Imperial College London

Compatible finite element methods for numerical weather prediction on moving meshes

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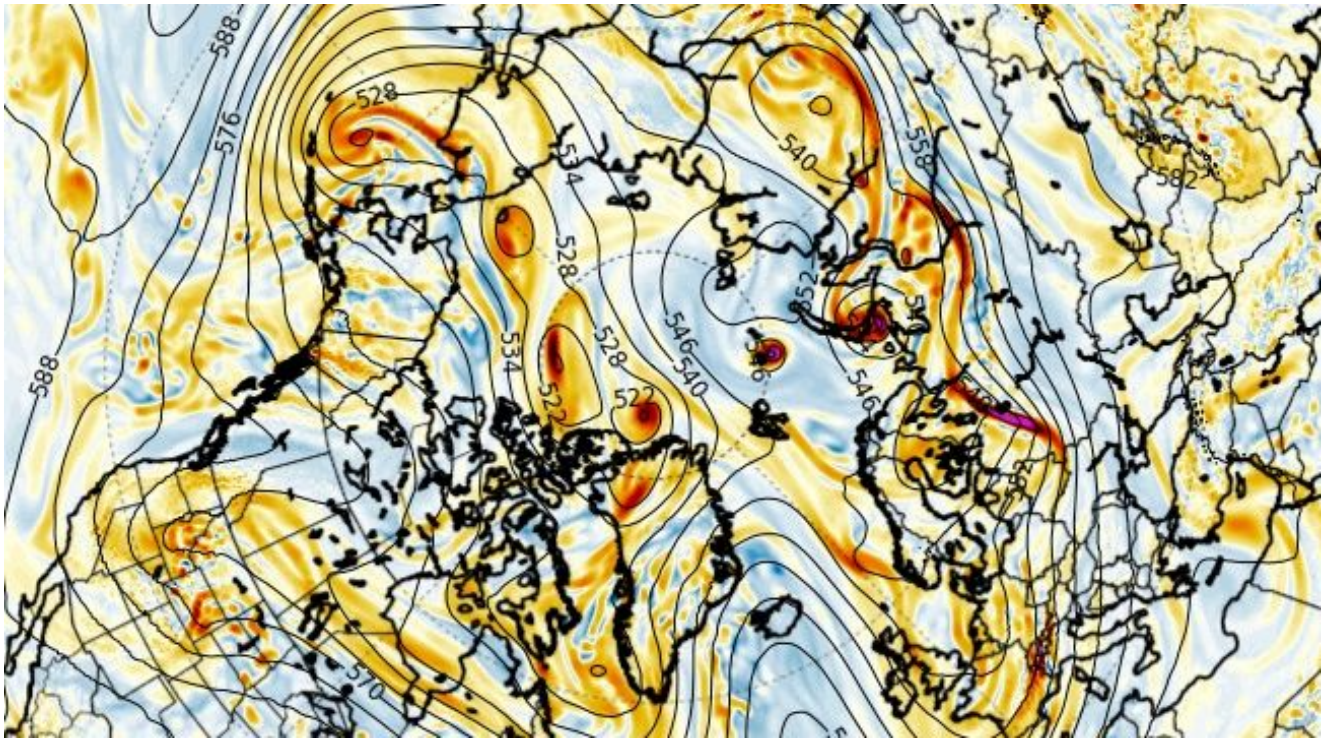


Motivation

Geophysical flows:

- development of small scale structure
- increasing resolution has a significant impact on model accuracy
- more mesh points where 'things are happening'

Why r-adaptivity?

- data structure unchanged
- interpolation between meshes unnecessary
- avoid sharp changes in resolution

Our approach

- Meshes provided using optimal transport ideas
  - The scaling and skewness of optimally transported meshes on the sphere. *Journal of Computational Physics* **375** (2018)
  - Optimal transport based mesh adaptivity on the place and sphere using finite elements. *SIAM Journal of Scientific Computing* **40** (2018)
- Solve GFD equations using compatible finite element ideas
- Coupling - this talk ...and future work!

Some choices:

- Solve for mesh and PDE separately
- Modify equations to solve in moving frame

Compatible finite element methods

$$\underbrace{\mathbb{V}_0}_{\text{Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1}_{\text{Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2}_{\text{Discontinuous}}$$

Mimetic property: $\nabla \cdot \nabla^\perp \boldsymbol{u} = 0$

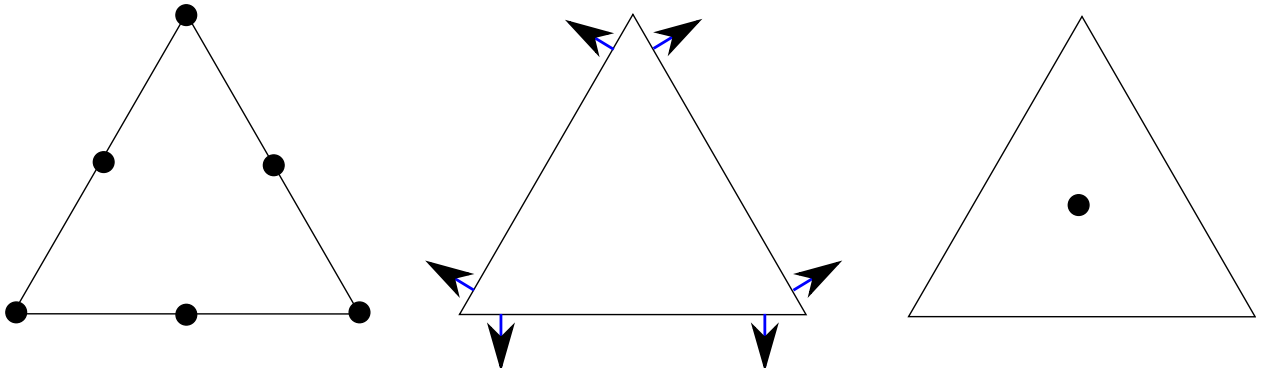
- $\nabla\cdot$ maps from $\mathbb{V}_1$ onto $\mathbb{V}_2$.
- $\nabla^\perp$ maps from $\mathbb{V}_0$ onto the kernal of $\nabla\cdot$ in $\mathbb{V}_1$.

Properties of compatible FEM discretisation of LSWE

- energy conservation
- mass conservation
- steady geostrophic states
- no spurious pressure or inertial modes

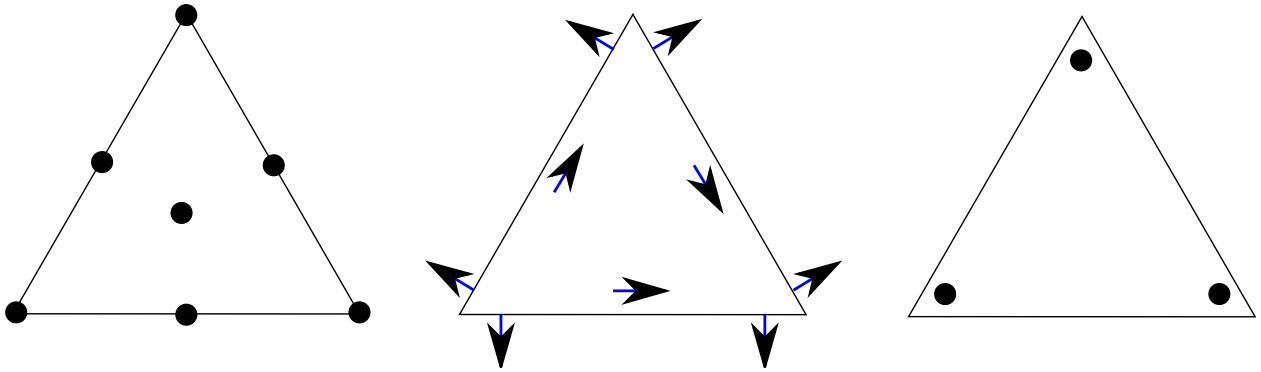
Compatible finite element methods

$$\underbrace{\mathbb{V}_0 = P2}_{\text{Quadratic, Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = BDM1}_{\text{Linear, Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P0}_{\text{Constant, Discontinuous}}$$



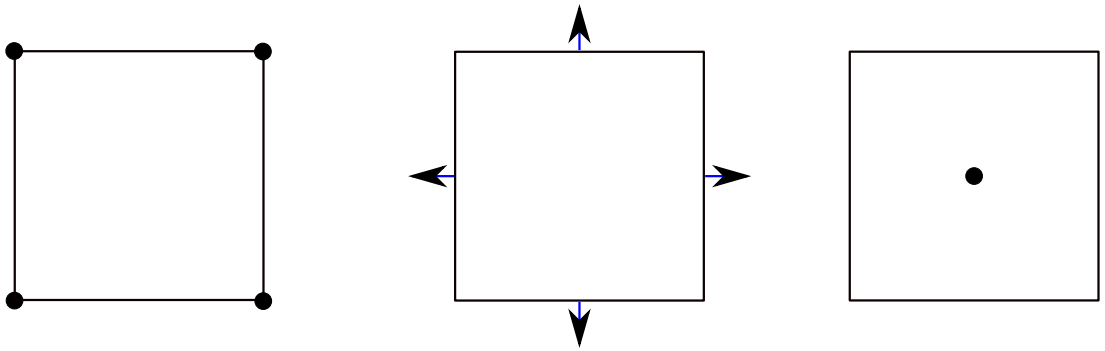
Compatible finite element methods

$$\underbrace{\mathbb{V}_0 = P2+}_{\text{Quadratic (+1 Cubic) Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = BDFM1}_{\text{Linear (+2 Quadratic) Cont. normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P1_{DG}}_{\text{Linear Discontinuous}}$$



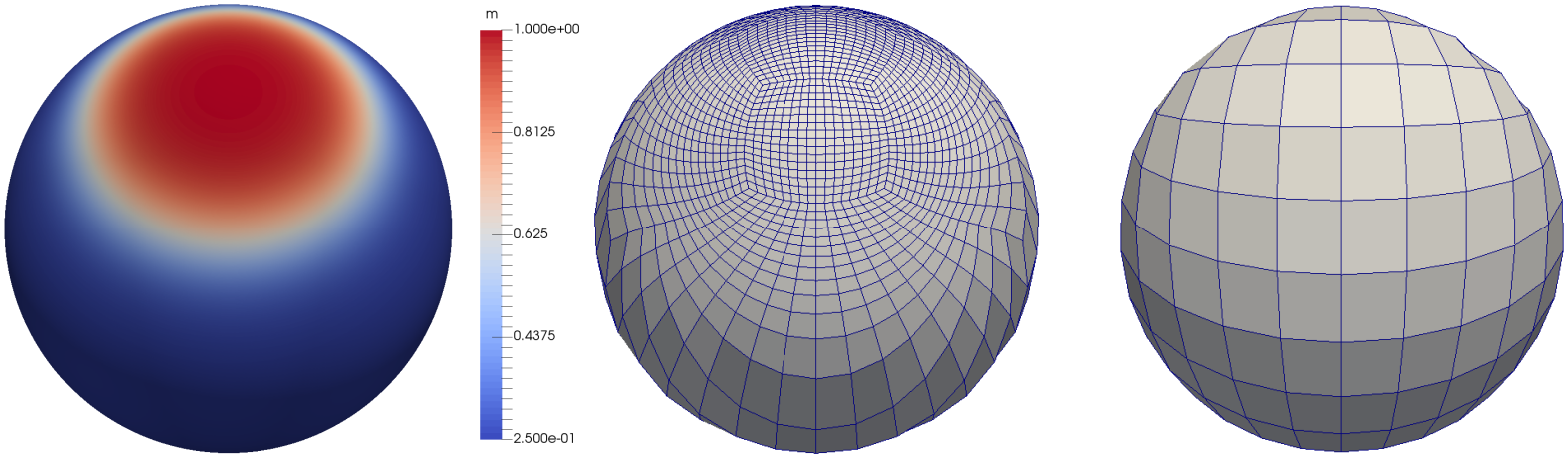
Compatible finite element methods

$$\underbrace{\mathbb{V}_0 = Q1}_{\text{Bilinear Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = RT0}_{\text{Constant/Linear, Cont. normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = Q0_{DG}}_{\text{Constant, Discontinuous}}$$



Mesh movement: optimal transport

- resolution is equidistributed with respect to a monitor function
- the monitor function is derived from the flow
- adjust monitor function to control mesh properties
- formulation guarantees that the mesh will not tangle
- we find that we require a higher-order mesh representation for convergence on the sphere



Scalar advection

Advection equation for scalar c with prescribed velocity $\bar{\boldsymbol{u}}$ and mesh velocity $\boldsymbol{v}$:

$$\frac{\partial c}{\partial t} + ((\bar{\boldsymbol{u}} - \boldsymbol{v}) \cdot \nabla)c + (\boldsymbol{v} \cdot \nabla)c = 0$$

Observe: moving the mesh with velocity $\boldsymbol{v}$, fixing DOF values, is exactly Lagrangian advection of c at velocity $\boldsymbol{v}$ for time Δt.

Therefore:

- $\frac{1}{2}\Delta t$ Eulerian advection $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on old mesh
- Move mesh, without changing values
- $\frac{1}{2}\Delta t$ Eulerian advection $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on new mesh

Scalar advection

Video:

- Nair–Lauritzen test case 4
- P_1^{DG} discretisation
- DG advection with upwinding
- SSPRK3 timestepping, no slope limiter

Solving the shallow water equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -g\nabla D - f\hat{\boldsymbol{k}} \times \boldsymbol{u}$$

$$\frac{\partial D}{\partial t} + \nabla \cdot (\boldsymbol{u}D) = 0$$

$\boldsymbol{u}^* \leftarrow \boldsymbol{u}^n + (1-\alpha)\Delta t \boldsymbol{F}^n$ ▷ explicit forcing step

for $k \leq k_{max}$ **do**

  $\boldsymbol{u}^p \leftarrow \boldsymbol{u}^* - \Delta t L^{\boldsymbol{u}}_{\bar{\boldsymbol{u}}}(\boldsymbol{u}^*)$ ▷ advection step

  $D^p \leftarrow D^* - \Delta t L^{D}_{\bar{\boldsymbol{u}}}(D^*)$

  $R[\boldsymbol{U}] \leftarrow 0$ ▷ initialise residual

  for $i \leq i_{max}$ **do**

    $\boldsymbol{u}^{n+1} \leftarrow \boldsymbol{u}^p + \alpha \Delta t \boldsymbol{F}^{n+1}$ ▷ implicit forcing step

    $R[\boldsymbol{U}] \leftarrow R[\boldsymbol{U}] - \boldsymbol{U}^{n+1}$ ▷ update residual

    Solve: $J[d\boldsymbol{U}] = -R[\boldsymbol{U}]$ ▷ elliptic solve

Solving the shallow water equations

$\boldsymbol{u}^* \leftarrow \boldsymbol{u}^n + (1-\alpha)\Delta t \boldsymbol{F}^n$ ▷ explicit forcing step

for $k \leq k_{max}$ **do**

  $\boldsymbol{u}^{p'} \leftarrow \boldsymbol{u}^* - \frac{\Delta t}{2} L^{\boldsymbol{u}}_{\bar{\boldsymbol{u}}-\boldsymbol{v}}(\boldsymbol{u}^*)$ ▷ advection step

  $D^{p'} \leftarrow D^* - \frac{\Delta t}{2} L^{D}_{\bar{\boldsymbol{u}}-\boldsymbol{v}}(D^*)$

  $\boldsymbol{x} \leftarrow \boldsymbol{x}_{new}$ with $\boldsymbol{v} = (\boldsymbol{x}_{new} - \boldsymbol{x}_{old})/\Delta t$ ▷ move mesh

  $\boldsymbol{u}^p \leftarrow \tilde{\boldsymbol{u}}^{p'} - \frac{\Delta t}{2} L^{\boldsymbol{u}}_{\bar{\boldsymbol{u}}-\boldsymbol{v}}(\tilde{\boldsymbol{u}}^{p'})$ ▷ advection step

  $D^p \leftarrow \tilde{D}^{p'} - \frac{\Delta t}{2} L^{D}_{\bar{\boldsymbol{u}}-\boldsymbol{v}}(\tilde{D}^{p'})$

  $R[\boldsymbol{U}] \leftarrow 0$ ▷ initialise residual

  for $i \leq i_{max}$ **do**

    $\boldsymbol{u}^{n+1} \leftarrow \boldsymbol{u}^p + \alpha \Delta t \boldsymbol{F}^{n+1}$ ▷ implicit forcing step

    $R[\boldsymbol{U}] \leftarrow R[\boldsymbol{U}] - \boldsymbol{U}^{n+1}$ ▷ update residual

    Solve: $J[d\boldsymbol{U}] = -R[\boldsymbol{U}]$ ▷ elliptic solve

Scalar continuity equation

Continuity equation for scalar D with prescribed velocity $\bar{\boldsymbol{u}}$ and mesh velocity $\boldsymbol{v}$:

$$\frac{\partial D}{\partial t} + \nabla \cdot (D\bar{\boldsymbol{u}}) = 0$$

Idea: split into

$$\frac{\partial D}{\partial t} = -\nabla \cdot (D(\bar{\boldsymbol{u}} - \boldsymbol{v})) - \nabla \cdot (D\boldsymbol{v})$$

- $\frac{1}{2}\Delta t$ Eulerian continuity $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on old mesh
- Move mesh, adjusting values of D [next slide]
- $\frac{1}{2}\Delta t$ Eulerian continuity $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on new mesh

Scalar continuity equation

How to solve

$$\frac{\partial D}{\partial t} = -\nabla \cdot (D\boldsymbol{v})$$

while moving the mesh?

Observe: for a standard test-function ϕ, we have $\phi_t + (\boldsymbol{v} \cdot \nabla)\phi = 0$.

Consider

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\Omega(t)} \phi D \,\mathrm{d}x &= \int_{\Omega(t)} \phi_t D \,\mathrm{d}x + \int_{\Omega(t)} \phi D_t \,\mathrm{d}x + \int_{\delta\Omega(t)} \phi D \boldsymbol{v} \cdot \boldsymbol{n} \,\mathrm{d}x \\
&= -\int_{\Omega(t)} D(\boldsymbol{v} \cdot \nabla)\phi \,\mathrm{d}x - \int_{\Omega(t)} \phi \nabla \cdot (\boldsymbol{v} D) \,\mathrm{d}x + \int_{\Omega(t)} \nabla \cdot (\phi D \boldsymbol{v}) \,\mathrm{d}x \\
&= 0
\end{aligned}$$

Scalar continuity equation

So, for the continuity equation

$$\frac{\partial D}{\partial t} + \nabla \cdot (D\bar{\boldsymbol{u}}) = 0,$$

we do

- $\frac{1}{2}\Delta t$ Eulerian continuity $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on old mesh
- Move mesh, and adjust D according to

$$\int_{\Omega(t^{n+1})} \phi D \, \mathrm{d}x = \int_{\Omega(t^{n})} \phi D \, \mathrm{d}x \qquad \forall \phi \in V_D$$

- $\frac{1}{2}\Delta t$ Eulerian continuity $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on new mesh

Velocity advection

Recall the shallow water momentum equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\bar{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -g\nabla D$$

Write as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{(\nabla \times \boldsymbol{u}) \times \bar{\boldsymbol{u}} + \nabla(\boldsymbol{u} \cdot \bar{\boldsymbol{u}})}_{\text{advection}} = \underbrace{-f\hat{\boldsymbol{k}} \times \boldsymbol{u} - g\nabla D + \frac{1}{2}\nabla|\boldsymbol{u}|^2}_{\text{forcing}}$$

Velocity advection

Velocity advection step:

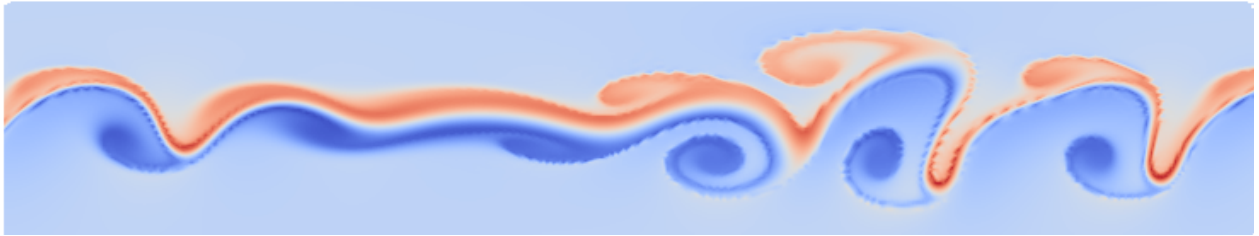
$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \bar{\boldsymbol{u}} + \nabla(\boldsymbol{u} \cdot \bar{\boldsymbol{u}}) = 0$$

Split into

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\left[(\nabla \times \boldsymbol{u}) \times (\bar{\boldsymbol{u}} - \boldsymbol{v}) + \nabla(\boldsymbol{u} \cdot (\bar{\boldsymbol{u}} - \boldsymbol{v}))\right] - \left[(\nabla \times \boldsymbol{u}) \times \boldsymbol{v} + \nabla(\boldsymbol{u} \cdot \boldsymbol{v})\right]$$

- $\frac{1}{2}\Delta t$ circulation-form advection $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on old mesh
- Move mesh, adjusting DOF values of $\boldsymbol{u}$ (similar to continuity equation)
- $\frac{1}{2}\Delta t$ circulation-form advection $(\bar{\boldsymbol{u}} - \boldsymbol{v})$ on new mesh

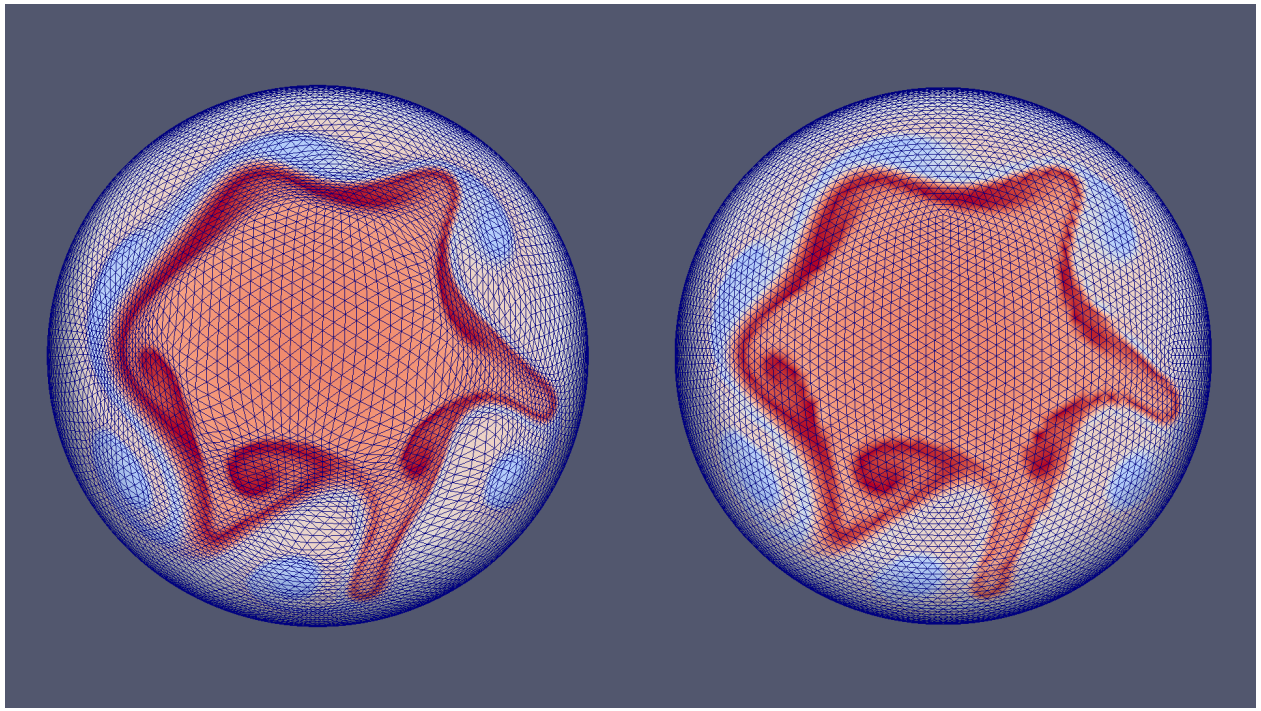
Does it work?



Testcase: barotropic jet (Galewsky)

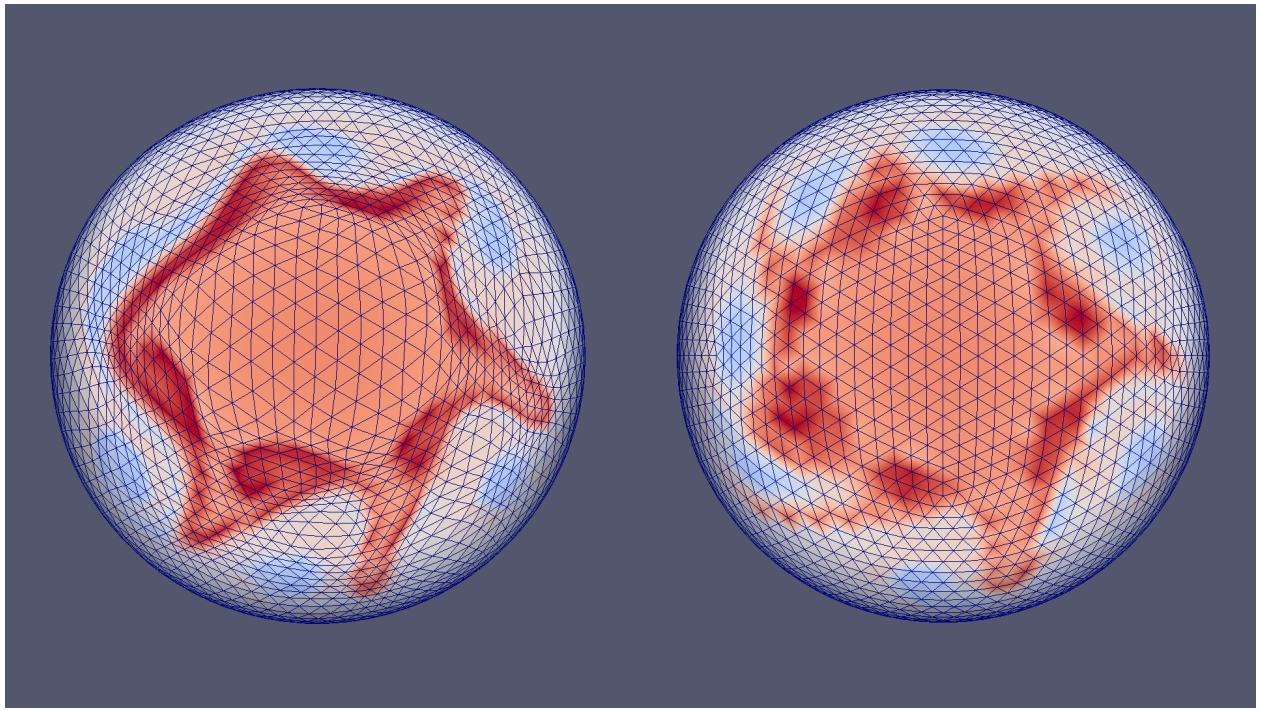
- correct wavenumber
- quiscent section
- grid imprinting has not triggered early development of instability

Does it work?



Moving mesh vs static mesh, reference level 5.

Does it work?



Moving mesh vs static mesh, reference level 4.

Summary

We can produce optimally transported meshes on the sphere and solve the shallow water equations on them, using a compatible finite element scheme.

Outlook

- Investigate:
  - parameters controlling mesh quality
  - different monitor functions
- More rigorous testing to check for:
  - correct wave propagation speeds
  - possible spurious effects
- Computation times