Compatible finite element methods for numerical weather prediction on moving meshes

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Motivation

Geophysical flows:
- development of small scale structure
- increasing resolution has a significant impact on model accuracy
- more mesh points where ‘things are happening’

Why r-adaptivity?
- data structure unchanged
- interpolation between meshes unnecessary
- avoid sharp changes in resolution
Our approach

• Meshes provided using optimal transport ideas
  • The scaling and skewness of optimally transported meshes on
  • Optimal transport based mesh adaptivity on the place and
    sphere using finite elements. *SIAM Journal of Scientific
• Solve GFD equations using compatible finite element ideas
• Coupling - this talk ...and future work!

Some choices:
• Solve for mesh and PDE separately
• Modify equations to solve in moving frame
Compatible finite element methods

\[ \nabla_0 \xrightarrow{\nabla \perp} \nabla_1 \xrightarrow{\nabla \cdot} \nabla_2 \]

Continuous \quad Continuous normals \quad Discontinuous

Mimetic property: \( \nabla \cdot \nabla \perp \mathbf{u} = 0 \)

- \( \nabla \cdot \) maps from \( \nabla_1 \) onto \( \nabla_2 \).
- \( \nabla \perp \) maps from \( \nabla_0 \) onto the kernel of \( \nabla \cdot \) in \( \nabla_1 \).

Properties of compatible FEM discretisation of LSWE

- energy conservation
- mass conservation
- steady geostrophic states
- no spurious pressure or inertial modes
Compatible finite element methods

$V_0 = P_2$
Quadratic, Continuous

$\nabla^\perp \rightarrow V_1 = \text{BDM1}$
Linear, Continuous normals

$\nabla \cdot \rightarrow V_2 = P_0$
Constant, Discontinuous
Compatible finite element methods

\[ \nabla_0 = P2^+ \]

Quadratic (+1 Cubic) Continuous

\[ \nabla \rightarrow \nabla_1 = BDFM1 \]

Linear (+2 Quadratic) Cont. normals

\[ \nabla \cdot \rightarrow \nabla_2 = P1_{DG} \]

Linear Discontinuous

Jemma Shipton
Compatible FEM for NWP on OT meshes
Compatible finite element methods

\[ \nabla_0 = Q1 \quad \nabla^\perp \quad \nabla_1 = RT0 \quad \nabla \cdot \quad \nabla_2 = Q0_{DG} \]

- Bilinear Continuous
- Constant/Linear, Cont. normals
- Constant, Discontinuous
Mesh movement: optimal transport

- resolution is equidistributed with respect to a monitor function
- the monitor function is derived from the flow
- adjust monitor function to control mesh properties
- formulation guarantees that the mesh will not tangle
- we find that we require a higher-order mesh representation for convergence on the sphere
Scalar advection

Advection equation for scalar $c$ with prescribed velocity $\bar{u}$ and mesh velocity $v$:

$$\frac{\partial c}{\partial t} + ((\bar{u} - v) \cdot \nabla)c + (v \cdot \nabla)c = 0$$

Observe: moving the mesh with velocity $v$, fixing DOF values, is exactly Lagrangian advection of $c$ at velocity $v$ for time $\Delta t$.

Therefore:
- $\frac{1}{2} \Delta t$ Eulerian advection $(\bar{u} - v)$ on old mesh
- Move mesh, without changing values
- $\frac{1}{2} \Delta t$ Eulerian advection $(\bar{u} - v)$ on new mesh
Scalar advection

Video:
- Nair–Lauritzen test case 4
- $P_{1}^{\text{DG}}$ discretisation
- DG advection with upwinding
- SSPRK3 timestepping, no slope limiter
Solving the shallow water equations

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -g \nabla D - f \hat{k} \times u
\]

\[
\frac{\partial D}{\partial t} + \nabla \cdot (uD) = 0
\]

\[u^* \leftarrow u^n + (1 - \alpha) \Delta t F^n\]

\[\text{for } k \leq k_{\text{max}} \text{ do}\]

\[u^p \leftarrow u^* - \Delta t L_u^u(u^*)\]

\[D^p \leftarrow D^* - \Delta t L_u^D(D^*)\]

\[R[U] \leftarrow 0\]

\[\text{for } i \leq i_{\text{max}} \text{ do}\]

\[u^{n+1} \leftarrow u^p + \alpha \Delta t F^{n+1}\]

\[R[U] \leftarrow R[U] - U^{n+1}\]

\[\text{Solve: } J[dU] = -R[U]\]

▷ explicit forcing step

▷ advection step

▷ initialise residual

▷ implicit forcing step

▷ update residual

▷ elliptic solve
Solving the shallow water equations

\( u^* \leftarrow u^n + (1 - \alpha) \Delta t F^n \)

for \( k \leq k_{\text{max}} \) do

\( u^p' \leftarrow u^* - \frac{\Delta t}{2} L_{\bar{u}-\nu} (u^*) \)

\( D^p' \leftarrow D^* - \frac{\Delta t}{2} L_{\bar{D}-\nu} (D^*) \)

\( x \leftarrow x_{\text{new}} \) with \( \nu = (x_{\text{new}} - x_{\text{old}})/\Delta t \)

\( u^p \leftarrow \tilde{u}^p' - \frac{\Delta t}{2} L_{\bar{u}-\nu} (\tilde{u}^p') \)

\( D^p \leftarrow \tilde{D}^p' - \frac{\Delta t}{2} L_{\bar{D}-\nu} (\tilde{D}^p') \)

\( R[U] \leftarrow 0 \)

for \( i \leq i_{\text{max}} \) do

\( u^{n+1} \leftarrow u^p + \alpha \Delta t F^{n+1} \)

\( R[U] \leftarrow R[U] - U^{n+1} \)

Solve: \( J[dU] = -R[U] \)
Scalar continuity equation

Continuity equation for scalar $D$ with prescribed velocity $\bar{u}$ and mesh velocity $v$:

$$\frac{\partial D}{\partial t} + \nabla \cdot (D \bar{u}) = 0$$

Idea: split into

$$\frac{\partial D}{\partial t} = -\nabla \cdot (D(\bar{u} - v)) - \nabla \cdot (Dv)$$

- $\frac{1}{2} \Delta t$ Eulerian continuity $(\bar{u} - v)$ on old mesh
- Move mesh, adjusting values of $D$ [next slide]
- $\frac{1}{2} \Delta t$ Eulerian continuity $(\bar{u} - v)$ on new mesh
Scalar continuity equation

How to solve

\[ \frac{\partial D}{\partial t} = -\nabla \cdot (D \mathbf{v}) \]

while moving the mesh?

Observe: for a standard test-function \( \phi \), we have \( \phi_t + (\mathbf{v} \cdot \nabla)\phi = 0 \).

Consider

\[
\frac{\partial}{\partial t} \int_{\Omega(t)} \phi D \, dx = \int_{\Omega(t)} \phi_t D \, dx + \int_{\Omega(t)} \phi D_t \, dx + \int_{\delta\Omega(t)} \phi D \mathbf{v} \cdot \mathbf{n} \, dx \\
= -\int_{\Omega(t)} D (\mathbf{v} \cdot \nabla) \phi \, dx - \int_{\Omega(t)} \phi \nabla \cdot (\mathbf{v} D) \, dx + \int_{\Omega(t)} \nabla \cdot (\phi D \mathbf{v}) \, dx \\
= 0
\]
Scalar continuity equation

So, for the continuity equation

\[
\frac{\partial D}{\partial t} + \nabla \cdot (D \bar{u}) = 0,
\]

we do

- \( \frac{1}{2} \Delta t \) Eulerian continuity \((\vec{u} - \vec{v})\) on old mesh
- Move mesh, and adjust \( D \) according to

\[
\int_{\Omega(t^{n+1})} \phi D \, dx = \int_{\Omega(t^n)} \phi D \, dx \quad \forall \phi \in V_D
\]

- \( \frac{1}{2} \Delta t \) Eulerian continuity \((\vec{u} - \vec{v})\) on new mesh
Velocity advection

Recall the shallow water momentum equation:

\[
\frac{\partial u}{\partial t} + (\bar{u} \cdot \nabla) u + f \hat{k} \times u = -g \nabla D
\]

Write as

\[
\frac{\partial u}{\partial t} + (\nabla \times u) \times \bar{u} + \nabla (u \cdot \bar{u}) = -f \hat{k} \times u - g \nabla D + \frac{1}{2} \nabla |u|^2
\]

- **Advection**
- **Forcing**
Velocity advection

Velocity advection step:

\[
\frac{\partial u}{\partial t} + (\nabla \times u) \times \bar{u} + \nabla (u \cdot \bar{u}) = 0
\]

Split into

\[
\frac{\partial u}{\partial t} = - \left[ (\nabla \times u) \times (\bar{u} - v) + \nabla (u \cdot (\bar{u} - v)) \right] \\
- \left[ (\nabla \times u) \times v + \nabla (u \cdot v) \right]
\]

- \( \frac{1}{2} \Delta t \) circulation-form advection \((\bar{u} - v)\) on old mesh
- Move mesh, adjusting DOF values of \( u \) (similar to continuity equation)
- \( \frac{1}{2} \Delta t \) circulation-form advection \((\bar{u} - v)\) on new mesh
Does it work?

Testcase: barotropic jet (Galewsky)

- correct wavenumber
- quiscent section
- grid imprinting has not triggered early development of instability
Does it work?

Moving mesh vs static mesh, reference level 5.
Does it work?

Moving mesh vs static mesh, reference level 4.
Summary
We can produce optimally transported meshes on the sphere and solve the shallow water equations on them, using a compatible finite element scheme.

Outlook
• Investigate:
  • parameters controlling mesh quality
  • different monitor functions
• More rigorous testing to check for:
  • correct wave propagation speeds
  • possible spurious effects
• Computation times