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Discrete Monge Ampere approaches to solving the semigeostrophic equations

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Tuesday 30th April 2019

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Abstract

In this talk we revisit Mike Cullen's geometric algorithm for solving the semigeostrophic equations using optimal transport, using a new iterative algorithm based on power/Laguerre diagrams.

We'll specialise to the vertical slice Eady model.

All of the material on OT for SG can be found in:
Cullen, M.J.P. A mathematical theory of large-scale atmosphere/ocean flow. Imperial College Press, 2006.

Deriving vertical slice Eady Boussinesq model

(3D incompressible, constant f, infinite channel with $\boldsymbol{u} \cdot \boldsymbol{n} = w = 0$ on top/bottom)

$$\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{\partial p}{\partial x}, \\
\frac{Dv}{Dt} + fu &= -\frac{\partial p}{\partial y}, \\
\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \frac{g\theta}{\theta_0}, \\
\frac{D\theta}{Dt} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}$$

Vertical slice solutions

$$
\begin{aligned}
u(x,z,t),\ v(x,z,t),\ w(x,z,t),\\
\theta = \theta'(x,z,t) + sy,\ p = \bar{p}(y,z) + \phi(x,z,t).
\end{aligned}
$$

Hydrostatic balance (plus symmetry BCs):

$$
\frac{\partial \bar{p}}{\partial z} = \frac{gsy}{\theta_0} \implies \bar{p} = \frac{gsy}{\theta_0}\left(z - \frac{H}{2}\right).
$$

Vertical slice Eady Boussinesq model

$$
\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{\partial \phi}{\partial x}, \\
\frac{Dv}{Dt} + fu &= -\frac{gs}{\theta_0}\left(z - \frac{H}{2}\right), \\
\frac{Dw}{Dt} &= -\frac{\partial \phi}{\partial z} + \frac{g\theta}{\theta_0}, \\
\frac{D\theta'}{Dt} + sv &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\
\frac{D}{Dt} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}.
\end{aligned}
$$

Aside

These equations have some interesting variational structure that explains their conservation properties (energy, potential vorticity, mass). See:

Cotter, C. J., and D. D. Holm. "A variational formulation of vertical slice models." Proc. R. Soc. A 469.2155 (2013): 20120678.

Cotter, C. J., and M. J. P. Cullen. "Particle relabelling symmetries and Noether's theorem for vertical slice models." arXiv preprint arXiv:1808.05486 (2018).

Semi-geostrophic (and hydrostatic) approximation

$$
\begin{aligned}
\cancel{\frac{Du}{Dt}} - fv &= -\frac{\partial \phi}{\partial x}, \\
\frac{Dv}{Dt} + fu &= -\frac{gs}{\theta_0}\left(z - \frac{H}{2}\right), \\
\cancel{\frac{Dw}{Dt}} &= -\frac{\partial \phi}{\partial z} + \frac{g\theta}{\theta_0}, \\
\frac{D\theta'}{Dt} + sv &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\
\frac{D}{Dt} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}.
\end{aligned}
$$

Geostrophic coordinates

Hoskins + Cullen/Shutts coordinate change:

$$X = x + \frac{v}{f}, \quad Z = \frac{g\theta'}{f^2\theta_0},$$

$$\frac{DX}{Dt} = -\frac{gs}{f\theta_0}\left(z - \frac{H}{2}\right),$$

$$\frac{DZ}{Dt} = \frac{gs}{f\theta_0}(x - X),$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

Geostrophic coordinates 2

Now take opposite viewpoint, $x(X,Z,t)$, $z(X,Z,t)$.

$$\dot{X} = U := -\frac{gs}{f\theta_0}\left(z - \frac{H}{2}\right),$$

$$\dot{Z} = W := \frac{gs}{f\theta_0}(x - X),$$

$$\frac{\partial \sigma}{\partial t} + \nabla_{X,Z}\cdot(\sigma \boldsymbol{U}) = 0,$$

$$\sigma = \det\left(\frac{\partial(x,z)}{\partial(X,Z)}\right),$$

$$\boldsymbol{U} = (U, W).$$

Optimal transport formulation

Theorem

The map $x, z \mapsto X, Z = \Phi(x, z)$, $\Phi : \Omega \to \mathbb{R}^2$, *minimises the squared distance*

$$\int (X - x)^2 + (Z - z)^2 \, \mathrm{d}x \, \mathrm{d}z,$$

subject to the constraint

$$\int_{\Phi(A)} \sigma \, \mathrm{d}X \, \mathrm{d}Z = \int_A \mathrm{d}x \, \mathrm{d}z,$$

for all measurable sets A *(i.e.,* $\sigma \, \mathrm{d}X \, \mathrm{d}Z = \Phi^*(\mathrm{d}x \, \mathrm{d}z)$*).*

Semidiscrete optimal transport

$$\sigma = \sum_{i=1}^{N} w_i \delta(\boldsymbol{X} - \boldsymbol{X}_i(t)),$$

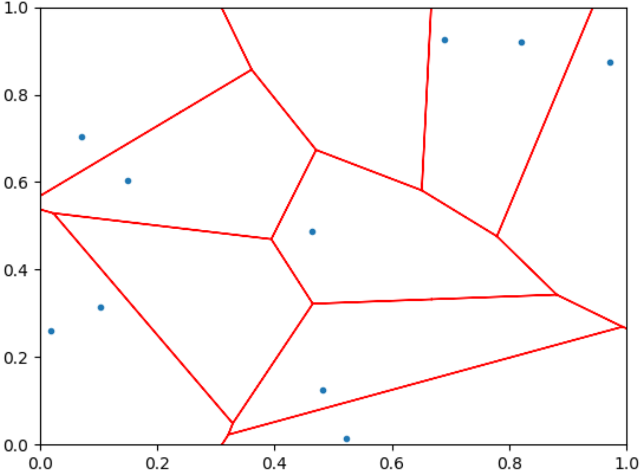
is a weak solution to the conservation equation for σ.

Semidiscrete OT problem

Find $\Phi : \Omega \to \{\boldsymbol{X}_i\}_{i=1}^{N}$ that minimises

$$\sum_{i=1}^{N} \int_{A_i} (X_i - x)^2 + (Z_i - z)^2 \, \mathrm{d}x \, \mathrm{d}z, \quad A_i = \Phi^{-1}(\boldsymbol{X}_i)$$

$$\text{such that } \int_{A_i} \mathrm{d}x = \nu_i, \quad i = 1, \ldots, N.$$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Semi-discrete semi-geostrophic scheme

Projection onto piecewise constants

x, z are multivalued for each X_i, Z_i. Replace x, z with their averages over A_i.

Set of ODES:

$$\dot{X}_i = -\frac{gs}{f\theta_0}\left(\frac{1}{|A_i|}\int_{A_i} z \,\mathrm{d}x\,\mathrm{d}z - \frac{H}{2}\right),$$
$$\dot{Z}_i = \frac{gs}{f\theta_0}\left(\frac{1}{|A_i|}\int_{A_i} x \,\mathrm{d}x\,\mathrm{d}z - X_i\right).$$

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Why is this useful?

- Mathematics of optimal transport provides explanation for front formation in the SG equations.
- Semi-discrete SG solution can resolve fronts without numerical dissipation, providing a reference for standard numerical codes by taking sequence of solutions in SG parameter limit (asymptotic limit solutions):

Cullen, Mike. "Modelling atmospheric flows." Acta Numerica (2007)
Visram, A. R., C. J. Cotter, and M. J. P. Cullen. "A framework for evaluating model error using asymptotic convergence in the Eady model." QJRMS (2014)
Yamazaki, Hiroe, et al. "Vertical slice modelling of nonlinear Eady waves using a compatible finite element method." JCP (2017)

Timestepping

Forward Euler scheme

1. Solve semidiscrete OT problem to get $\{A_i^n\}_{i=1}^N$ corresponding to $\{\boldsymbol{X}_i^n\}_{i=1}^N$.
2. $$X_i^{n+1} = X_i^n - \Delta t \frac{gs}{f\theta_0}\left(\frac{1}{|A_i^n|}\int_{A_i^n} z \, \mathrm{d}x \, \mathrm{d}z - \frac{H}{2}\right),$$
$$Z_i^{n+1} = Z_i^n + \Delta t \frac{gs}{f\theta_0}\left(\frac{1}{|A_i^n|}\int_{A_i^n} x \, \mathrm{d}x \, \mathrm{d}z - X_i^n\right).$$

Easily extended to explicit RK schemes etc.

Solving the semidiscrete OT problem

- Cullen/Shutts and Cullen/Roulstone used a direct solver, as described in Cullen (2006).
- Here we use a damped Newton algorithm of Mèrigot and colleagues based on characterisation of solutions to semidiscrete OT problems as Laguerre/power diagrams.

Laguerre diagrams

Voronoi diagram

For a set of points $\{\boldsymbol{X}_i\}_{i=1}^N$, the Voronoi cell corresponding to $\boldsymbol{X}_j$ is

$$V_j = \left\{ x \in \Omega : |x - X_j|^2 \leq |x - X_i|^2 \quad \forall i \neq j \right\}.$$

Laguerre diagram

For a set of points $\{\boldsymbol{X}_i\}_{i=1}^N$ and scalar parameters $\{\psi_i\}_{i=1}^N$, the Laguerre cell corresponding to $\boldsymbol{X}_j$ is

$$V_j = \left\{ x \in \Omega : |x - X_j|^2 + \psi_j \leq |x - X_i|^2 + \psi_i \quad \forall i \neq j \right\}.$$

Laguerre diagrams are optimal

Theorem

Let $\{\boldsymbol{X}_i\}_{i=1}^N$, $\{\psi_i\}_{i=1}^N$ be a set of points and scalar parameters with corresponding Laguerre diagram $\{L_i\}_{i=1}^N$. Then $A_i = L_i$ solves the semi-discrete OT problem with

$$\nu_i = |A_i|.$$

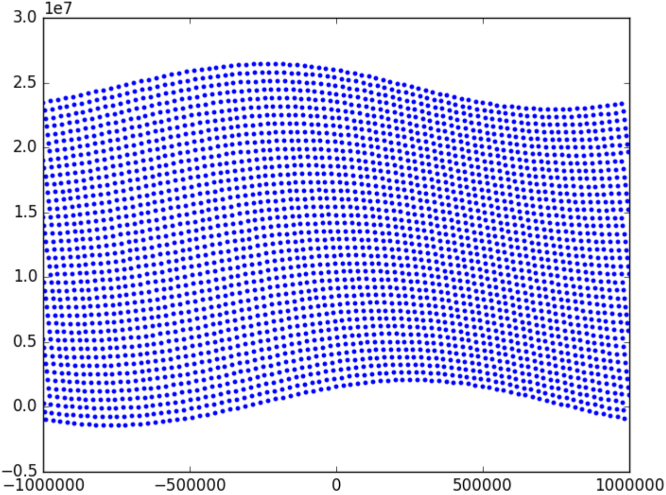
- Given $\{\psi_i\})_{i=1}^N$, Laguerre diagram produces the $\{\nu_i\}_{i=1}^N$ it is optimal for, $G_i(\boldsymbol{\psi}) = \nu_i$, $i = 1, \ldots, N$.
- We want $\nu_i = w_i$, $i = 1, \ldots, N$, so need to solve inverse problem $G_i(\boldsymbol{\psi}) = w_i$, $i = 1, \ldots, N$, to get the correct optimal Laguerre diagram.

Damped Newton algorithm

Kitagawa/Mèrigot/Thibert:

- provided an explicit formula for the Jacobian of $\boldsymbol{G}(\psi)$,
- showed that it is sparse and local (and hence parallelisable),
- showed that it is invertible provided that $\nu_i > 0$, $i = 1, \ldots, N$,
- proved that the resulting damped Newton algorithm converges.

Mèrigot provides an open source C++ implementation on Github with Python wrappers!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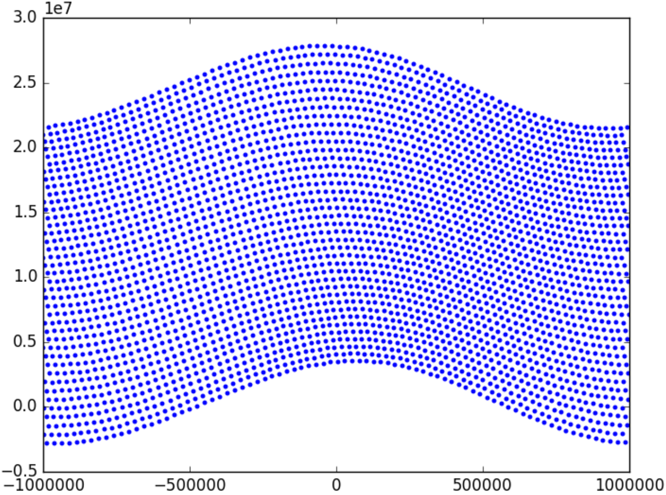
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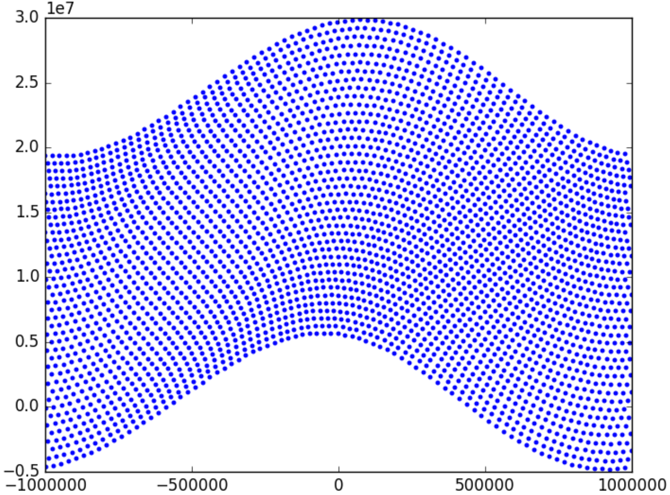
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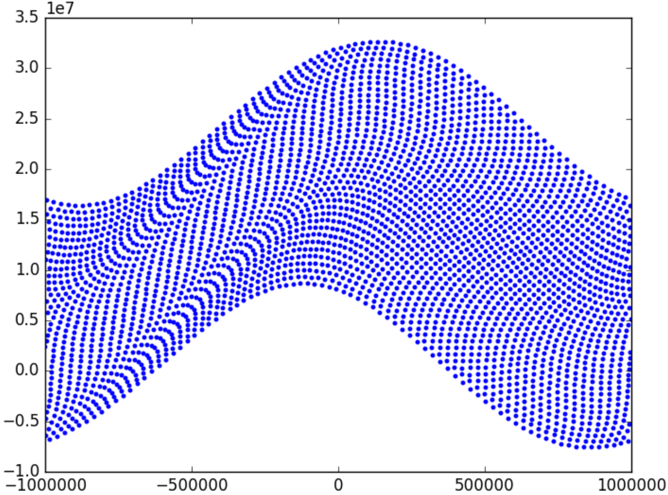

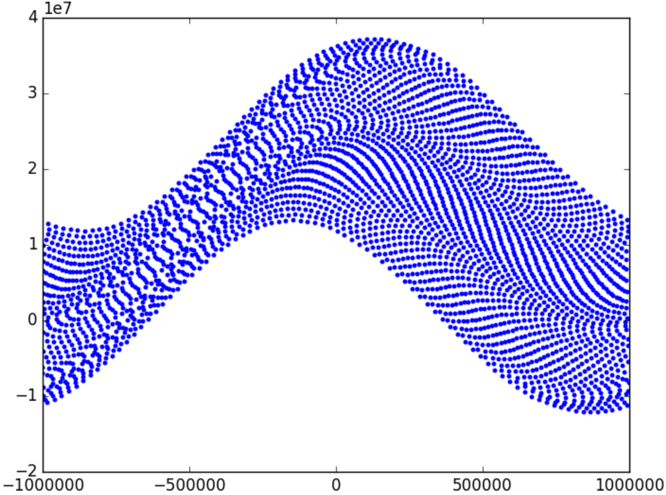
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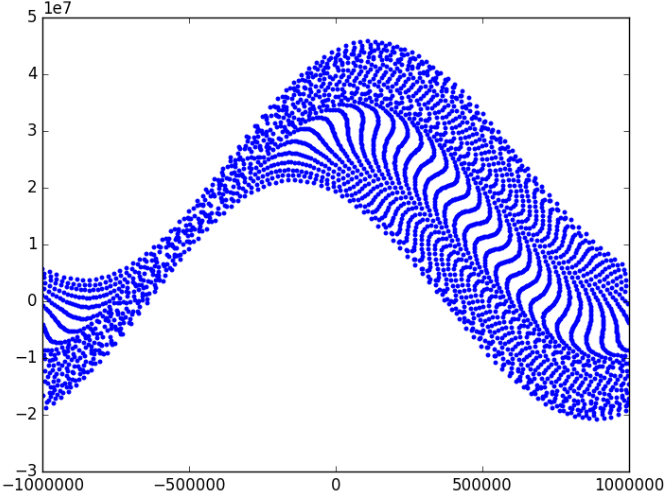

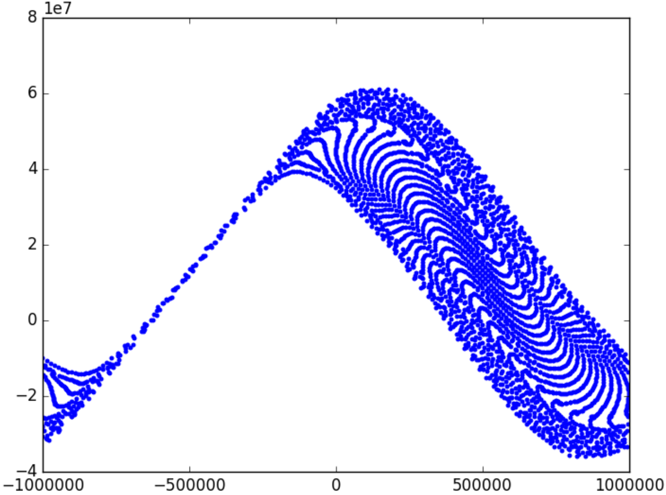
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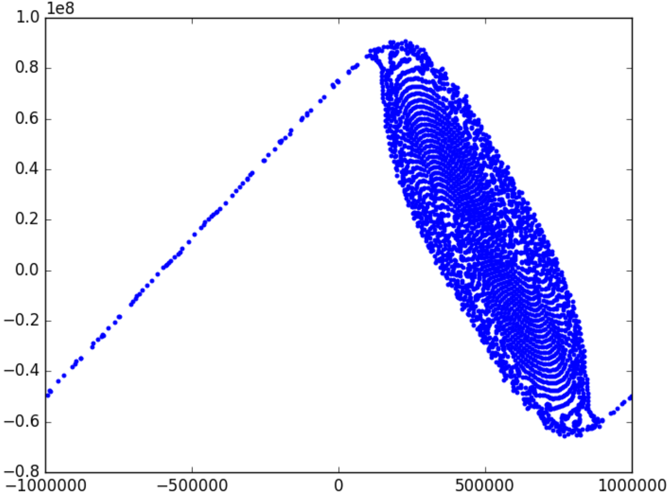
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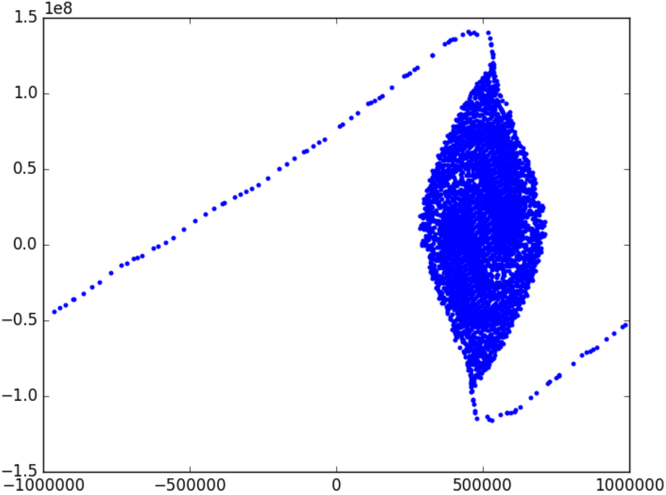
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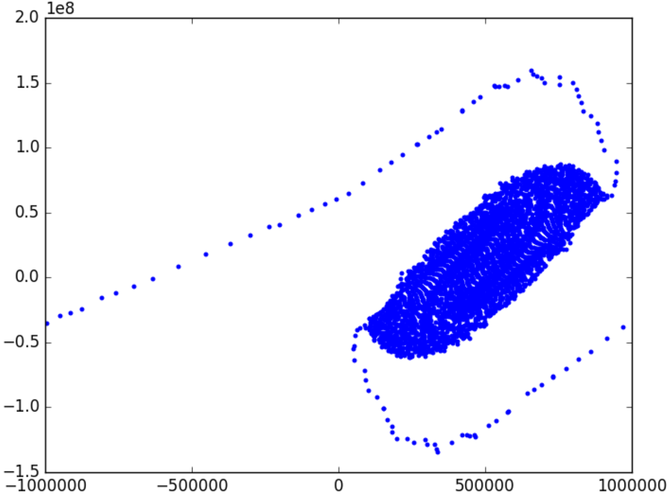
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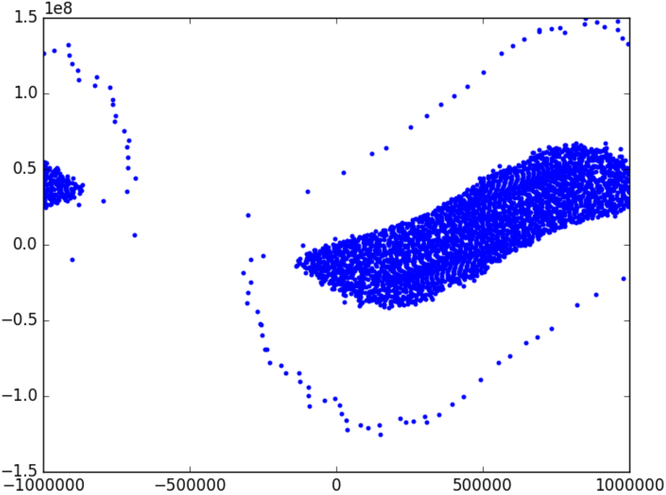
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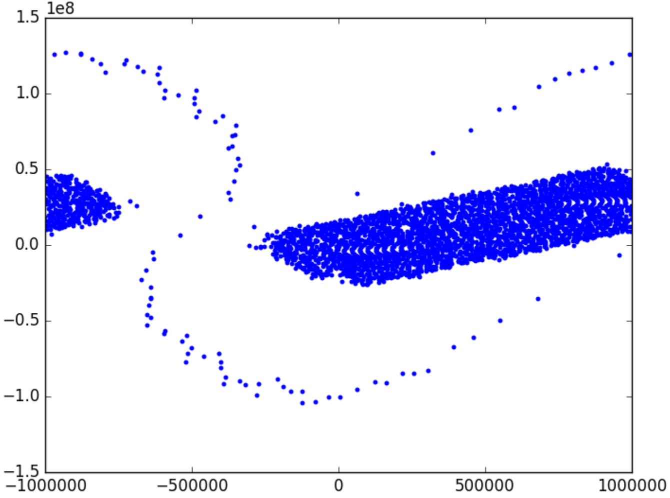
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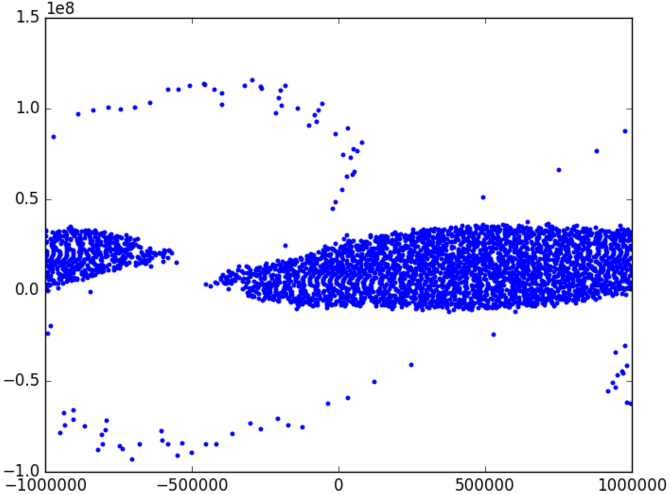
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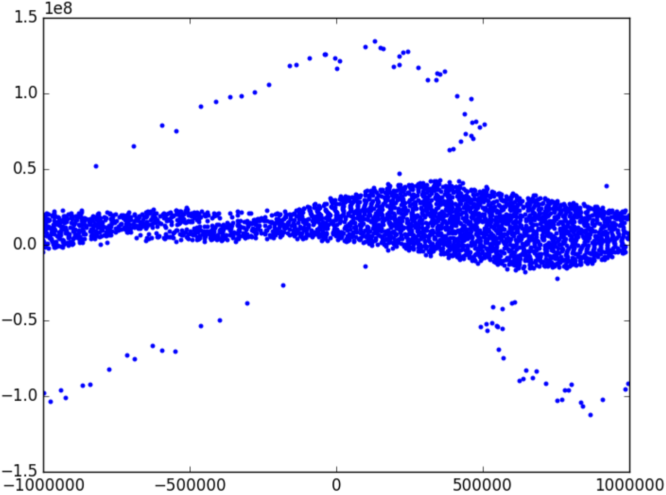
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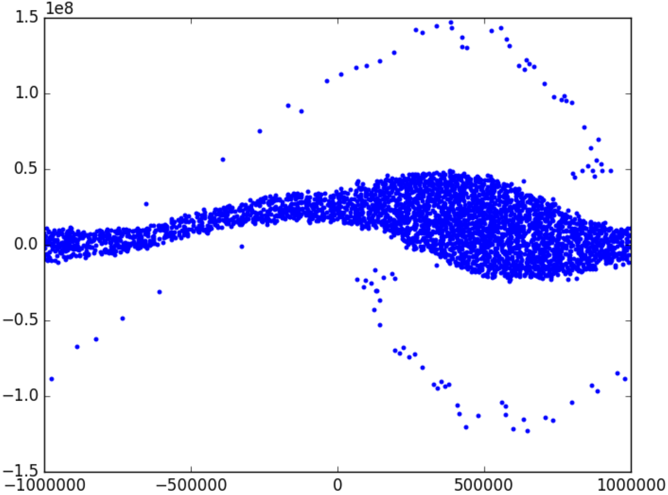
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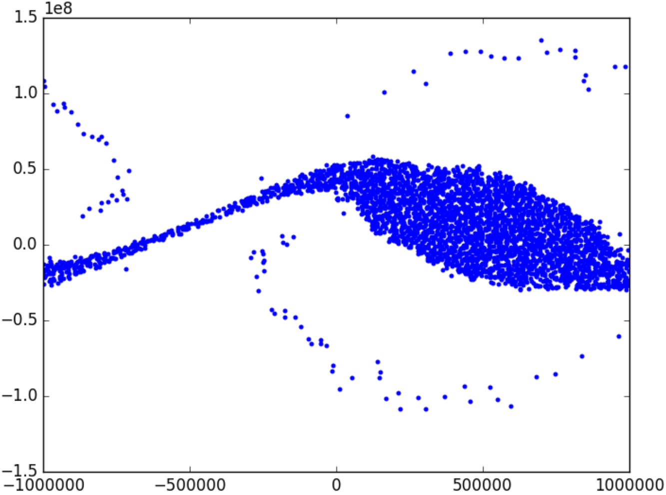
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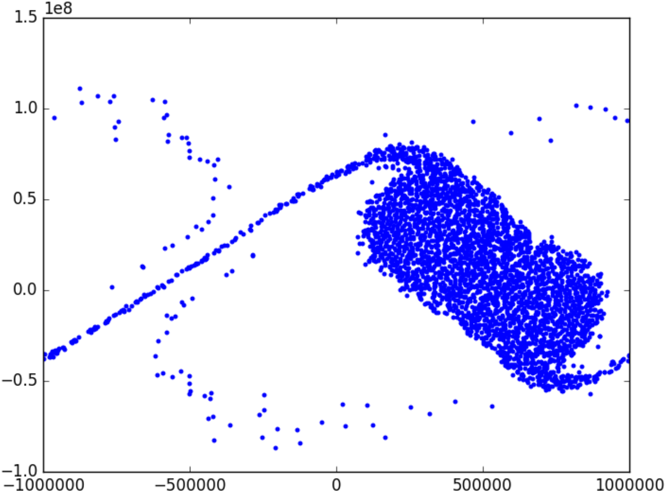
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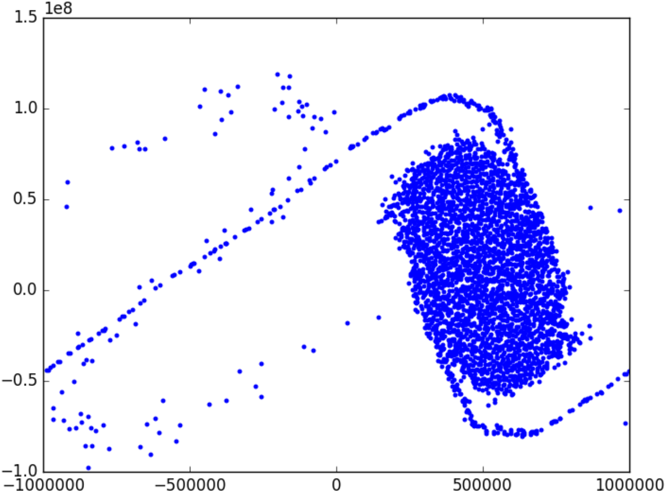
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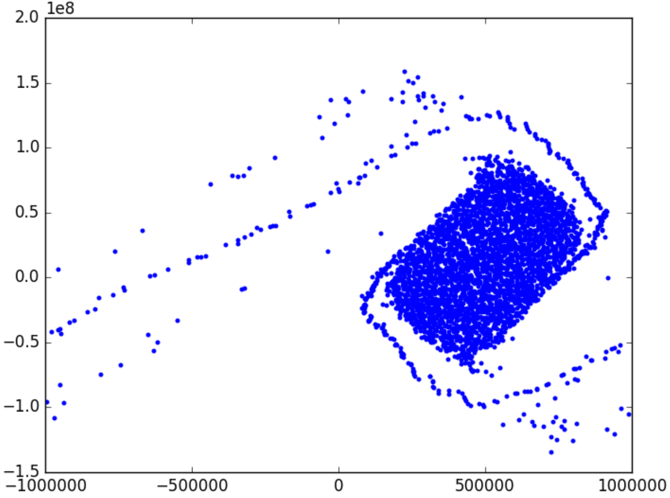

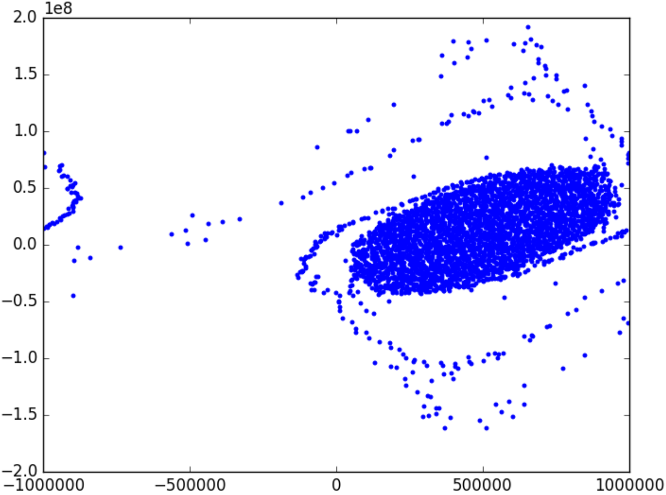
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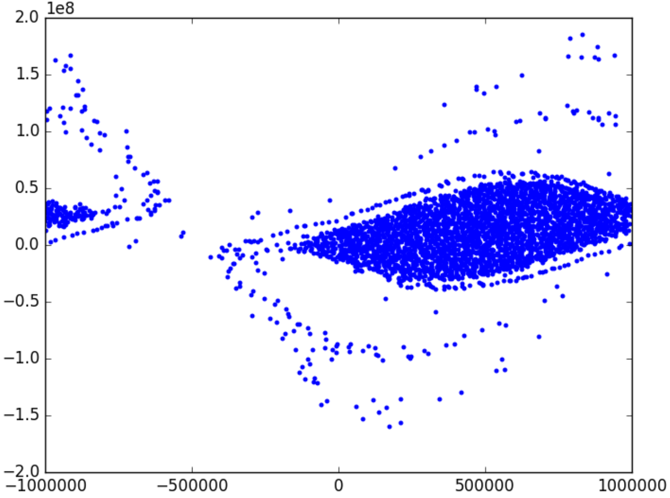
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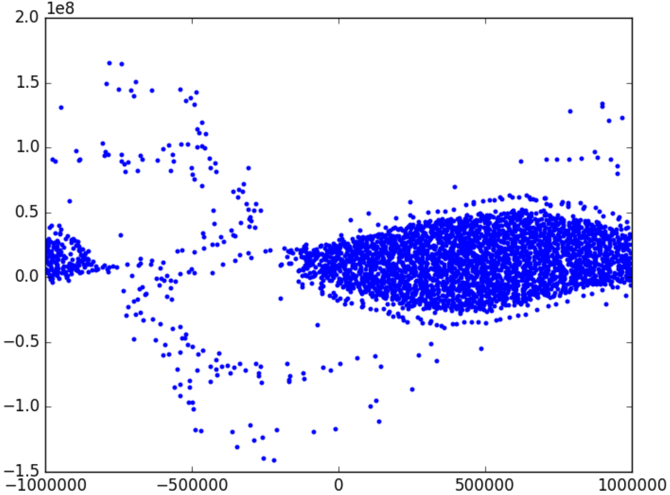
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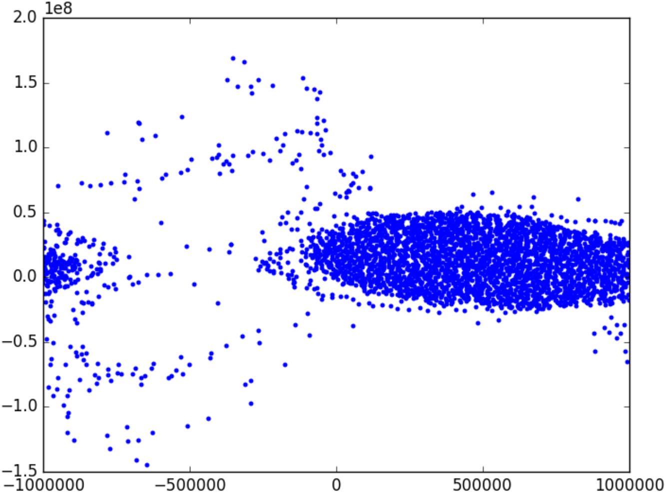
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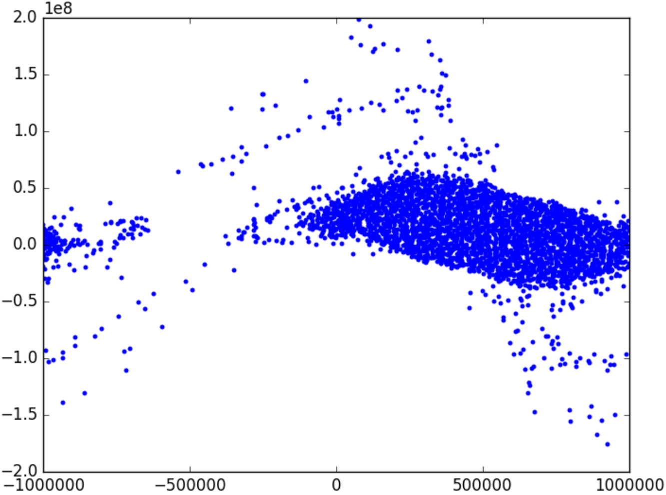
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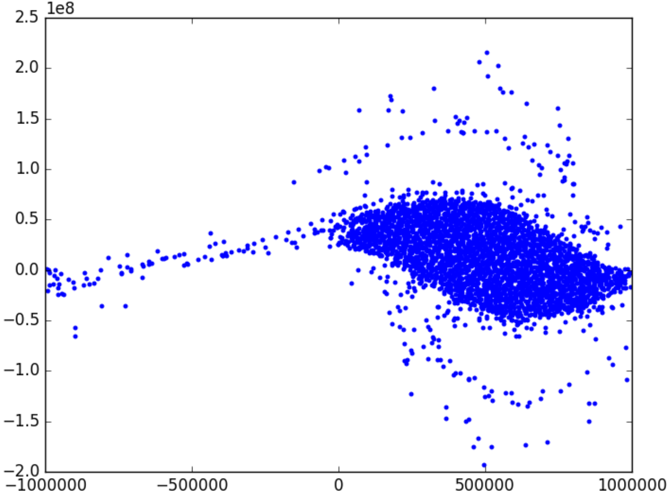

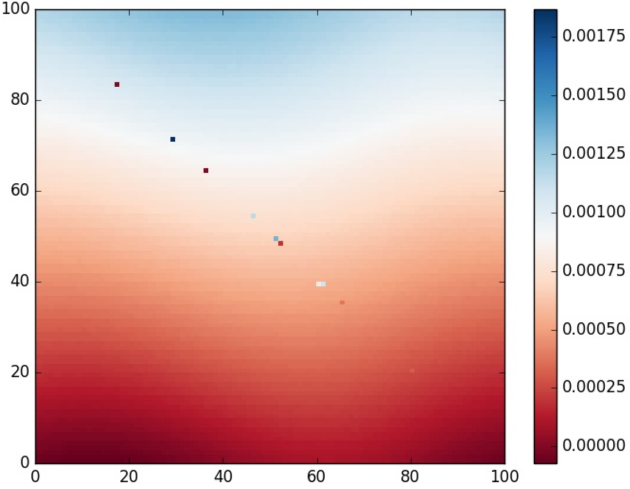
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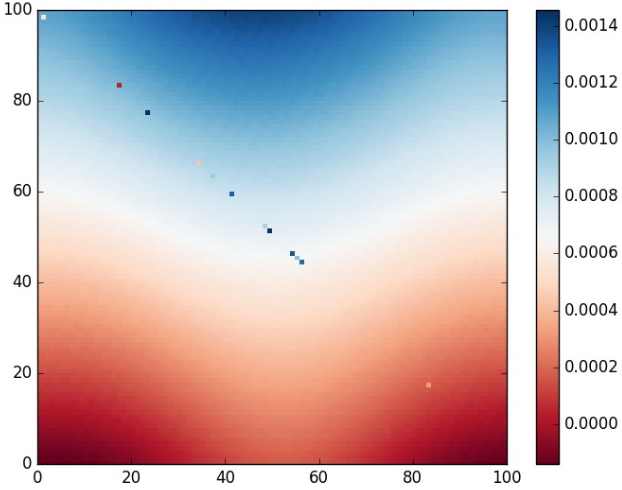
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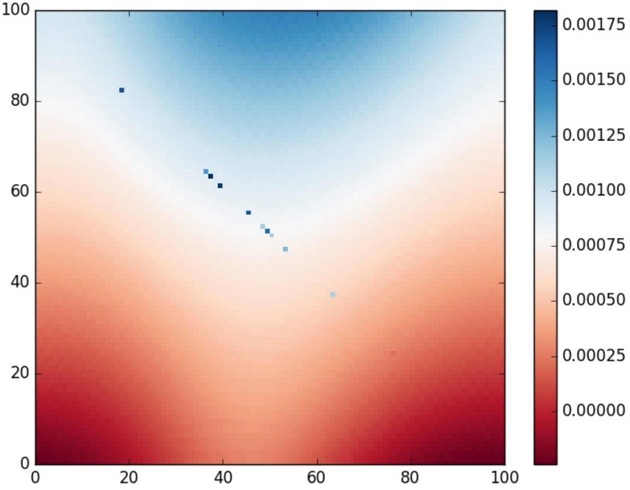

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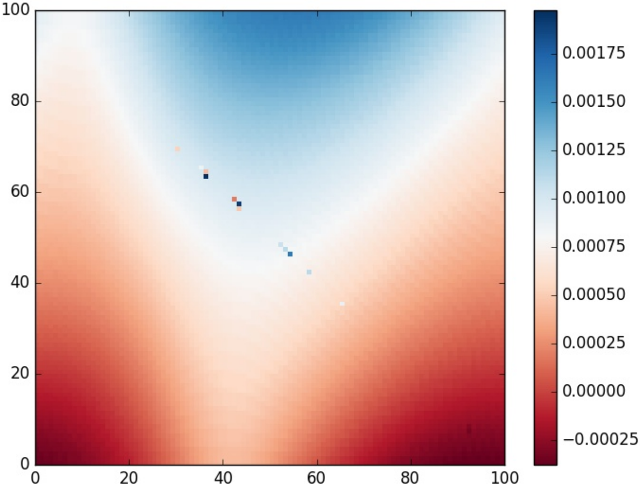

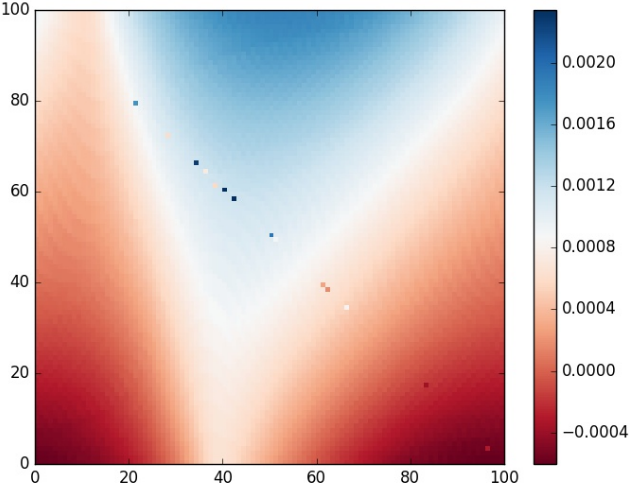
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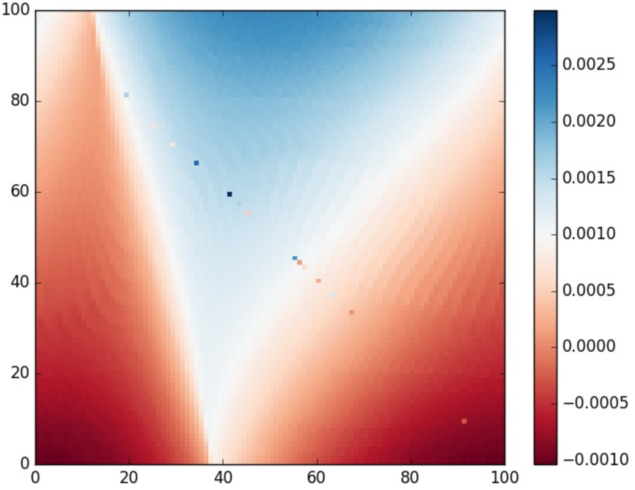

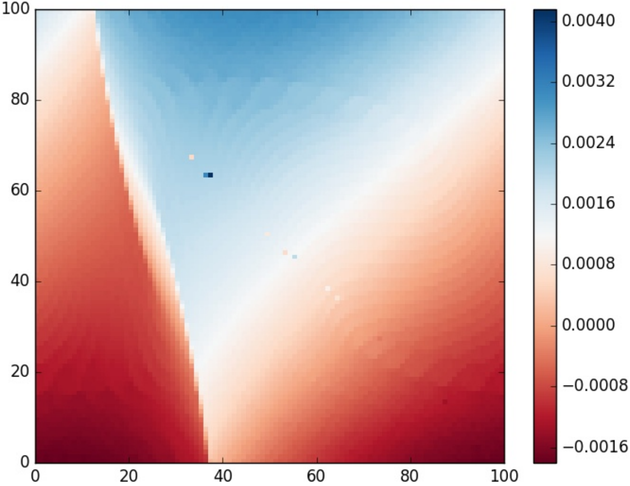
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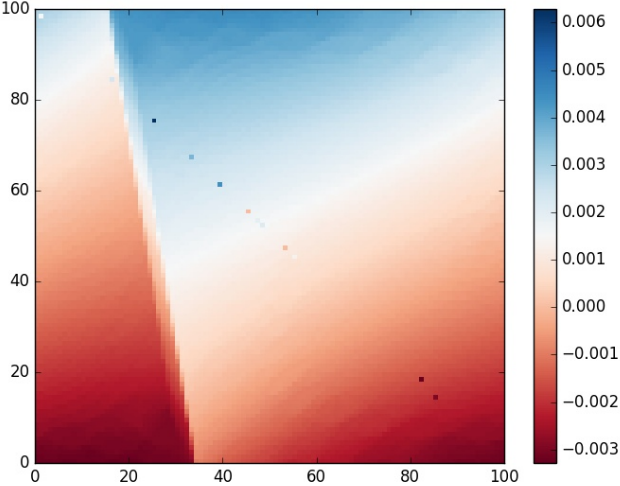
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0.000
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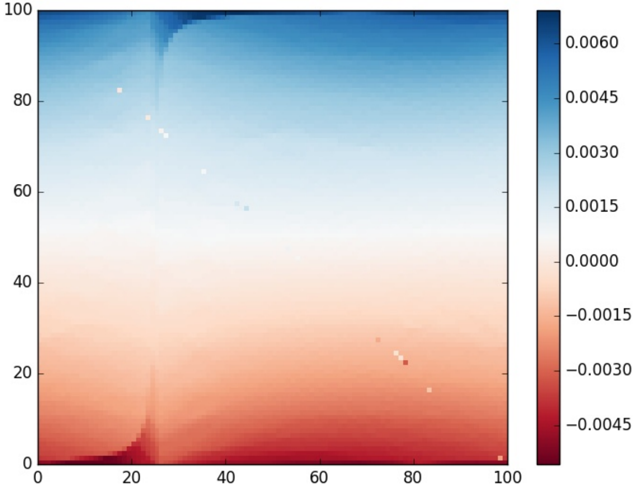
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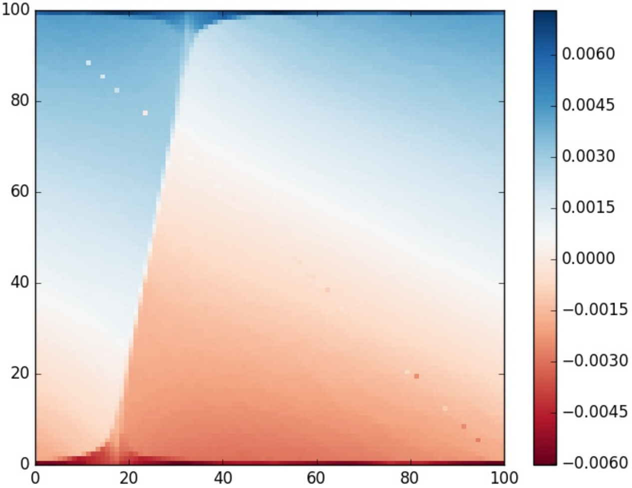
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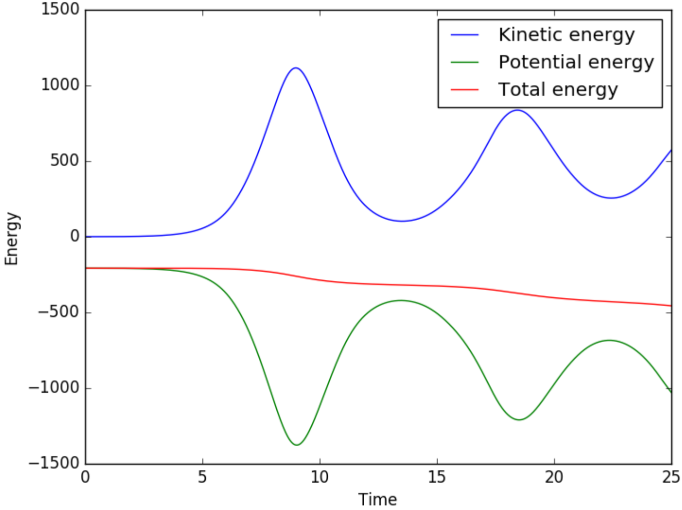
1500
1000
500
0
−500
−1000
−1500
Energy
0
5
10
15
20
25
Time
Kinetic energy
Potential energy
Total energy

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