Adaptive volcanic modeling using Discontinuous Galerkin Methods

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Introduction to volcanic eruptions

Eyjafjallajökull eruption 2010

Predicted ash cloud over the northern hemisphere

April 20th, 00:00 UTC

⇒ Big parts above Europe, Russia and the Atlantic are covered

⇒ Good forecasts necessary
Introduction to volcanic eruptions and motivation

Strong volcanic plume
Why model a volcanic plume?
Mass flux $\propto$ height$^{1/4}$

Main issues with existing plume models:
- resolution $\rightarrow$ Adaptive Mesh Refinement
- shock modeling $\rightarrow$ correct numerics
Quick outline:

• start with a weak formulation (Galerkin Method)
• use (piecewise continuous) polynomials as test functions
• coupling with neighboring elements through flux

Why use DGM (for volcanic settings)?

• well-suited for complex geometries
• local conservation
• good for capturing shocks
• local structure lends itself for parallelization and AMR
• easier to implement higher order schemes (compared to FVM)
Timestepping and Accuracy

Spatial discretization (nodal DGM):

Results presented for piecewise linear elements and without limiter ⇒ 2nd order accurate

Two choices for timestepping:

(implicit) Runge-Kutta methods e.g. SSPRK2 ⇒ 2nd order accurate
Rosenbrock-type methods (implicit RK) e.g. ROS2 ⇒ 2nd order accurate
Gas dynamics modeled with (compressible) Euler equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

\[
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + P I_2) = -\rho g \mathbf{k}
\]

\[
\frac{\partial \rho e}{\partial t} + \nabla \cdot [(\rho e + P) \mathbf{u}] = -\rho w g
\]

Equation of state:

\[
P = (\gamma - 1) \left( \rho e - \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right)
\]

- one-phase model
- gas only (dry air)
- 2D only
- inviscous
- no Coriolis force
⇒ simple model
DGM and compressible Euler equation allow shock capturing
RHS = −ρg is unstable for (nearly) hydrostatic states (\(\nabla P^h = −ρg k\))

⇒ need for well-balanced source term


RHS = −ρg ⇒ \(\int_{\Omega_e} \psi_i(x) \nabla P^h \, dx + \int_{\Gamma_e} \psi_i(x) \mathbf{n} \cdot P^h \, dx\)

\(\Omega_e\)         \(\Gamma_e\)

interior          edges

hydrostatic pressure \(P^h\) is calculated with averaging inside the element
Well-balancing

Rising warm air bubble test case
Well-balancing

Results with interior pressure reconstruction

\[\Delta x = \Delta z = 10\text{m}\]

⇒ Well-balancing has to be improved
Adaptive Mesh Refinement

Grid is adapted according to error estimator/indicator:

- small error  \(\Rightarrow\) grid is coarsened
- large error  \(\Rightarrow\) grid is refined

Reason: saving CPU time while maintaining high resolution

AMR library: AMATOS (h-adaptivity)
Shock wave test case again

Take very simple error indicator: density gradient
CPU time saved: 85%

In general: finding error estimators/indicators for more complex CFD can be quite challenging
Outlook

To-do list:

• Implement (better) well-balanced scheme
• Find good error estimation for AMR
• Include ash phases
• 3D?
• ...

M. Bänsch, Adaptive volcanic modeling, PDEs on the sphere 2019
Questions?

Thank you!

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## Appendix: Existing plume models

<table>
<thead>
<tr>
<th>Spatial discretization</th>
<th>ASHEE</th>
<th>ATHAM</th>
<th>PDAC</th>
<th>SK-3D</th>
<th>New model</th>
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<table>
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<td>shocks</td>
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### Example test case “weak plume” from intercomparison study (height \(\approx\) 12 km)

<table>
<thead>
<tr>
<th>(\Delta x_{\text{min}})</th>
<th>ASHEE</th>
<th>ATHAM</th>
<th>PDAC</th>
<th>SK-3D</th>
<th>New model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 m</td>
<td>18 m</td>
<td>18 m</td>
<td>3 m</td>
<td>adaptive</td>
</tr>
</tbody>
</table>

| \(\Delta x_{\text{min}}\) | 70 m  | 600 m | 200 m | 27 m  | adaptive  |
DGM weak form:

\[
\int_{\Omega_e} \left( \frac{\partial q}{\partial t} - \mathbf{F} \cdot \nabla - \mathbf{S} \right) \psi_i(\mathbf{x}) \, d\mathbf{x} = -\int_{\Gamma_e} \psi_i(\mathbf{x}) \mathbf{n} \cdot \mathbf{F}^* \, d\mathbf{x}
\]

for differential equations of the form:

\[
\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{F}(q) = \mathbf{S}(q)
\]

and numerical flux (Rusanov)

\[
\mathbf{F}^* = \frac{1}{2} \left[ \mathbf{F}(q^L) + \mathbf{F}(q^R) - |\lambda| (q^R - q^L) \mathbf{n} \right]
\]
Appendix: Convergence study

DGM with piecewise linear elements

Timestepping: Rosenbrock (ROS2)
linear profile for hydrostatic pressure reconstruction

\[ P^h = (\gamma - 1)\overline{\rho e} - g\overline{\rho}(z - \overline{z}) \]
Appendix: Warm air bubble with AMR

deviation

case 1

gradient

case 2

case 3