-1}$</td>
</tr>
<tr>
<td>w</td>
<td>$2.96 \times 10^{-2}$</td>
<td>$7.66 \times 10^{-2}$</td>
<td>$7.51 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>$5.49 \times 10^{-2}$</td>
<td>$6.78 \times 10^{-2}$</td>
<td>$5.84 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

(data from Giraldo and Restelli, JCP, 2008)
Benchmark: moist rising bubble

Bubble top: about 7km

$$\Delta x = \Delta z = 100 \text{ m}$$

Moistures transport by the 4th MCV3-WENO+FCT scheme

See Table 1
(Bryan and Fritsch, 2002)
set A

$$\frac{D \ln \theta}{Dt} = \frac{L_v \dot{q}}{c_p T}$$

Ignore the diabatic contribution to the pressure equation, the specific heats of water substances
Summary

✧ The 3-point MCV scheme with BGS limiting has uniform 4th-order accuracy and effectively suppresses unphysical oscillations.

✧ Unlike other existing high-order local reconstruction methods, MCV3-BGS does not need any ad hoc trouble-cell indicator.

✧ The MCV nonhydrostatic model with BGS limiter can resolve both smooth and less smooth (large gradient) solutions without any use of artificial diffusion or filtering.

✧ The numerical results of benchmark tests indicate that the nonhydrostatic model based on MCV3-BGS scheme has excellent solution quality competitive to other existing high-order models.

✧ With significant advantages in efficiency and flexibility as well, the present platform is very promising and has great potential as a dynamic core of atmospheric models.
Thanks for your attention