Towards Model Adaptivity
Localized Nonhydrostatic Wave Modeling

Jörn Behrens
University of Hamburg, Germany

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Anja Jeschke, Leila Wegener
and
Nicole Beisiegel, Geir Kleivstul Petersen, Stefan Vater
Motivation for Dispersion

Dispersive model

A. Fuchs (2014)
Governing Equations

Conventions

\[ \nabla \times \begin{pmatrix} \partial_x, \partial_y, \partial_z \end{pmatrix} \text{ curl operator} \]
\[ \nabla \cdot \begin{pmatrix} \partial_x, \partial_y, \partial_z \end{pmatrix} \text{ divergence operator} \]
\[ W = \begin{pmatrix} (u, v, w) \end{pmatrix} \text{ velocity vector} \]
\[ P \text{ pressure} \]
\[ g \text{ gravitational acceleration} \]
\[ E \text{ vertical unit vector} \]
\[ \bar{h} \text{ total water depth} \]

Non-hydrostatic extension to shallow water eq.

\[ \partial_t \xi + \nabla \cdot (h \mathbf{u}) = 0, \]
\[ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -g \nabla \xi - \frac{1}{\rho_0} (\nabla (h \rho m) - f_0 \nabla d), \]
\[ \partial_t w + (\mathbf{u} \cdot \nabla) w = \frac{1}{\rho_0} (f_0 \rho_0 m^h + f_0), \]
\[ h (\nabla \cdot \mathbf{u}) = -2 (\mathbf{w} \cdot \nabla d) \]

Linear Pressure Profile

\[ P^h_d = 2 \rho^h \]

Quadratic Pressure Profile

\[ P^h_d = \frac{3}{2} \rho^h + \frac{1}{4} \rho^h \Phi \]
\[ \Phi \equiv -\nabla \cdot (\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) - m \nabla (\nabla \cdot \mathbf{u}) \]

Generalized Pressure Profile

\[ P^h_d = f_0 \rho_0 \rho^h + f_0 \]

- Start from incompressible Euler eq.
- Use kinematic boundary conditions
- Split pressure \( P = P^{hy} + P^{nh} = \rho g (\xi - z) + P^{nh} \)
- Integrate over depth

\[ \partial_t \xi + \nabla \cdot (h \mathbf{u}) = 0, \]
\[ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -g \nabla \xi - \frac{1}{\rho_0} (\nabla (h \rho m) - P^{nh} \nabla d), \]
\[ \partial_t w + (\mathbf{u} \cdot \nabla) w = \frac{1}{\rho_0} P^{nh}, \]
\[ h (\nabla \cdot \mathbf{u}) = -2 (\mathbf{w} \cdot \nabla d) \]
Conventions

Water height decomposition

\[ \nabla_3 = (\partial_x, \partial_y, \partial_z)^\top \] 3D gradient operator

\[ \nabla = (\partial_x, \partial_y)^\top \] horizontal gradient operator

\[ \mathbf{V} = (U, V, W)^\top = (U, W)^\top \] velocity vector

\[ \rho \] density

\[ P \] pressure

\[ g \] gravitational acceleration

\[ \mathbf{E}_z \] vertical unit vector

\[ h = d + \xi \] total water depth
• Start from incompressible Euler eq.
• Use kinematic boundary conditions
• Split pressure \( P = P^{hy} + P^{nh} = \rho g (\xi - z) + P^{nh} \)
• Integrate over depth

\[
\partial_t \xi + \nabla \cdot (hu) = 0,
\]
\[
\partial_t u + (u \cdot \nabla) u = -g \nabla \xi - \frac{1}{\rho h} (\nabla (hp^{nh}) - P^{nh}_{-d} \nabla d),
\]
\[
\partial_t w + (u \cdot \nabla) w = \frac{1}{\rho h} P^{nh}_{-d},
\]
\[
h(\nabla \cdot u) = -2(w + u \cdot \nabla d)
\]
Linear Pressure Profile

\[ P_{-d}^{nh} = 2p^{nh} \]

Quadratic Pressure Profile

\[ P_{-d}^{nh} = \frac{3}{2}p^{nh} + \frac{1}{4}\rho h\Phi \]

\[ \Phi = -\nabla d \cdot (\partial_t u + (u \cdot \nabla) u - u \cdot \nabla (\nabla d) \cdot u) \]

Generalized Pressure Profile

\[ P_{-d}^{nh} = f_{nh} p^{nh} + f_d \]
Non-hydrostatic extension to shallow water eq.

\[ \partial_t \xi + \nabla \cdot (hu) = 0, \]

\[ \partial_t u + (u \cdot \nabla) u = -g \nabla \xi - \frac{1}{\rho h} \left( \nabla (hp^{nh}) - (f_{nh}p^{nh} + f_d) \nabla d \right), \]

\[ \partial_t w + (u \cdot \nabla) w = \frac{1}{\rho h} (f_{nh}p^{nh} + f_d), \]

\[ h(\nabla \cdot u) = -2(w + u \cdot \nabla d) \]
Simple Numerical Examples

Standing Wave

\[ \xi(x, t) = a \cosh^{-2}(K(x - ct - x_0)), \]
\[ u(x, t) = \frac{\xi(x, t)}{d + \xi(x, t)}, \]
\[ w(x, t) = -0.5h(x, t) \partial_x u(x, t) \]

Experimental Dispersion Relation

Solitary Wave

\[ \xi(x, t) = a \cosh^{-2}(K(x - ct - x_0)), \]
\[ u(x, t) = \frac{\xi(x, t)}{d + \xi(x, t)}, \]
\[ w(x, t) = -0.5h(x, t) \partial_x u(x, t) \]
Standing Wave

\[ \xi(x, t) = -a \sin(\kappa x) \cos(\kappa ct), \]

\[ u(x, t) = \frac{c}{d} \cos(\kappa x) \sin(\kappa ct), \]

\[ v(x, t) = 0, \quad \forall \mathbf{x} = (x, y)^T \in \Omega, \forall t \in \mathbb{R}, \]

\( c = \frac{\omega}{\kappa} \) hydrostatic

\( c = c_{\text{eff}} \) non-hydrostatic

Experimental Dispersion Relation
Experimental Dispersion Relation
Solitary Wave

\[\xi(x, t) = a \cosh^{-2}(K(x - ct - x_0)),\]

\[u(x, t) = c \frac{\xi(x, t)}{d + \xi(x, t)},\]

\[w(x, t) = -0.5h(x, t) \partial_x u(x, t)\]
References


Linear System of Equations

Recall Equations

1D for simplicity

\[
\frac{\partial}{\partial t} \left( \begin{array}{c} h \\ hu \\ h \end{array} \right) + \frac{\partial}{\partial x} \left( \begin{array}{c} \frac{h}{2} \frac{\partial (hu^2)}{\partial x} \\ \frac{h}{2} \frac{\partial (hu^2)}{\partial x} + \frac{1}{3} \lambda_h \frac{\partial (hu^2)}{\partial x} + B \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right)
\]

With an initial guess:
- \( \lambda_h \) is the horizontal velocity, gravitational acceleration, and frictional (resistible).
- \( h, \lambda_h \) are horizontal pressure, coefficients for vertical pressure
- \( u \) is the vertical velocity

Algorithm (Projection Method)

1. Predictor Step:
   (a) With \( h^n \), \( (hu)^n \) solve shallow water equations, obtain \( h^{n+1}, (hu)^{n+1} \).
   (b) With \( (hu)^n \) solve vertical velocity advection, obtain \( (hu)^{n+1} \).

2. Corrector Step:
   (a) Solve linear system of eq. for \( (h^p h)^n \).
   (b) Update \( (hu)^{n+1}, (hu)^{n+1} \) with \( (h^p h)^n \), obtain \( (hu)^{n+1}, (hu)^{n+1} \).

Linear System:

Construction:

\[
\begin{align*}
(h^p h)^{n+1} + \frac{\Delta t}{h} (h^p h)^n + \frac{1}{2 \Delta t} (h^p h)^{n+1} - \frac{1}{2} h u^{n+1} - \frac{\Delta t}{h} &= s_1 \\
(hu)^{n+1} + 2 \Delta t \left( \frac{1}{h^2} (h^p h)^n - (hu)^{n+1} + \frac{(hu)^{n+1}}{h} (h + \Delta h) - \frac{2 \Delta t}{h} (hu)^{n+1} + \frac{\Delta t}{h} \right) &= s_2
\end{align*}
\]

Linear system of equations:

\[
\begin{pmatrix}
\frac{1}{h} & \frac{\Delta t}{h} + \frac{1}{3} \lambda_h \\
\frac{1}{h} + \frac{1}{3} \lambda_h & \frac{1}{2 \Delta t}
\end{pmatrix}
\begin{pmatrix}
(hu)^{n+1} \\
(h^p h)^{n+1}
\end{pmatrix}
= \begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
Recall Equations

1D for simplicity

\[
\frac{\partial}{\partial t} \begin{pmatrix} h \\ hu \\ hw \end{pmatrix} + \frac{\partial}{\partial x} \left( hu^2 + \frac{1}{2} gh^2 + \frac{1}{\rho} (hp^{nh}) \right) = \begin{pmatrix} 0 \\ -ghb_x \frac{1}{\rho} (A_{h} \frac{1}{h} (hp^{nh}) + B) b_x \\ \frac{1}{\rho} (A_{h} \frac{1}{h} (hp^{nh}) + B) \end{pmatrix}
\]

We have used:

- \(h, u, g, b\) the height, horizontal velocity, gravitational acceleration, and bathymetry (resp.)
- \(p^{nh}, A, B\) the non-hydrostatic pressure, coefficients for vertical profile
- \(w\) the vertical velocity
Algorithm (Projection Method)

1. Predictor Step:
   - (a) With $h^n, (hu)^n$ solve shallow water equations, obtain $\tilde{h}^{n+1}, (\tilde{h}u)^{n+1}$.
   - (b) With $(hw)^n$ solve vertical velocity advection, obtain $(\tilde{h}w)^{n+1}$.

2. Corrector Step:
   - (a) Solve linear system of eq. for $(hp^{nh})^n$.
   - (b) Update $(\tilde{h}u)^{n+1}, (\tilde{h}w)^{n+1}$ with $(hp^{nh})^n$, obtain $(hu)^{n+1}, (hw)^{n+1}$.
Linear System:

Construction:

\[
(hp^{nh})^{n+1}_x + \frac{A}{h} (hp^{nh}) b_x + \frac{1}{\Delta t} (h\tilde{u})^{n+1} = \frac{1}{\Delta t} (hu)^{n+1} - \frac{\beta}{\rho} b_x =: s_1
\]

\[
(hu)^{n+1} + 2\Delta t \frac{A}{h^2} (hp^{nh}) - \frac{(hu)^{n+1}}{h} (h + 2b)_x = -\frac{2}{h} (h\tilde{u})^{n+1} - \frac{2\Delta t}{h} \beta =: s_2
\]

Linear system of equations:

\[
\begin{pmatrix}
\frac{1}{\Delta t} & (\cdot)_x + \frac{A}{h} (\cdot) b_x \\
(\cdot)_x - \frac{1}{h} (h + 2b)_x & 2\Delta t \frac{A}{h^2}
\end{pmatrix}
\begin{pmatrix}
(hu)^{n+1} \\
(hp^{nh})^{n+1}
\end{pmatrix}
= 
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix}
\]
Idea of Model Adaptivity

- Apply non-hydrostatic approach locally
- Solve system only on few grid points

Questions:
- Where to apply?
- How to interface?
Local Application on Non-Hydrostatic Approach

**Considerations**
- **Splitting** of domain: $\Omega = \Omega_{\text{inh}} \cup \Omega_{\text{hyd}}$ with $\Omega_{\text{inh}} \cap \Omega_{\text{hyd}} = \emptyset$
- **Criterion** for selecting $\Omega_{\text{inh}}$ from hydrostatic variables!
- Example: Sea surface elevation $\xi$: use NH, where $\left| \frac{\xi}{h} \right| \ll 1$
- **Boundary Conditions** for elliptic problem: Dirichlet Motivation $(h_u) = 0$ and $(p^\text{nh}) = 0$ outside of $\Omega_{\text{inh}}$.

**Fixing the Interface Problem**
- Previously: Set $(h_u) = 0$ and $(p^\text{nh}) = 0$ in $\Omega_{\text{hyd}}$.
- New: $(h_u) = (\hat{h}u)$ from predictor step.

**Selection Criteria**

**Preliminary Results**
- Global non-hydrostatic
- Criterion: $\frac{\xi}{h}$
- Criterion: $\omega_r$
Considerations

- **Splitting** of domain: $\Omega = \Omega_{\text{nh}} \cup \Omega_{\text{hyd}}$ with $\Omega_{\text{nh}} \cap \Omega_{\text{hyd}} = \emptyset$

- **Criterion** for selecting $\Omega_{\text{nh}}$ from hydrostatic variables!

- **Example**: Sea surface elevation $\tilde{\xi}$: use NH, where $\left| \frac{\tilde{\xi}}{d} \right| \ll 1$

- **Boundary Conditions** for elliptic problem: Dirichlet Motivation $(hw) = 0$ and $(\rho^{nh}) = 0$ outside of $\Omega_{\text{nh}}$. 
Selection Criteria

![Graph showing error in \( h \) vs. \( \sum \text{corrected degrees of freedom} \). Multiple lines represent different parameters such as \( \tilde{u} \), \( \tilde{w} \), \( (p^{nh})^{ex} \), \( h^z \), \( \xi / d \), and \( \tilde{u}_x \).]
Preliminary Results

Global non-hydrostatic

Criterion: $\tilde{\xi}/d$  Criterion: $\tilde{w}_x$
Fixing the Interface Problem

- Previously: Set \((hu) = 0\) and \((p^{nh}) = 0\) in \(\Omega_{hyd}\).
- New: \((hu) = \(\tilde{h}u\)\) from predictor step.
Result

Independence of #DOFs

Final result
Independence of #DOFs
Final result

400 cells, $\Delta t=0.1$, $c_{nh} = 0.0002$

- hydrostatic region
- non-hydrostatic region

Error $|h - h_{ana}|$ vs $x$

400 cells, $\Delta t=0.1$, global approach

- non-hydrostatic region
Conclusions

Shown:
- Discontinuous Galerkin scheme for SWE
- Dispersive model by non-hydrostatic pressure correction (projection)
- Local application on non-hydrostatic correction
- Criteria for selecting NH region

Future:
- Implement non-hydrostatic DG in 2D
- Further investigation on criteria and tests

References: