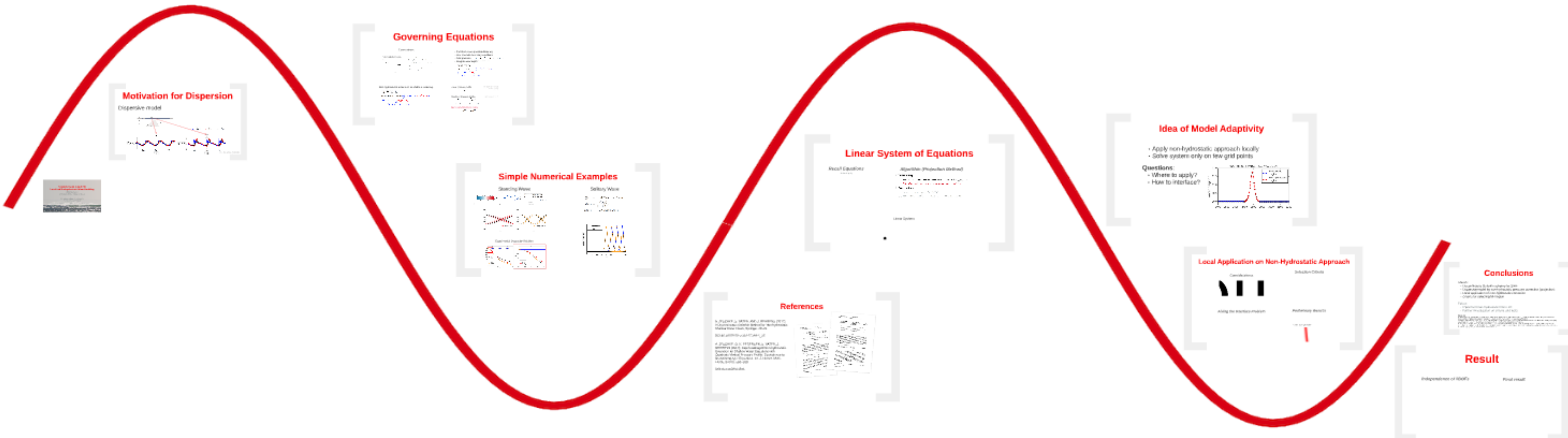


Motivation for Dispersion
Dispersive model
Governing Equations
Simple Numerical Examples
Standing Wave
Solitary Wave
References
Linear System of Equations
Idea of Model Adaptivity
- Apply non-hydrostatic approach locally
- Solve system only on few grid points
Questions:
- Where to apply?
- How to interface?
Local Application on Non-Hydrostatic Approach
Conclusions
Result

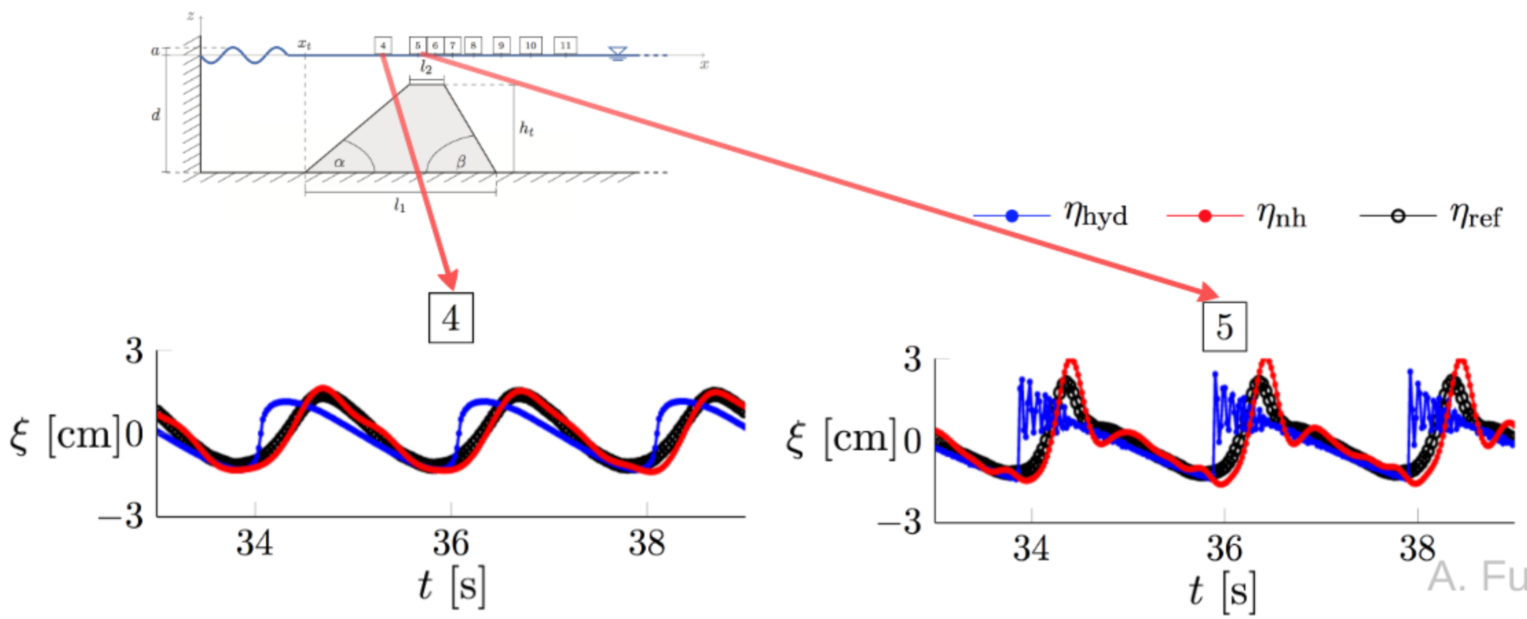
Towards Model Adaptivity
Localized Nonhydrostatic Wave Modeling

Jörn Behrens
University of Hamburg, Germany

Acknowledging Major Contributions by:
Anja Jeschke, Leila Wegener
and
Nicole Beisiegel, Geir Kleivstul Petersen, Stefan Vater

Motivation for Dispersion

Dispersive model

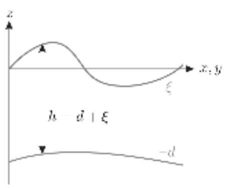


A. Fuchs (2014)

Governing Equations

Conventions

Water height decomposition



$\nabla_3 = (\partial_x, \partial_y, \partial_z)^\top$ 3D gradient operator

$\nabla = (\partial_x, \partial_y)^\top$ horizontal gradient operator

$V = (U, V, W)^\top = (\boldsymbol{U}, W)^\top$ velocity vector

ρ density

P pressure

g gravitational acceleration

$\boldsymbol{E}_z$ vertical unit vector

$h = d + \xi$ total water depth

- Start from incompressible Euler eq.
- Use kinematic boundary conditions
- Split pressure $P = P^{hy} + P^{nh} = \rho g(\xi - z) + P^{nh}$
- Integrate over depth

$$\partial_t \xi + \nabla \cdot (h\boldsymbol{u}) = 0,$$
$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -g\nabla\xi - \frac{1}{\rho h}\left(\nabla\left(hp^{nh}\right) - P^{nh}_{-d}\nabla d\right),$$
$$\partial_t w + (\boldsymbol{u} \cdot \nabla) w = \frac{1}{\rho h} P^{nh}_{-d},$$
$$h(\nabla \cdot \boldsymbol{u}) = -2\left(w + \boldsymbol{u} \cdot \nabla d\right)$$

Non-hydrostatic extension to shallow water eq.

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Linear Pressure Profile

Stelling/Zijlema (2003)
Walters (2005)

$$P^{nh}_{-d} = 2p^{nh}$$

Quadratic Pressure Profile

Jeschke et al. (2017)

$$P^{nh}_{-d} = \frac{3}{2}p^{nh} + \frac{1}{4}\rho h \Phi$$

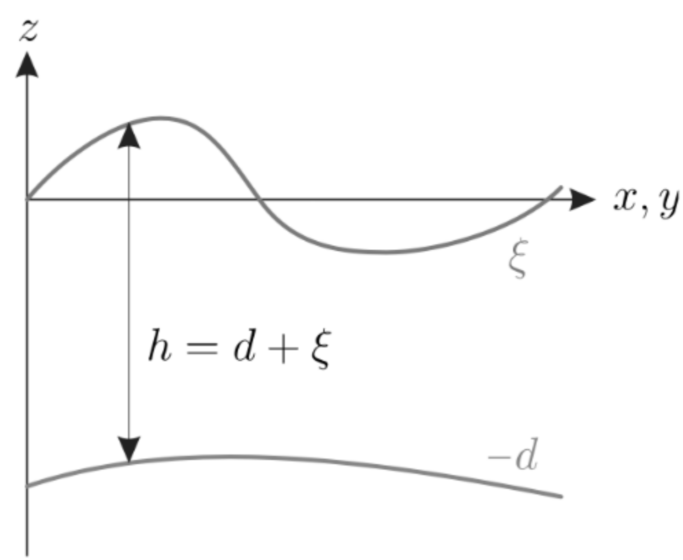
$$\Phi = -\nabla d \cdot (\partial_t u + (u \cdot \nabla) u) - u \cdot \nabla(\nabla d) \cdot u$$

Generalized Pressure Profile

$$P^{nh}_{-d} = f_{\mathrm{nh}} p^{nh} + f_{\mathrm{d}}$$

Conventions

Water height decomposition



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\end{aligned}$$

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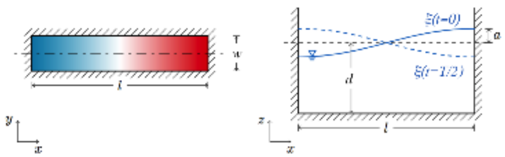
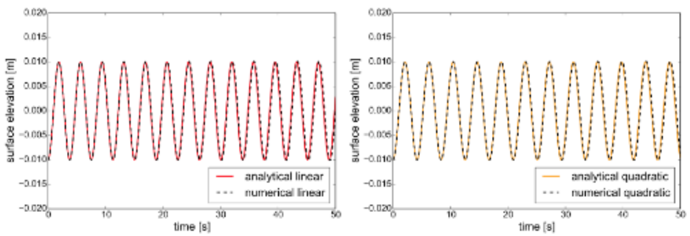
\partial_t w + (\boldsymbol{u} \cdot \boldsymbol{\nabla})w &= \frac{1}{\rho h}(\boldsymbol{f_{nh}p^{nh}} + \boldsymbol{f_d}), \\
h(\boldsymbol{\nabla} \cdot \boldsymbol{u}) &= -2\left(w + \boldsymbol{u} \cdot \boldsymbol{\nabla} d\right)
\end{aligned}$$

Simple Numerical Examples

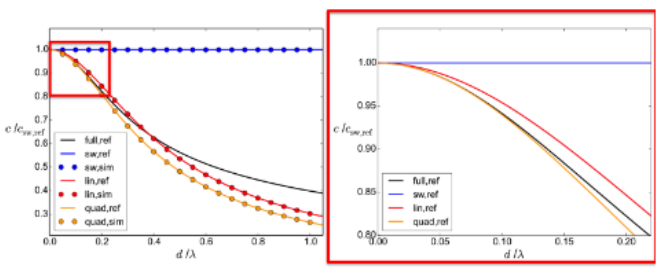
Standing Wave



$$\xi(\boldsymbol{x},t) = -a \sin(\kappa x)\cos(\kappa c t),$$
$$u(\boldsymbol{x},t) = a\frac{c}{d}\cos(\kappa x)\sin(\kappa c t),$$
$$w(\boldsymbol{x},t) = 0, \quad \forall\, \boldsymbol{x} = (x,y)^T \in \Omega, \forall t \in \mathbb{R}.$$

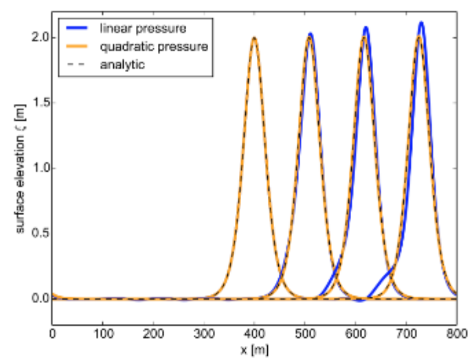


Experimental Dispersion Relation



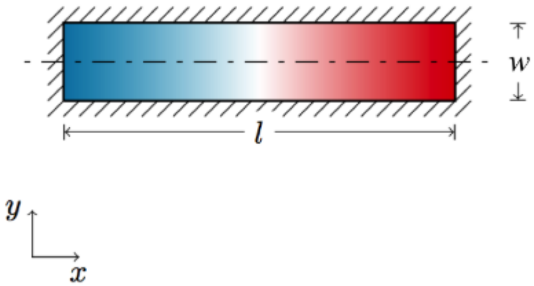
Solitary Wave

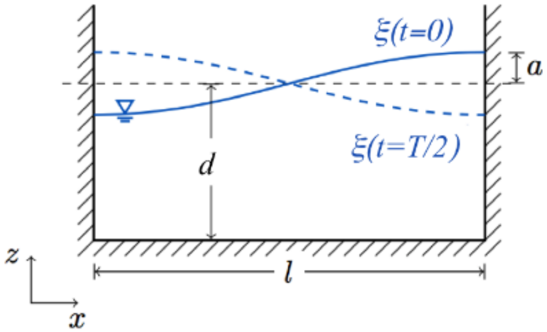
$$\xi(\boldsymbol{x},t) = a\cosh^{-2}(K(x - ct - x_0)),$$
$$u(\boldsymbol{x},t) = c\frac{\xi(\boldsymbol{x},t)}{d + \xi(\boldsymbol{x},t)},$$
$$w(\boldsymbol{x},t) = -0.5h(\boldsymbol{x},t)\partial_x u(\boldsymbol{x},t)$$



Simple Numeri

Standing Wave

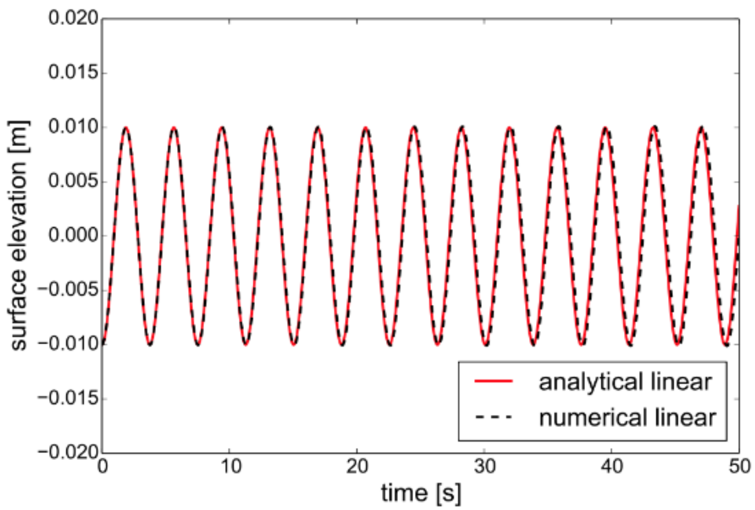


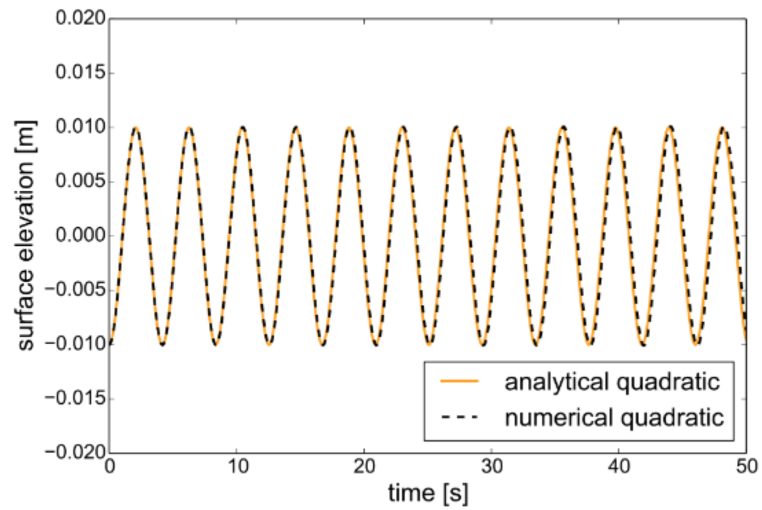


$$\xi(\boldsymbol{x},t) = -a\sin(\kappa x)\cos(\kappa c t)\,,$$
$$u(\boldsymbol{x},t) = a\frac{c}{d}\cos(\kappa x)\sin(\kappa c t)\,,$$
$$v(\boldsymbol{x},t) = 0, \qquad \forall\, \boldsymbol{x} = (x,y)^T \in \Omega, \forall t \in \mathbb{R},$$

$c = \frac{\omega_{nh,f_{nh}}}{\kappa}$ hydrostatic

$c = c_{sw}$ non-hydrostatic

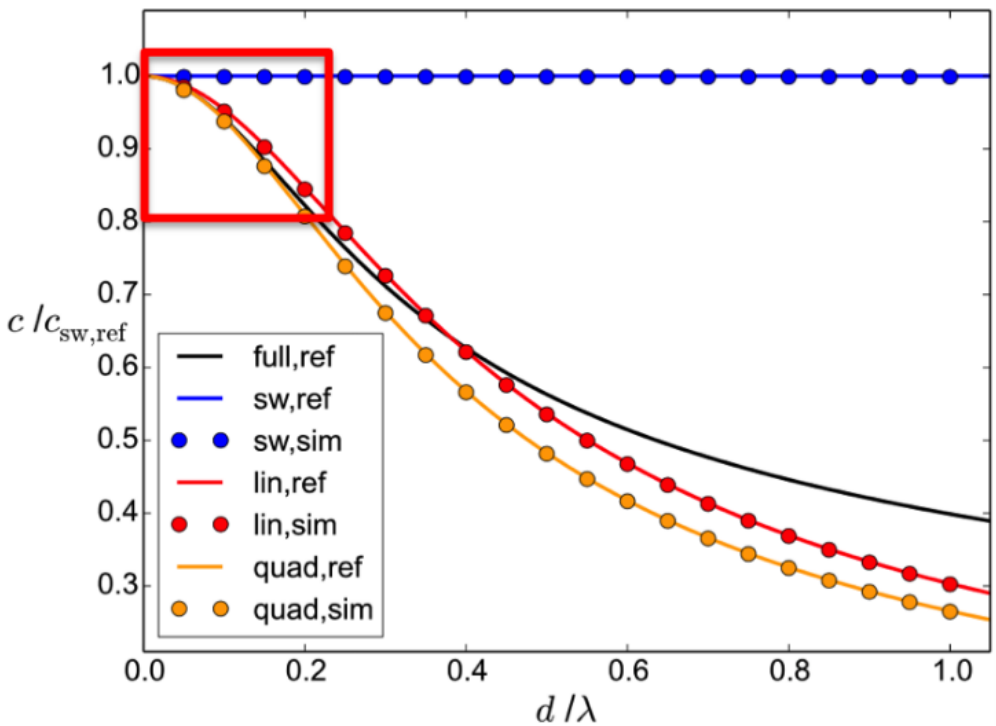


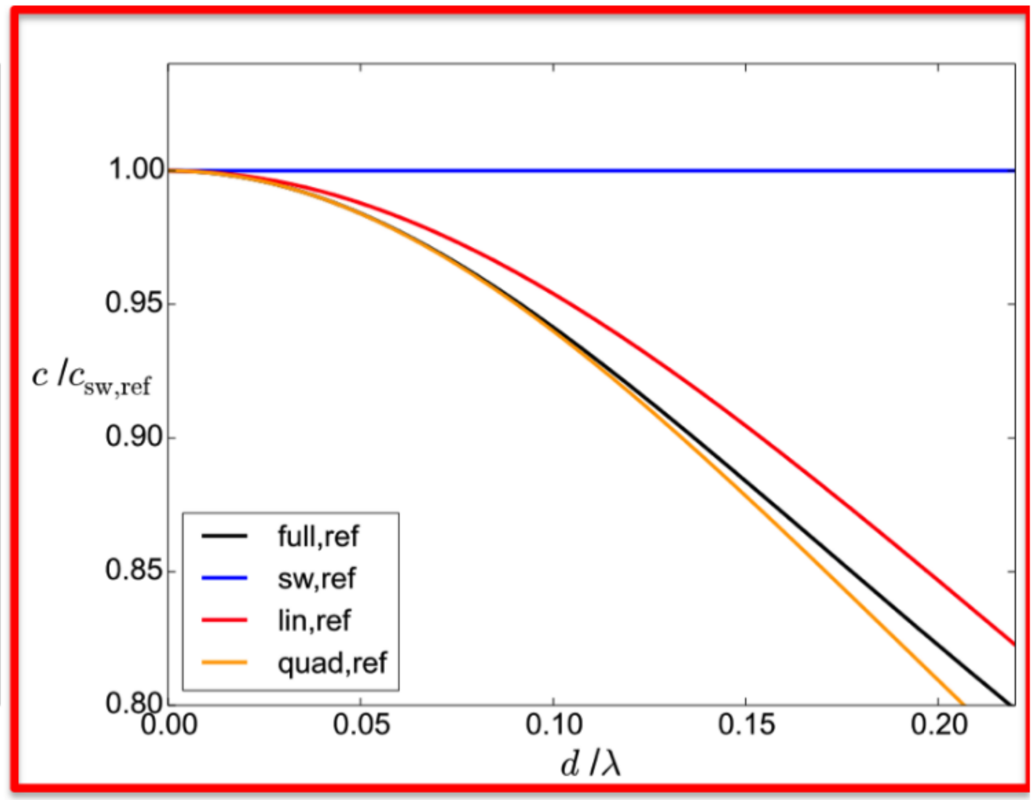


Experimental Dispersion Relation

time [s] time [s]

Experimental Dispersion Relation





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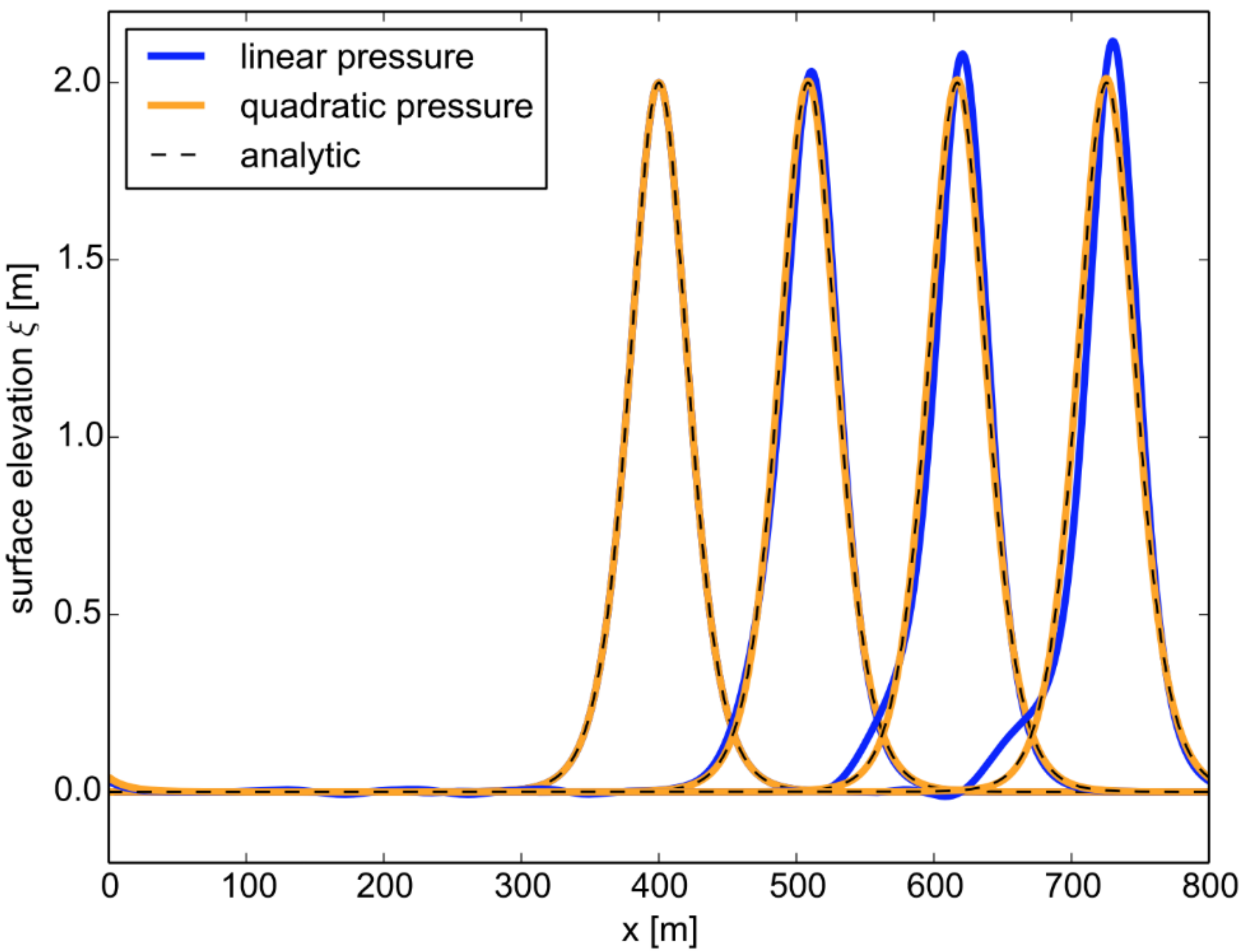
linear pressure
quadratic pressure
analytic
surface elevation ξ [m]
2.0
1.5
1.0
0.5
0.0
0
100
200
300
400
500
600
700
800
x [m]

References

A. JESCHKE, S. VATER, AND J. BEHRENS (2017): A Discontinuous Galerkin Method for Non-hydrostatic Shallow Water Flows, Springer, Cham,

DOI:10.1007/978-3-319-57394-6_27.

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Linear System of Equations

Recall Equations

1D for simplicity

$$\partial t \begin{pmatrix} h \\ hu \\ hw \end{pmatrix} + \partial x \begin{pmatrix} hu \\ hu^2 + \frac{1}{2} g h^2 + \frac{1}{\rho}(hp^{\mathrm{nh}}) \\ hwu \end{pmatrix} = \begin{pmatrix} 0 \\ -ghb_x \frac{1}{\rho}(A\frac{1}{h}(hp^{\mathrm{nh}}) + B) b_x \\ \frac{1}{\rho}(A\frac{1}{h}(hp^{\mathrm{nh}}) + B) \end{pmatrix}$$

We have used:

- h, u, g, b the height, horizontal velocity, gravitational acceleration, and bathymetry (resp.)
- p^{nh}, A, B the non-hydrostatic pressure, coefficients for vertical profile
- w the vertical velocity

Algorithm (Projection Method)

1. **Predictor Step**:
   (a) With $h^n, (hu)^n$ solve shallow water equations, obtain $\tilde{h}^{n+1}, (\tilde{hu})^{n+1}$.
   (b) With $(hw)^n$ solve vertical velocity advection, obtain $(\tilde{hw})^{n+1}$.
2. **Corrector Step**:
   (a) Solve linear system of eq. for $(hp^{\mathrm{nh}})^n$.
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Linear System:

Construction:

$$(hp^{\mathrm{nh}})_x^{n+1} + \frac{A}{h}(hp^{\mathrm{nh}}) b_x + \frac{1}{\Delta t}(h\bar{u})^{n+1} = \frac{1}{\Delta t}(hu)^{n+1} - \frac{\beta}{\rho} b_x \qquad =: s_1$$

$$(hu)^{n+1} + 2\Delta t \frac{A}{h^2}(hp^{\mathrm{nh}}) - \frac{(hu)^{n+1}}{h}(h + 2b)_x = -\frac{2}{h}(h\bar{u})^{n+1} - \frac{2\Delta t}{h}\beta \qquad =: s_2$$

Linear system of equations:

$$\begin{pmatrix} \frac{1}{\Delta t} & (\cdot)_x + \frac{A}{h}(\cdot) b_x \\ (\cdot)_x - \frac{1}{h}(h + 2b)_x & 2\Delta t \frac{A}{h^2} \end{pmatrix} \begin{pmatrix} (hu)^{n+1} \\ (hp^{\mathrm{nh}})^{n+1} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

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Idea of Model Adaptivity

- Apply non-hydrostatic approach locally
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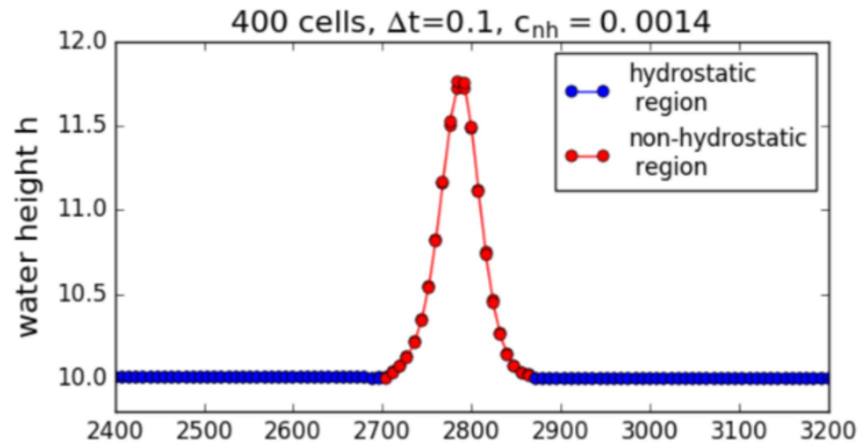
Questions:

- Where to apply?
- How to interface?

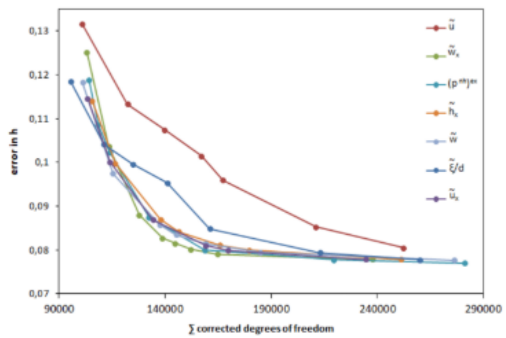


Local Application on Non-Hydrostatic Approach

Considerations

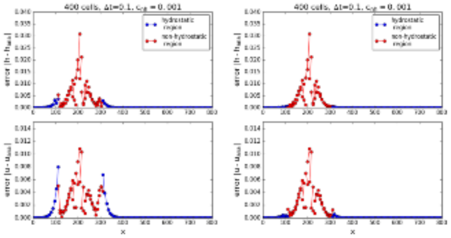
- Splitting of domain: $\Omega = \Omega_{\text{nh}} \cup \Omega_{\text{hyd}}$ with $\Omega_{\text{nh}} \cap \Omega_{\text{hyd}} = \emptyset$
- Criterion for selecting Ω_{nh} from hydrostatic variables!
- Example: Sea surface elevation $\tilde{\xi}$: use NH, where $\left|\frac{\tilde{\xi}}{d}\right| \ll 1$
- Boundary Conditions for elliptic problem: Dirichlet
  Motivation $(hw) = 0$ and $(p^{nh}) = 0$ outside of Ω_{nh}.

Selection Criteria



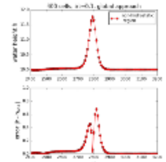
Fixing the Interface Problem

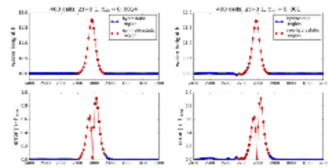
- Previosly: Set $(hu) = 0$ and $(p^{nh}) = 0$ in Ω_{hyd}.
- New: $(hu) = (\bar{hu})$ from predictor step.



Preliminary Results

Global non-hydrostatic





Criterion: $\frac{\tilde{\xi}}{d}$ Criterion: $\tilde{w}_x$

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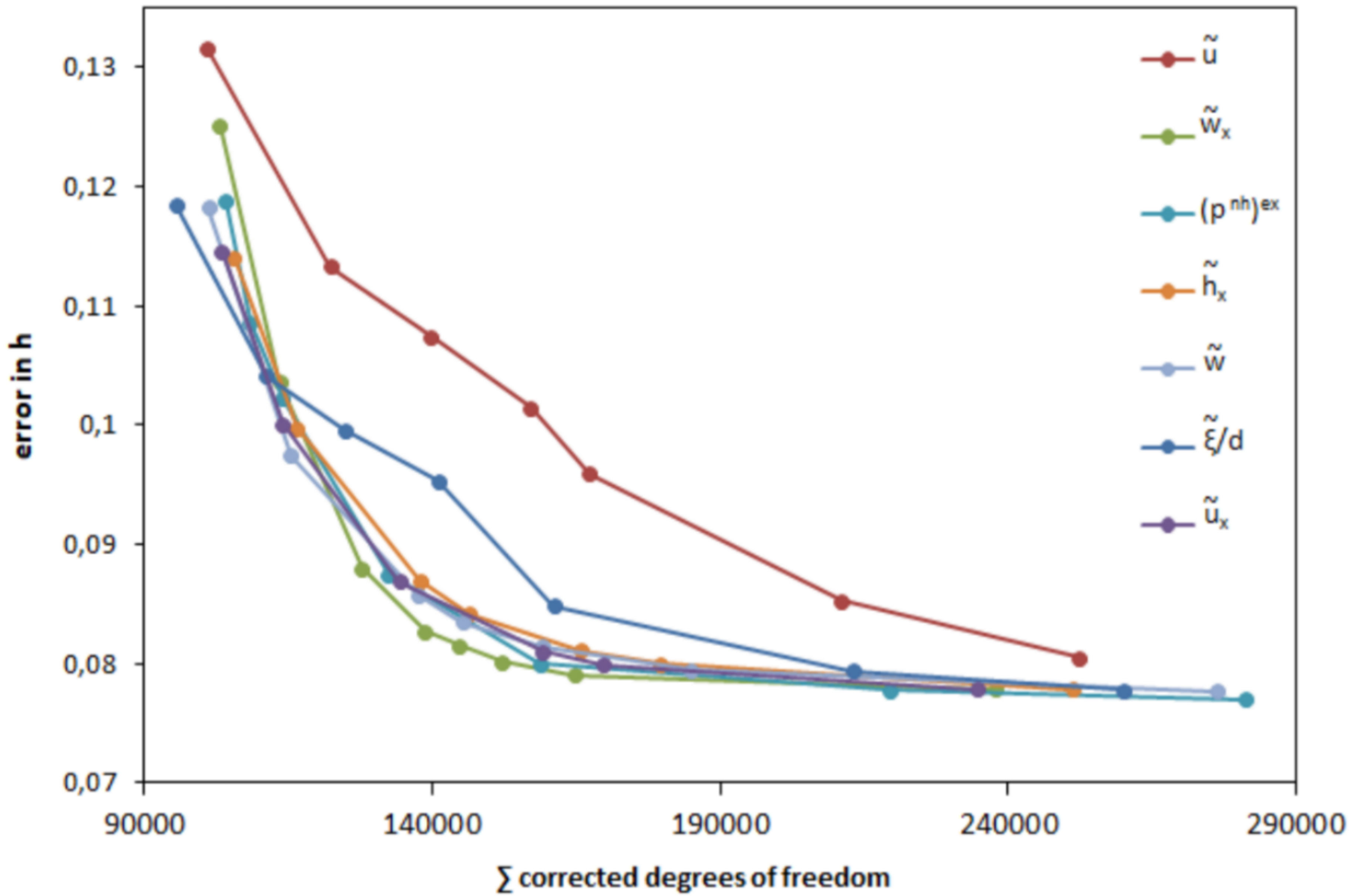

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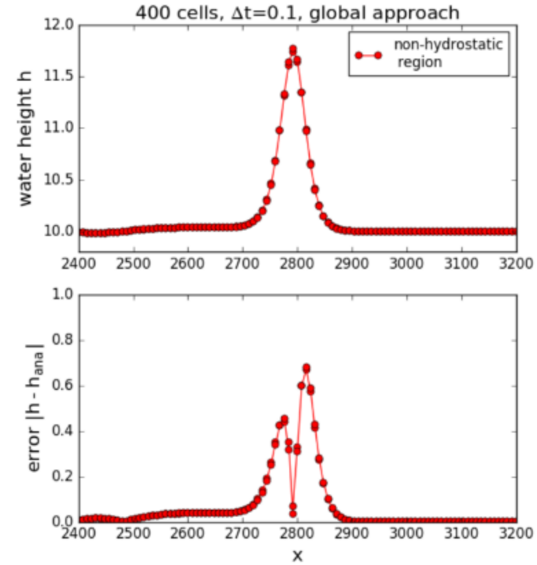

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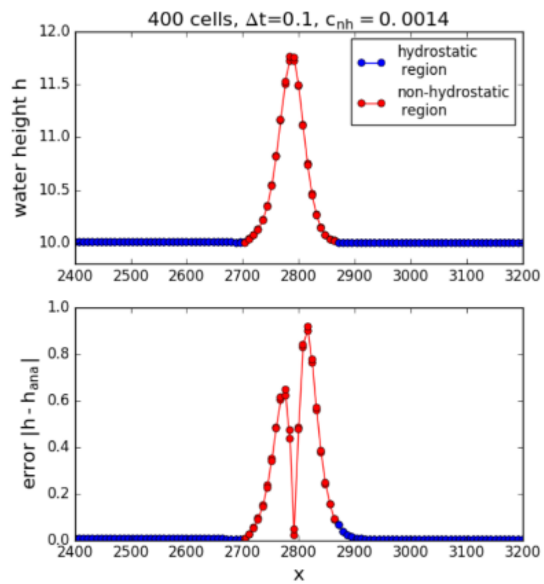

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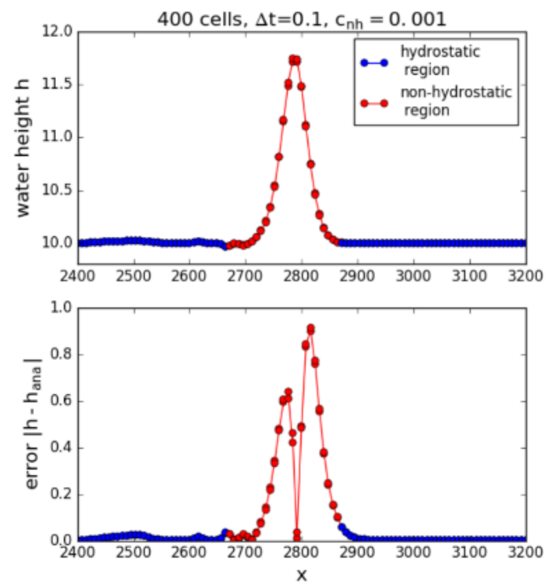

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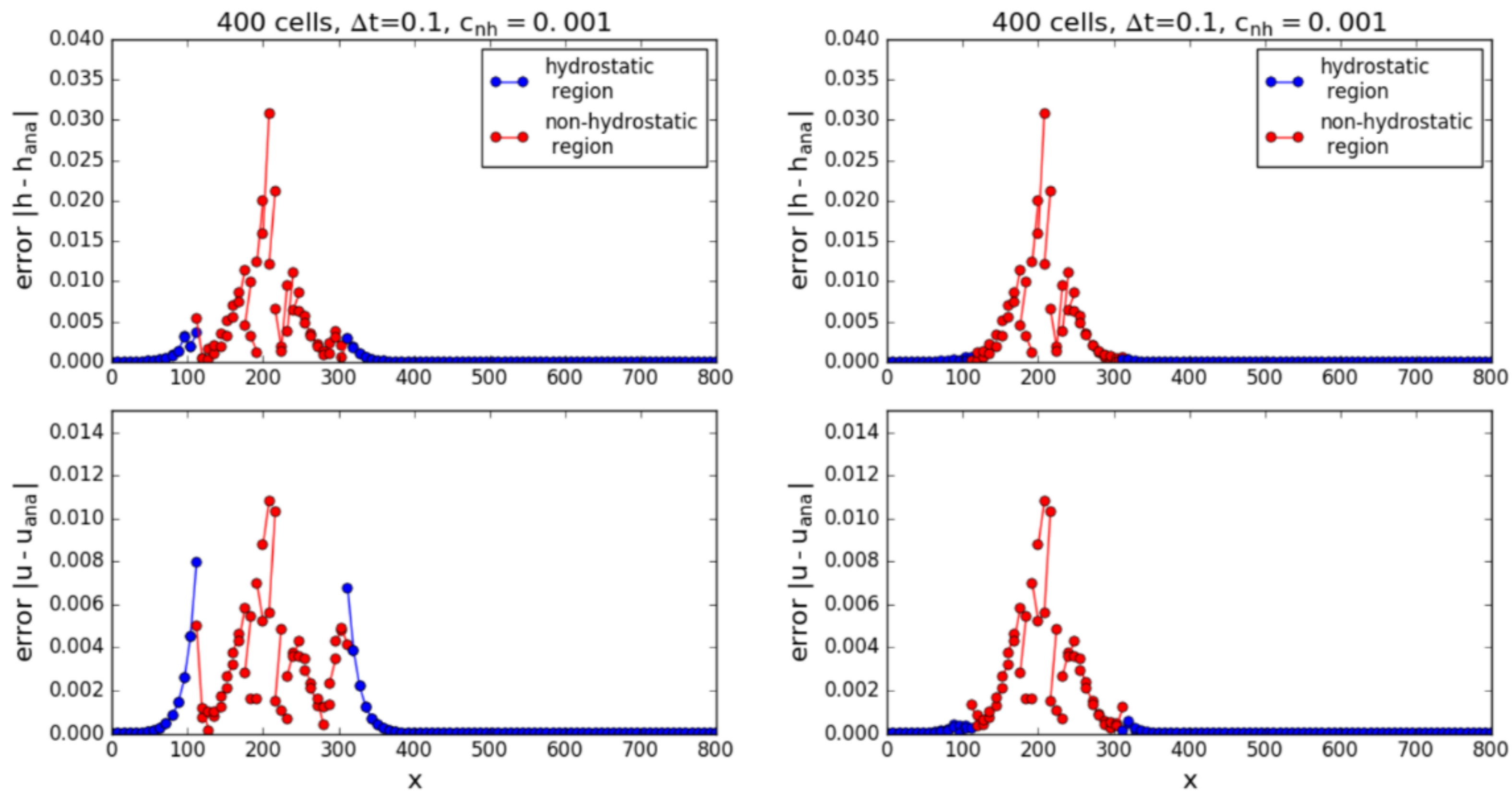
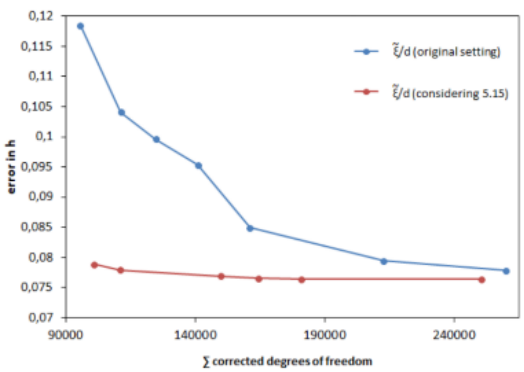
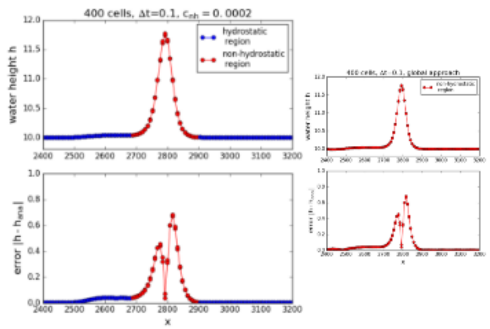


Result

Independence of #DOFs



Final result



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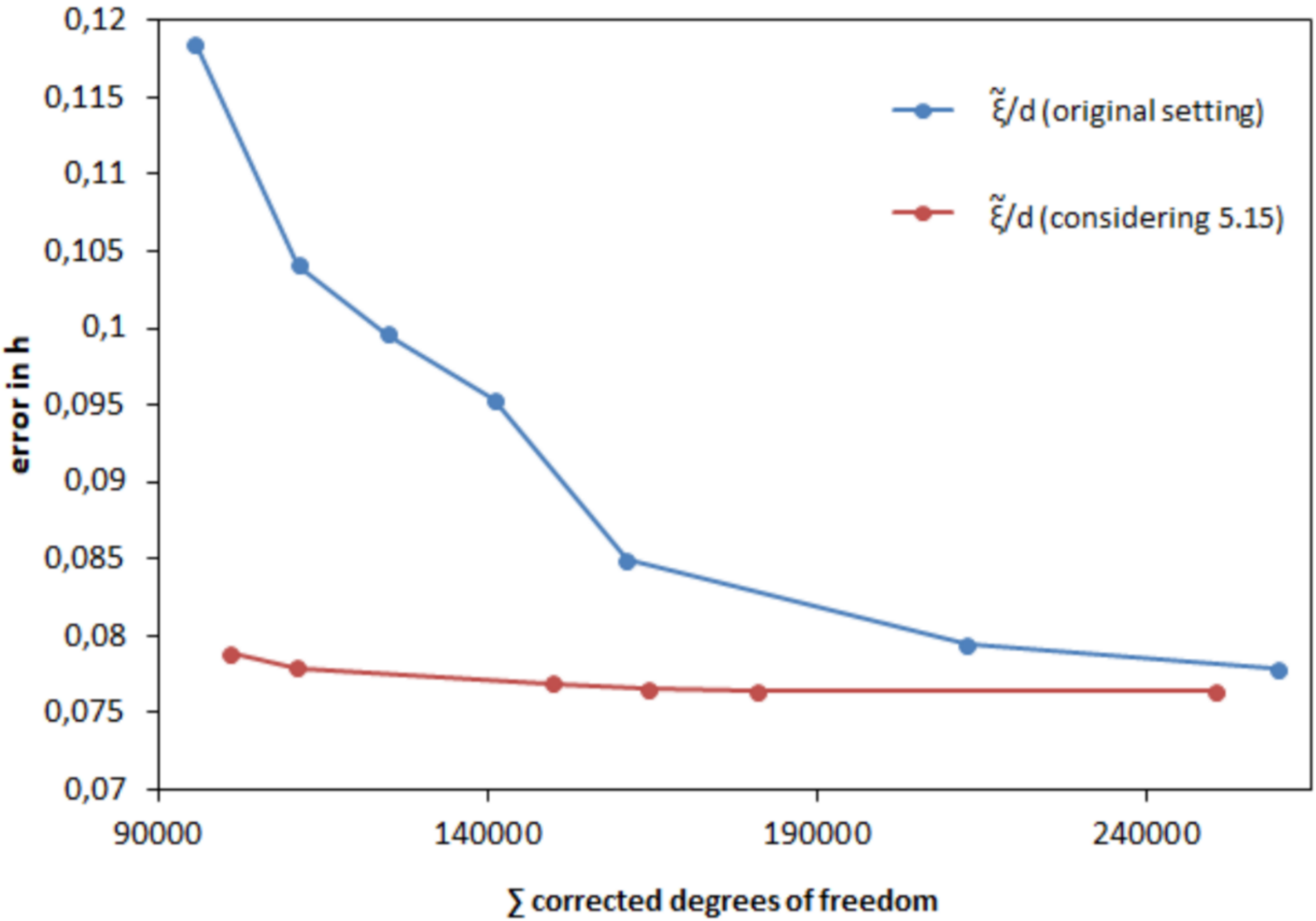

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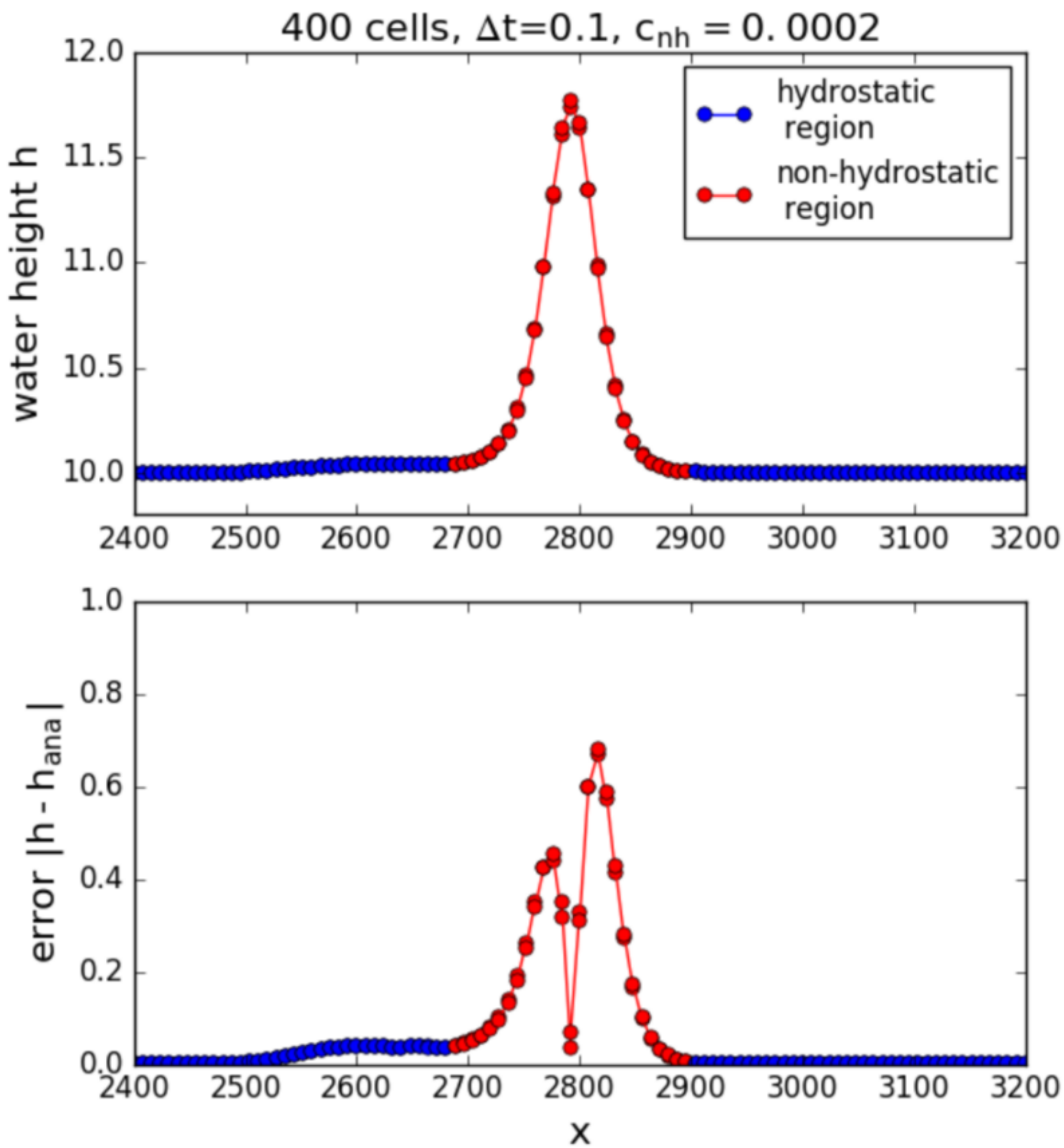
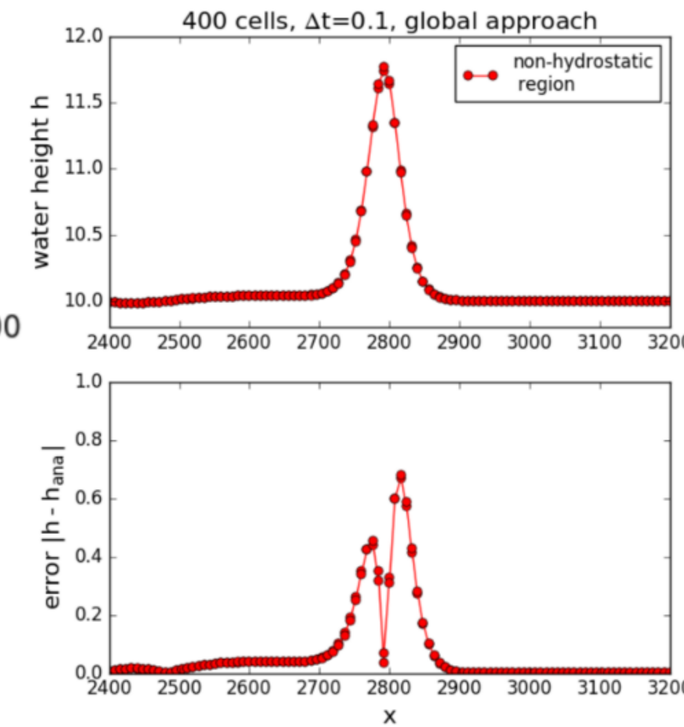




Conclusions

Shown:

- Discontinuous Galerkin scheme for SWE
- Dispersive model by non-hydrostatic pressure correction (projection)
- Local application on non-hydrostatic correction
- Criteria for selecting NH region

Future:

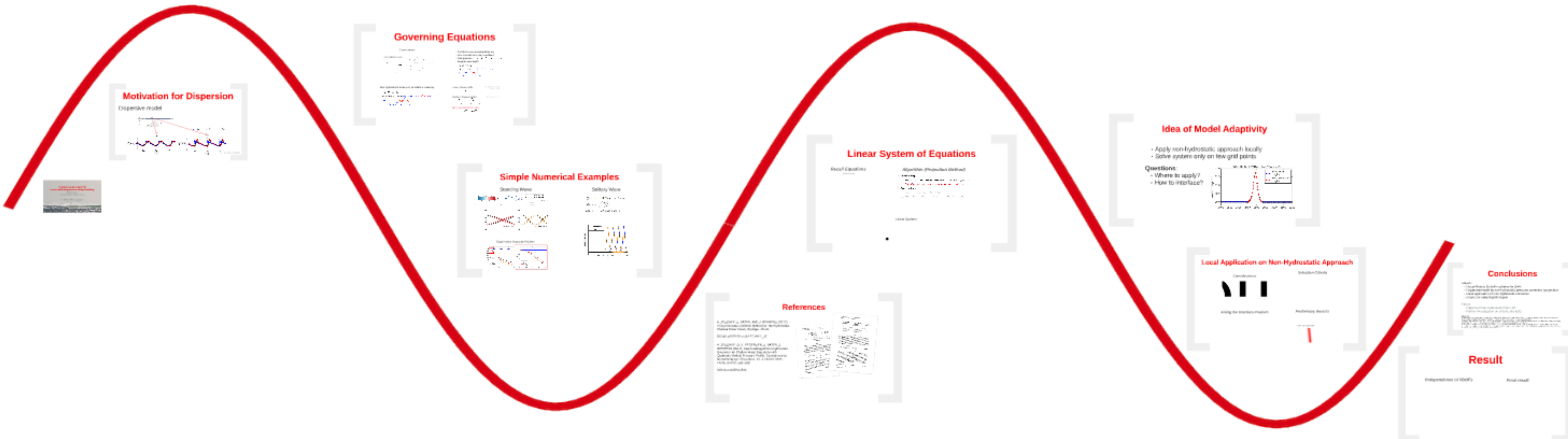
- Implement non-hydrostatic DG in 2D
- Further investigation on criteria and tests

References:

A. JESCHKE, G. K. PEDERSEN, S. VATER, J. B. (2017): Depth-averaged Non-hydrostatic Extension for Shallow Water Equations with Quadratic Vertical Pressure Profile: Equivalence to Boussinesq-type Equations, Int. J. Numer. Meth. Fluids, *DOI:10.1002/fld.4361*.

S. VATER, N. BEISIEGEL, and J. B. (2015): A Limiter-Based Well-Balanced Discontinuous Galerkin Method for Shallow-Water Flows with Wetting and Drying: One-Dimensional Case, Advances in Water Resources, 85, 1-13, *DOI:10.1016/j.advwatres.2015.08.008*.

A. JESCHKE, S. VATER, and J. B. (2017): A Discontinuous Galerkin Method for Non-hydrostatic Shallow Water Flows, In: In: Finite Volumes for Complex Applications VIII - Hyperbolic, Elliptic and Parabolic Problems (Eds. C. Cancès and P. Omnes) . FVCA 2017. Springer Proceedings in Mathematics & Statistics, vol 200. Springer, Cham, *DOI:10.1007/978-3-319-57394-6_27*.



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