Shallow-Atmosphere Form for Deep-Atmosphere Dynamics Equations

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Introduction

- NCEP/SWPC has space weather models which need atmosphere model top at around 600km to support winds etc, as whole atmosphere modeling (WAM).

- Deep-Atmosphere dynamics (DAD) can improve accuracy not only for WAM but also for weather/climate modeling beyond nonhydrostatic.

- To look for an easy implementation of DAD into existed GFS model, such as FV3, instead of developing a new deep-atmosphere dynamics model, is the purpose of this presentation.
Opr GFS vs WAM

64 layers
GFS hybrid vertical grid
(every 2nd level)

150 layers
WAM hybrid vertical grid
(every 3rd level)

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Earth radius $a=6371$ km, $r=a+z$

**Shallow Atmosphere**
Assumption $r=a$

Op GFS $z \sim 60$ km
1% of $a$

**Deep Atmosphere**
No assumption
$r=a+z$
WAM $z \sim 600$ km, 10% of $a$

Most of NWP models are shallow-atmosphere dynamics
Deep-Atmosphere Dynamics (DAD)

While \( r = a + z \)

Equations tend to be more accurate but complicated

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + u \frac{\partial A}{r \cos \phi \partial \lambda} + v \frac{\partial A}{r \partial \phi} + w \frac{\partial A}{\partial r} \quad \text{where} \quad u = r \cos \phi \frac{d \lambda}{dt} \quad ; \quad v = r \frac{d \phi}{dt} \quad ; \quad w = \frac{dr}{dt}
\]

\[
\begin{align*}
\frac{du}{dt} - \frac{uv \tan \phi}{r} + \frac{uw}{r} & - (2 \Omega \sin \phi) v + \frac{(2 \Omega \cos \phi) w}{\rho r \cos \phi} + \frac{1}{\rho} \frac{\partial p}{\partial \lambda} = F_u \\
\frac{dv}{dt} + \frac{u^2 \tan \phi}{r} + \frac{vw}{r} & + (2 \Omega \sin \phi) u + \frac{1}{\rho} \frac{\partial p}{r \partial \phi} = F_v \\
\frac{dw}{dt} - \frac{u^2 + v^2}{r} & - (2 \Omega \cos \phi) u + \frac{1}{\rho} \frac{\partial p}{\partial r} + g = F_w
\end{align*}
\]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{r \cos \phi \partial \lambda} + \frac{\partial \rho v \cos \phi}{r \cos \phi \partial \phi} + \frac{\partial \rho r^2 w}{r^2 \partial r} = F_\rho
\]
let’s start from deep atmosphere continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial \lambda} r \cos \phi \frac{\partial \phi}{\partial \rho} + \frac{\partial \rho \cos \phi}{\partial r} r \cos \phi \frac{\partial \phi}{\partial \rho} + \frac{\partial \rho r^2 w}{\partial r} = 0$$

let $u^* = u \cos \phi$; $v^* = v \cos \phi$; $m = \frac{1}{\cos \phi}$; $\frac{\partial}{\partial \phi} = \cos \phi \frac{\partial}{\partial \phi} = \frac{1}{m} \frac{\partial}{\partial \phi}$

We can derive it into generalized vertical coordinate, we have (Juang 2014, ON#477, Appendix A)

$$\frac{\partial \rho^*}{\partial t} + m^2 \frac{\partial \rho^* u^*}{\partial \lambda} r + m^2 \frac{\partial \rho^* v^*}{\partial \phi} r + \frac{\partial \rho^* \xi}{\partial \xi} = 0$$

where $\rho^* = \rho \frac{r^2}{a^2} \frac{\partial r}{\partial \xi}$
So the DAD equation in a generalized vertical coordinate

\[
\frac{du^*}{dt} + \frac{u^* w}{r} - f_s v^* + f_c^* w + \frac{1}{\rho} \left( \frac{\partial p}{r \partial \lambda} - \frac{\partial p}{\partial \xi} \frac{\partial \xi}{\partial r} \frac{\partial r}{r \partial \lambda} \right) = F_u
\]

\[
\frac{dv^*}{dt} + \frac{v^* w}{r} + f_s u^* + m^2 \frac{s^*}{r} \sin \phi + \frac{1}{\rho} \left( \frac{\partial p}{r \partial \phi} - \frac{\partial p}{\partial \xi} \frac{\partial \xi}{\partial r} \frac{\partial r}{r \partial \phi} \right) = F_v
\]

\[
\frac{dw}{dt} - m^2 \frac{s^*}{r} - m^2 f_c^* u^* + \frac{1}{\rho} \frac{\partial p}{\partial \xi} \frac{\partial \xi}{\partial r} + g = F_w
\]

\[
\frac{\partial \rho^*}{\partial t} + m^2 \frac{u^*}{r \partial \lambda} + m^2 \frac{v^*}{r \partial \phi} + \frac{\partial \rho^*}{\partial \xi} = F_\rho
\]

where

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + m^2 \frac{u^*}{r \partial \lambda} + m^2 \frac{v^*}{r \partial \phi} + \frac{\partial A}{\partial \xi}
\]

\[
s^* = u^* + v^2
\]

\[
f_s = 2\Omega \sin \phi ; \quad f_c = 2\Omega \cos^2 \phi
\]

and

\[
g = g_0 \frac{a^2}{r^2}
\]
Follow the same definition traditionally

\[
\text{let } \frac{\partial \tilde{p}}{\partial \zeta} = -\rho^* g_0
\]

so

\[
\frac{\partial \frac{\partial \tilde{p}}{\partial \zeta}}{\partial t} + m^2 \frac{\partial \frac{\partial \tilde{p}}{\partial \zeta} \tilde{u}}{\partial \lambda} + m^2 \frac{\partial \frac{\partial \tilde{p}}{\partial \zeta} \tilde{v}}{\partial \varphi} + \frac{\partial \frac{\partial \tilde{p}}{\partial \zeta} \cdot \zeta}{\partial \zeta} = 0
\]

let

\[
\tilde{u} = \frac{a}{r} \frac{\partial \tilde{u}^*}{\partial \varphi} = \frac{a}{r} u \cos \phi; \quad \tilde{v} = \frac{a}{r} \frac{\partial \tilde{v}^*}{\partial \varphi} = \frac{a}{r} v \cos \phi
\]

The above continuity becomes a same form as Shallow-atmosphere continuity eqn

\[
\frac{\partial \frac{\partial \tilde{p}}{\partial \zeta}}{\partial t} + m^2 \frac{\partial \frac{\partial \tilde{p}}{\partial \zeta} \tilde{u}}{a \partial \lambda} + m^2 \frac{\partial \frac{\partial \tilde{p}}{\partial \zeta} \tilde{v}}{a \partial \varphi} + \frac{\partial \frac{\partial \tilde{p}}{\partial \zeta} \cdot \zeta}{\partial \zeta} = 0
\]
Next, let see advection w.r.t. scaled horizontal wind

The advection in generalized coordinate, can be easily to use scaled wind as

$$\frac{d()}{dt} = \frac{\partial()}{\partial t} + m^2 u^* \frac{\partial()}{r \partial \lambda} + m^2 v^* \frac{\partial()}{r \partial \phi} + \zeta \frac{\partial()}{\partial \zeta}$$

$$= \frac{\partial()}{\partial t} + m^2 \tilde{u} \frac{\partial()}{a \partial \lambda} + m^2 \tilde{v} \frac{\partial()}{a \partial \phi} + \zeta \frac{\partial \tilde{p}}{\partial \zeta} \frac{\partial()}{\partial \tilde{p}}$$

At this moment, continuity equation and advection terms are the same form as shallow-atmosphere equations with horizontal winds are represented by different definition, can be called as smily winds due to the symbol we give.

$$\tilde{u} = a \frac{u \cos \phi}{r} ; \quad \tilde{v} = a \frac{v \cos \phi}{r}$$
Since horizontal wind is scaled, how about vertical wind

We can start from definition of

\[
\frac{\partial \tilde{p}}{\partial \zeta} = -\rho^* g_0 = -\rho \frac{r^2}{a^2} \frac{\partial r}{\partial \zeta} = g_0 = -\rho \frac{\partial \tilde{\Phi}}{\partial \zeta}
\]

Same form as S.A.

so

\[
\frac{\partial \tilde{\Phi}}{\partial \zeta} = \frac{r^2}{a^2} \frac{\partial r}{\partial \zeta} = \frac{g_0}{3a^2} \frac{\partial r^3}{\partial \zeta}
\]

The general solution can be

thus the vertical motion w.r.t new geopotential height

\[
\frac{d \tilde{\Phi}}{dt} = \frac{g_0 r^2}{a^2} \frac{dr}{dt} = g_0 \frac{r^2}{a^2} \tilde{w}
\]

In shallowness

\[
\Phi = g_0 \left(r - a\right) \quad \text{and} \quad \frac{d \Phi}{dt} = g_0 \frac{dr}{dt} = g_0 \tilde{w}
\]

In analogy, we let

\[
\frac{d \tilde{\Phi}}{dt} = g_0 \tilde{w} \quad \text{so we get} \quad \tilde{w} = \frac{r^2}{a^2} w
\]
With 3D scaled winds as

\[
\tilde{u} = \frac{a}{r} u^* ; \quad \tilde{v} = \frac{a}{r} v^* ; \quad \tilde{w} = \frac{r^2}{a^2} w
\]

Their time derivatives are

\[
\frac{d\tilde{u}}{dt} = \frac{a}{r} \frac{du^*}{dt} - \frac{au^*}{r^2} \frac{dr}{dt} = \frac{1}{\epsilon} \frac{du^*}{dt} - \tilde{u}\tilde{w} \\
\frac{d\tilde{v}}{dt} = \frac{a}{r} \frac{dv^*}{dt} - \frac{av^*}{r^2} \frac{dr}{dt} = \frac{1}{\epsilon} \frac{dv^*}{dt} - \tilde{v}\tilde{w} \\
\frac{d\tilde{w}}{dt} = \frac{r^2}{a^2} \frac{dw}{dt} + \frac{2r}{a^2} w^2 = \epsilon^2 \frac{dw}{dt} + \frac{2\tilde{w}^2}{a\epsilon^3}
\]

where

\[
\epsilon = \frac{r}{a}
\]
The DAD scaled momentum eqns in shallowness form become

\[
\frac{du}{dt} = -2 \frac{\bar{w}}{\varepsilon^3 a} \delta - f_c \frac{\bar{w}}{\varepsilon^3} \delta + f_s \bar{v} - \frac{1}{\varepsilon^2} \left( \frac{1}{\rho} \frac{\partial p}{a \partial \lambda} + \frac{\partial p}{\bar{p}} \frac{\partial \Phi}{a \partial \lambda} \right)
\]

\[
\frac{d\bar{v}}{dt} = -2 \frac{\bar{v}}{\varepsilon^3 a} \delta - f_s \bar{u} - m^2 \frac{\bar{s}}{a} \sin \phi - \frac{1}{\varepsilon^2} \left( \frac{1}{\rho} \frac{\partial p}{a \partial \varphi} + \frac{\partial p}{\bar{p}} \frac{\partial \Phi}{a \partial \varphi} \right)
\]

\[
\frac{d\bar{w}}{dt} = 2 \frac{\bar{w}}{\varepsilon^3 a} \delta + m^2 \varepsilon^3 \frac{\bar{s}}{a} \delta + m^2 \varepsilon^3 f_c^* \bar{v} \delta + g_o \left( \frac{\partial p}{\bar{p}} \varepsilon^4 - 1 \right)
\]

All terms with \( \varepsilon = r / a \) and \( \delta = 1 \) are additions to shallow atmosphere dynamics while \( \varepsilon = 1 \) and \( \delta = 0 \) All eqns are back to shallowness

And it is possible to define hydrostatic pressure as \( \frac{\partial p}{\bar{p}} \varepsilon^4 - 1 = 0 \)
Apply vertical Lagrangian, we have

\[ D_L \ddot{u} + m^2 \vec{V} \cdot \nabla \ddot{u} = -\frac{\ddot{w}}{\varepsilon^3} \left( \frac{2 \ddot{u}}{a} - f_c^* \right) \delta + f_s \ddot{v} - \frac{1}{\varepsilon^2} \left( \frac{1}{\rho} \frac{\partial p}{a \partial \lambda} + \frac{\partial p}{\partial \rho} \frac{\partial \Phi}{a \partial \lambda} \right) \]

\[ D_L \ddot{v} + m^2 \vec{V} \cdot \nabla \ddot{v} = -\frac{\ddot{w}}{\varepsilon^3} \frac{2 \ddot{v}}{a} \delta - f_s \ddot{u} - m^2 \frac{\dot{s}^2}{a} \sin \phi - \frac{1}{\varepsilon^2} \left( \frac{1}{\rho} \frac{\partial p}{a \partial \varphi} + \frac{\partial p}{\partial \rho} \frac{\partial \Phi}{a \partial \varphi} \right) \]

\[ D_L \Delta \tilde{p} \ddot{w} + m^2 \nabla \cdot (\vec{V} \Delta \tilde{p} \ddot{w}) = \left( 2 \frac{\ddot{w}^2}{\varepsilon^3 a} + m^2 \varepsilon^3 \frac{\dot{s}^2}{a} + m^2 \varepsilon^3 f_c^* \ddot{u} \right) \Delta \tilde{p} \delta + g_0 \Delta \tilde{p} \frac{\partial p'}{\partial \tilde{p}} \varepsilon^4 \]

Add two previous obtained

\[ D_L \Delta \tilde{p} + m^2 \nabla \cdot (\vec{V} \Delta \tilde{p}) = 0 \]

\[ D_L \Delta \tilde{p} \theta_v + m^2 \nabla \cdot (\vec{V} \Delta \tilde{p} \theta_v) = 0 \]

We can see some terms are added to momentum eqns with facts of epsilon and delta, which won’t influence numerical techniques inside FV3 core
From continuity equation and 3D momentum equations, we know DAD equations are in the shallow-atmosphere form without any approximation. It can be seen as real coordinates projecting into smily space:

\[ \tilde{u}; ~ \tilde{v}; ~ \tilde{w}; ~ \tilde{p}; ~ \tilde{\Phi} \]

Including some extra terms in the momentum equation and projection coefficient as epsilon. And thermodynamics variables, not related to space, are unchanged.

\[ p = \rho RT \]

\[ \frac{dC_p T}{dt} - \frac{RT}{p} \frac{dp}{dt} = F_T \]

except the advection terms should use smily wind in S. A. form.
FV3WAM Initial Condition
at \( \text{lat}=0 \) and \( \text{lon}=180 \)
cold start with standard T, P, O, and O2
other fields, wind, and q are gradually
decreasing to zero at top
top pressure is about 1.E-7 Pa
close to 500~600 km
Ideal gas law with multi gas constituents should be considered

Multi gas constituents can exist under a common temperature

\[ p_i = \rho_i R_i T \quad p = \rho RT \]

\[ p = \sum_{i=0}^{n} p_i \quad \rho = \sum_{i=0}^{n} \rho_i \quad \text{Base air with } n \text{ gas tracers} \]

\[ p = \sum_{i=0}^{n} p_i = \left( \sum_{i=0}^{n} \rho_i R_i \right) T = \rho \left( \sum_{i=0}^{n} \frac{\rho_i R_i}{\rho} \right) T = \rho RT \]

\[ q_i = \frac{\rho_i}{\rho} \quad R = \sum_{i=0}^{n} \frac{\rho_i}{\rho} R_i = \sum_{i=0}^{n} q_i R_i \]

\[ Cp = \sum_{i=0}^{n} q_i C_{pi} \quad Cv = \sum_{i=0}^{n} q_i C_{vi} \]
How to implement?

From IO or physics components, we have \( u, v, w, T, p, qi, \) and \( Ps \) or \( dz \)

Use \( Ri, Cpi, \) and \( qi \) to obtain \( R, Cp, \pi, T_v, \theta_v \)

Use \( Ps \) to get \( dp, \) and density to get \( dphi \) then get \( r, \) \( \frac{\partial \rho}{\partial \Phi} = -\rho \)

compute all scaled values \( \tilde{u}, \tilde{v}, \tilde{w} \)

Pass to dynamics to get prognostic values at next time step

Use new scaled prognostic values and others \( \tilde{u}, \tilde{v}, \tilde{w}, \delta \Phi, q_i, \theta_v \)

Convert to obtain \( R, Cp, \pi, T, \theta \)

Get \( r \) from \( dz \)

Transform scaled wind into real wind

Pass to physics or write components
Conclusion

• DAD equation set can be derived into shallow-atmosphere form with some extra terms and scale coefficients, without any approximation, thus implementation is straightforward.

• Only height related prognostic variables are scaled from real space into scaled space (smily space), thermodynamics variables are not scaled.

• The variable conversions for rescaling between DAD and IO and model physics are required. Careful rescaling like vertical remapping may considers conservation.

• Multi-gases option has been implemented into FV3 for WAM, and the adding DAD is in progress.

• Details can be found in NCEP ON#477 and 488.