

Generalised Perturbation Equations for All-Scale Atmospheric Dynamics

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Aim: extend functionality of numerical models to the whole weather active atmosphere

Objective: develop effective solvers for all-scale PDEs formulated in terms of perturbation variables about an arbitrary state of the atmosphere

Formulating governing equations in terms of perturbation variables has a long history and comes in many flavours, from reduced PDEs, through turbulence closures and asymptotic expansions, to numerical facilitators. We advance the latter.

We start with formulating governing PDEs about a hypothetical *Ambient State* (AS) that satisfies the generic PDEs at hand. AS' role is to analytically precondition semi-implicit solvers and enhance simulation accuracy by selectively controlling discretization errors.



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Governing equations

Generic form

$$\frac{d\rho}{dt} = -\frac{\rho}{\mathcal{G}} \nabla \cdot (\mathcal{G}\mathbf{v}), \quad \nabla = (\partial_x, \partial_y, \partial_z)$$

$$\frac{d\theta}{dt} = \mathcal{H}, \quad d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla \quad \mathbf{v} = \tilde{\mathbf{G}}^T \mathbf{u}$$

$$\frac{d\mathbf{u}}{dt} = -\frac{\theta}{\theta_0} \tilde{\mathbf{G}} \nabla \phi + \mathbf{g} - \mathbf{f} \times \mathbf{u} + \mathbf{M}(\mathbf{u}) + \mathcal{D} \quad \phi = c_p \theta_0 \pi \quad \phi = c_p \theta_0 \left[\left(\frac{R}{p_0} \rho \theta \right)^{R/c_v} \right]$$

Defining $\theta' := \theta - \theta_a$, $\mathbf{u}' := \mathbf{u} - \mathbf{u}_a$, $\phi' := \phi - \phi_a \rightarrow$

$$\frac{d\theta'}{dt} = -\mathbf{v}' \cdot \nabla \theta_a + \mathcal{H}' - \left(\frac{d_a \theta_a}{dt} - \mathcal{H}_a \right),$$

$$\begin{aligned} \frac{d\mathbf{u}'}{dt} = & -\mathbf{v}' \cdot \nabla \mathbf{u}_a - \frac{\theta}{\theta_0} \tilde{\mathbf{G}} \nabla \phi' - \frac{\theta'}{\theta_0} \tilde{\mathbf{G}} \nabla \phi_a \\ & - \mathbf{f} \times \mathbf{u}' + \mathbf{M}'(\mathbf{u}', \mathbf{u}_a) + \mathcal{D}'(\mathbf{u}', \mathbf{u}_a) \\ & - \left(\frac{d_a \mathbf{u}_a}{dt} + \frac{\theta_a}{\theta_0} \tilde{\mathbf{G}} \nabla \phi_a - \mathbf{g} + \mathbf{f} \times \mathbf{u}_a - \mathbf{M}(\mathbf{u}_a) - \mathcal{D}(\mathbf{u}_a) \right) \end{aligned}$$

where:

$$\mathcal{H}' = \mathcal{H} - \mathcal{H}_a,$$

$$\mathbf{v}' = \tilde{\mathbf{G}}^T \mathbf{u}',$$

$$\mathbf{M}'(\mathbf{u}, \mathbf{u}_a) = \mathbf{M}(\mathbf{u}' + \mathbf{u}_a) - \mathbf{M}(\mathbf{u}_a),$$

$$\mathcal{D}'(\mathbf{u}, \mathbf{u}_a) = \mathcal{D}(\mathbf{u}' + \mathbf{u}_a) - \mathcal{D}(\mathbf{u}_a),$$

$$\rightarrow \mathcal{D}' = \mathcal{D}(\mathbf{u}')$$

for flow independent viscosity

Discussion

implicit, centred advection of the ambient state

$$\frac{d\theta'}{dt} = \boxed{-\mathbf{v}' \cdot \nabla \theta_a} + \mathcal{H}' ,$$

3D buoyancy

$$\frac{d\mathbf{u}'}{dt} = \boxed{-\mathbf{v}' \cdot \nabla \mathbf{u}_a} - \frac{\theta}{\theta_0} \tilde{\mathbf{G}} \nabla \phi' - \frac{\theta'}{\theta_0} \boxed{\tilde{\mathbf{G}} \nabla \phi_a} - \mathbf{f} \times \mathbf{u}' + \mathcal{M}'(\mathbf{u}, \mathbf{u}_a) + \mathcal{D}'(\mathbf{u}, \mathbf{u}_a)$$

GBIS

$$\tilde{\mathbf{G}} \nabla \phi_a = \frac{\theta_0}{\theta_a} \left(\mathbf{g} - \mathbf{f} \times \mathbf{u}_a + \mathcal{M}(\mathbf{u}_a) + \mathcal{D}(\mathbf{u}_a) - \frac{d_a \mathbf{u}_a}{dt} \right)$$

$$\begin{aligned} \frac{d\mathbf{u}}{dt} = & -\frac{\theta}{\theta_0} \tilde{\mathbf{G}} \nabla \phi' - \mathbf{g} \frac{\theta'}{\theta_a} - \mathbf{f} \times \left(\mathbf{u} - \frac{\theta}{\theta_a} \mathbf{u}_a \right) \\ & + \left(\mathcal{M}(\mathbf{u}) - \frac{\theta}{\theta_a} \mathcal{M}(\mathbf{u}_a) \right) + \left(\mathcal{D}(\mathbf{u}) - \frac{\theta}{\theta_a} \mathcal{D}(\mathbf{u}_a) \right) + \frac{\theta}{\theta_a} \frac{d_a \mathbf{u}_a}{dt} \end{aligned}$$

REF

employed in
IFS-FVM

Numerical solutions

Conservation form of perturbational PDEs (GBIS):

$$\frac{\partial \mathcal{G}\rho}{\partial t} + \nabla \cdot (\mathcal{G}\rho\mathbf{v}) = 0 ,$$

$$\frac{\partial \mathcal{G}\rho\theta'}{\partial t} + \nabla \cdot (\mathcal{G}\rho\mathbf{v}\theta') = -\mathcal{G}\rho \left(\tilde{\mathbf{G}}^T \mathbf{u}' \cdot \nabla \theta_a - \mathcal{H}' - \alpha^\theta \theta' \right) ,$$

$$\begin{aligned} \frac{\partial \mathcal{G}\rho\mathbf{u}'}{\partial t} + \nabla \cdot (\mathcal{G}\rho\mathbf{v} \otimes \mathbf{u}') = & -\mathcal{G}\rho \left(\tilde{\mathbf{G}}^T \mathbf{u}' \cdot \nabla \mathbf{u}_a + \frac{\theta}{\theta_0} \tilde{\mathbf{G}} \nabla \phi' + \frac{\theta'}{\theta_0} \tilde{\mathbf{G}} \nabla \phi_a \right. \\ & \left. + \mathbf{f} \times \mathbf{u}' - \mathcal{M}'(\mathbf{u}, \mathbf{u}_a) - \mathcal{D}'(\mathbf{u}, \mathbf{u}_a) - \alpha^u \mathbf{u}' \right) \end{aligned}$$

Template NFT algorithm:

$$\boldsymbol{\psi}_i^{n+1} = \mathcal{A}_i \left(\boldsymbol{\psi}^n + 0.5\delta t \mathcal{R}(\boldsymbol{\psi}^n) \right) + 0.5\delta t \mathcal{R}_i(\boldsymbol{\psi}^{n+1}) \equiv \hat{\boldsymbol{\psi}}_i + 0.5\delta t \mathcal{R}_i(\boldsymbol{\psi}^{n+1})$$

Step 1: mass continuity & advectors for specific variables

$$\rho_i^{n+1} = \mathcal{A}_i \left(\rho^n, (\mathbf{v}\mathcal{G})^{n+1/2}, \mathcal{G}^n, \mathcal{G}^{n+1} \right) \implies \mathbf{V}^{n+1/2} = \overline{\mathbf{v}^\perp \mathcal{G} \rho}^{n+1/2}$$

Step 2: advecting specific variables & formulating semi-implicit solver

$$\theta'_i{}^{n+1} = \widehat{\theta}'_i - 0.5\delta t \left(\widetilde{\mathbf{G}}^T \mathbf{u}'^{n+1} \cdot \nabla \theta_a^{n+1} + \alpha^\theta \theta'^{n+1} \right)_i$$

$$\begin{aligned} \mathbf{u}'_i{}^{n+1} = & \widehat{\mathbf{u}}'_i - 0.5\delta t \left(\widetilde{\mathbf{G}}^T \mathbf{u}'^{n+1} \cdot \nabla \mathbf{u}_a^{n+1} \right)_i \\ & - 0.5\delta t \left(\frac{\theta^\star}{\theta_0} \widetilde{\mathbf{G}} \nabla \phi'^{n+1} + \frac{\theta'^{n+1}}{\theta_0} \widetilde{\mathbf{G}} \nabla \phi_a^{n+1} \right)_i \\ & - 0.5\delta t \left(\mathbf{f} \times \mathbf{u}'^{n+1} - \mathcal{M}'(\mathbf{u}^\star, \mathbf{u}_a^{n+1}) + \alpha^u \mathbf{u}'^{n+1} \right)_i \quad \rightarrow \end{aligned}$$

$$\mathbf{L} \mathbf{u}' = \widehat{\widehat{\mathbf{u}}}' - \tau^u \Theta^\star \widetilde{\nabla} \phi' \implies$$

$$\mathbf{u}' = \check{\mathbf{u}}' - \mathbf{C} \nabla \phi' ; \quad \check{\mathbf{u}}' = \mathbf{L}^{-1} \widehat{\widehat{\mathbf{u}}}' , \quad \mathbf{C} = \tau^u \Theta^\star \mathbf{L}^{-1} \widetilde{\mathbf{G}} ,$$

Step 3: d/dt of the perturbation form of the gas law, and integrating it consistently with other dependent variables leads to the elliptic Helmholtz equation for pressure perturbation

$$0 = - \sum_{\ell=1}^3 \left(\frac{A_\ell^\star}{\zeta_\ell} \nabla \cdot \zeta_\ell (\check{\mathbf{v}} - \widetilde{\mathbf{G}}^T \mathbf{C} \nabla \varphi) \right) - B^\star (\varphi - \widehat{\varphi})$$

the solution of which essentially completes the time step.

Results for $[\rho_a, \theta_a, \phi_a, \mathbf{u}_a] = [\rho_a(y, z), \theta_a(y, z), \phi_a(y, z), (u_a(t, y, z), 0, 0)]$

Extended-range baroclinic-instability evolution

C. Kühnlein, W. Deconinck, R. Klein, S. Malardel, Z. Piotrowski, P.K. Smolarkiewicz, J. Szmelter, N.P. Wedi, FVM 1.0: a nonhydrostatic finite-volume dynamical core for the IFS, Geosci. Model Dev. (2019) 12, 651–676.

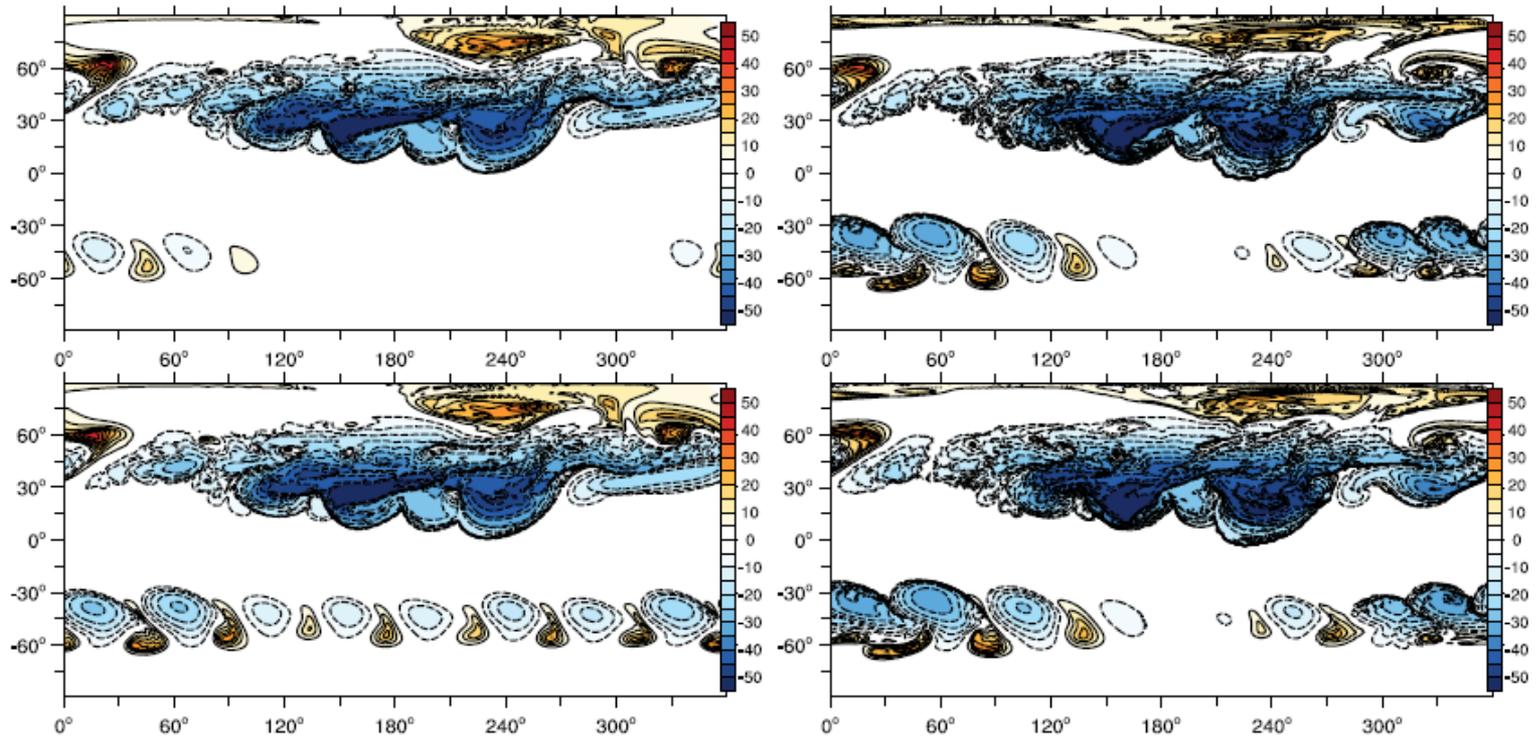


Fig. 1. Surface potential temperature perturbation θ' after 18 simulated days. The solutions GBIS and REF, corresponding to perturbation equations (7b) and (11) are shown at the top and bottom, respectively. The results from the O180 and O640 mesh are shown on the left and right, correspondingly.

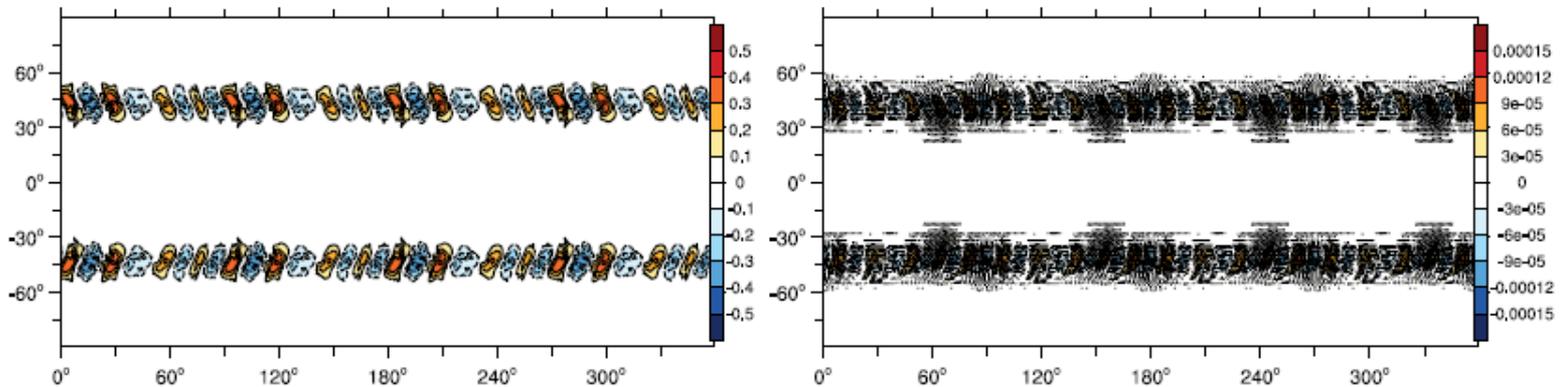


Fig. 2. The 18 days surface θ' REF (left) and GBIS (right) solutions on the O180 grid without the initial perturbation.

The corresponding O640 mesh results are undisplayable.

(average; standard deviation) measures

$(2.2 \times 10^{-4}; 2.4 \times 10^{-2})$

$(2.7 \times 10^{-11}; 7.4 \times 10^{-6})$

Overall, perturbations are at least two and five orders of magnitude smaller, for REF and GBIS respectively, than those in the runs with the initial perturbations applied.

Stratified orographic flow with critical level

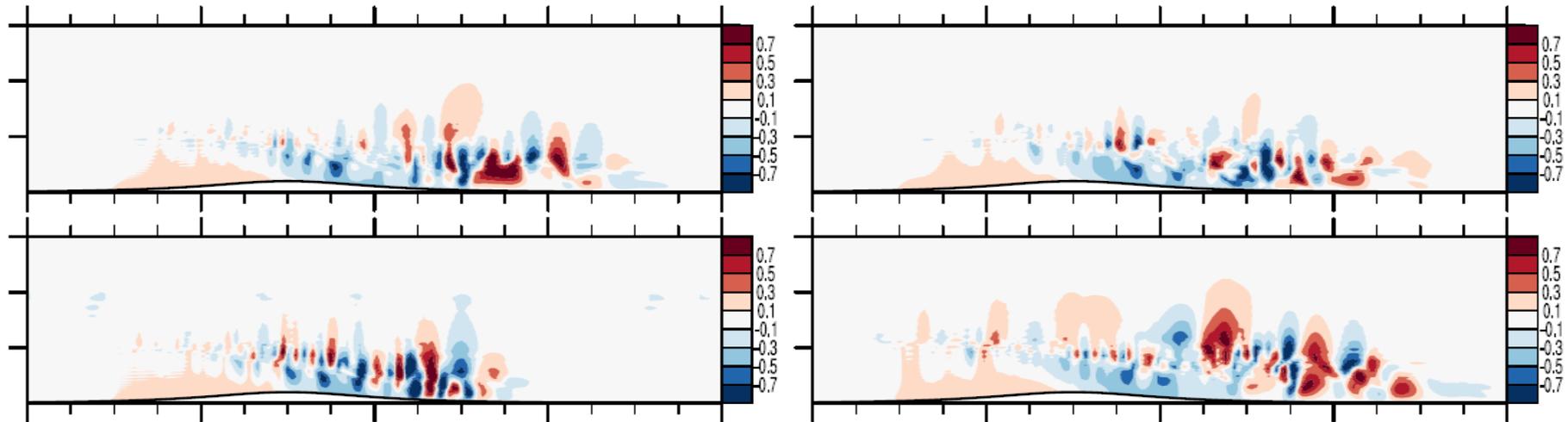


Fig. 5. Simulation of a marginally stable critical-level orographic flow (LS4 case in [37]) on a small planet [44,11]; vertical velocity [m/s] is shown in the central xz cross-section over the horizontal extent 60° - 140° and the full model depth 3 km at dimensionless time $T = 18$, using REF (left) and GBIS (right) integrators with the stationary (top) and evolutionary (bottom) ambient flow.

The figure shows REF and GBIS results for a stratified orographic flow with critical level (CL) in a non rotating shallow atmosphere, using pseudo-incompressible PDEs and hydrostatic ambient state with $u_a(t, y, z) = F(t) U_0 (1 - z/z_c) \cos(y/a)$, where $F(t) = 1$ for $T (= tU_0/\sigma) \leq 6$, and afterwards $F(t) = \cos[\pi(T - 6)/6]$; for the stationary case $F(t) \equiv 1$. The planet radius $a = 20$ km, $U_0 = 10$ ms $^{-1}$, CL height $z_c = 1$ km; $\sigma = 5$ km marks hill width; $Ri = (N z_c/U_0)^2 = 1$ and Brunt-Väisälä frequency $N = 0.01$ s $^{-1}$.

CONCLUDING REMARKS

The new formulation requires a learning effort in real applications in order to be used beneficially. The perturbation equations are formulated about a principally arbitrary ambient state that satisfies the generic equations (from which the perturbation forms derive) and is assumed to be externally known.

The selection of an ambient state is optional, and even a poor selection can deliver a reasonable result, provided the full solution satisfies the required initial and boundary conditions. On the other hand, an able selection can significantly improve the results. Consequently, a separate research effort is required to identify suitable ambient states for weather simulations with full complexity.

The expediting assumption (of ambient states satisfying the generic equations) **is by no means limiting**. Otherwise, the hypothetically vanishing terms represent auxiliary forcings or residuals suitable for minimisation, depending on their proximity to the full problem.

The developed apparatus is equally applicable to approximate ambient states, constructed based on alternate models, resolutions or data. The minimisation would seek suitable φ_a , whereas inclusion of additional forcing terms (bias correction) would model the ambient state errors as opposed to bias correcting the entire state evolution (in the spirit of turbulence closures or continuous data assimilation). As this will not affect the machinery of the semi-implicit integrators, the proposed solvers form the basis for further developments; e.g. multilevel methods, time parallel and/or hardware-failure resilient algorithms, and data assimilators.



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