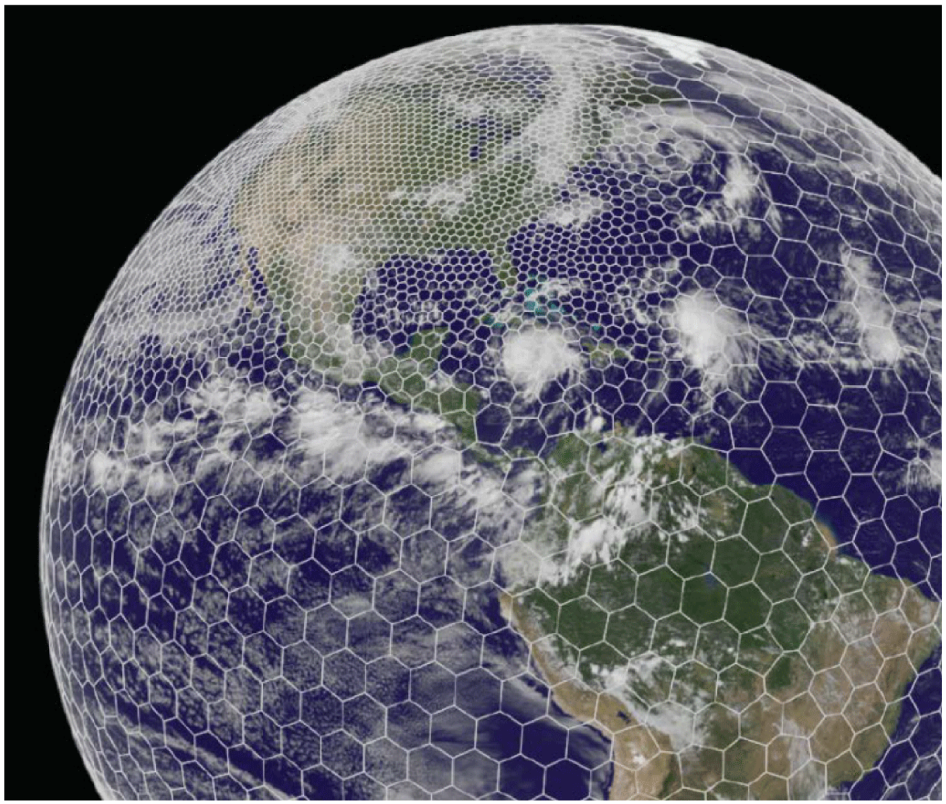
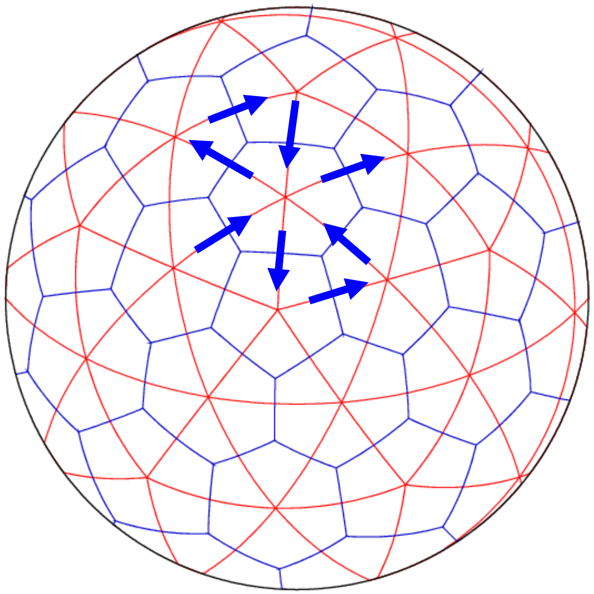
Discretization of generalized Coriolis and friction terms on the deformed hexagonal C-grid

Almut Gassmann, IAP Kühlungsborn, Germany



←Figure: MPAS model: variable mesh hexagonal C-grid

hexagonal C-grid

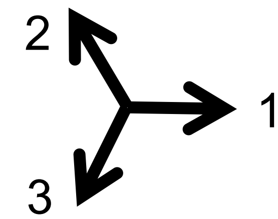


trivariate coordinate system

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} = \frac{2}{3}(u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3)$$

$$u_i = \mathbf{v} \cdot \mathbf{i}_i$$

$$u_1 + u_2 + u_3 = 0$$



discretization: $\boxed{\widetilde{u_1}^1 + \widetilde{u_2}^2 + \widetilde{u_3}^3 = 0}$

John Thuburn (2008): $\widetilde{u_1}^1 = \frac{1}{3}\left(\overline{u_1}^1 + 2\overline{\overline{u_1}}^{23}\right)$

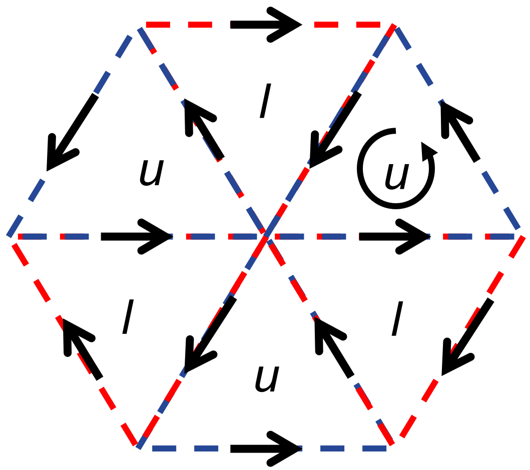
<u>Goal of this talk:</u>

Prove $\boxed{\widetilde{\partial_t u_1}^1 + \widetilde{\partial_t u_2}^2 + \widetilde{\partial_t u_3}^3 = 0}$ **for linearized SWEs around a zonal flow.**

Highlight implications of this constraint on vorticity flux term!

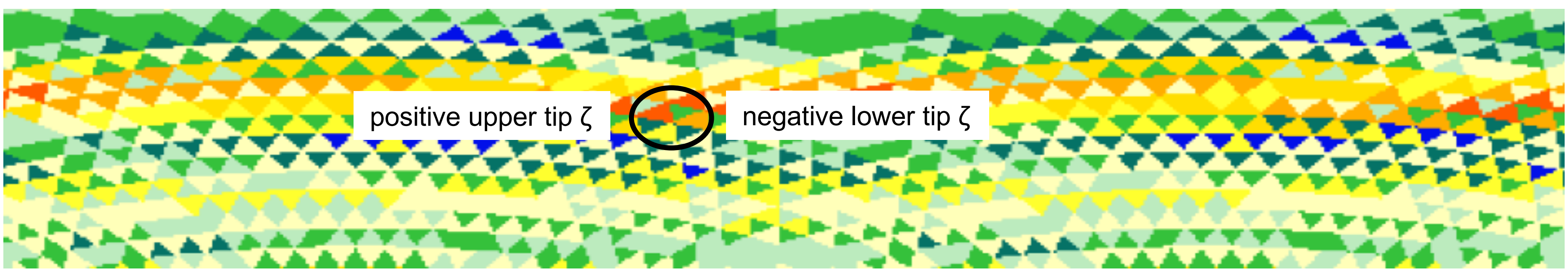
If $\boxed{\widetilde{u_1}^1 + \widetilde{u_2}^2 + \widetilde{u_3}^3 = 0}$ **is not fulfilled ...**

$\tilde{u}_1^1 + \tilde{u}_2^2 + \tilde{u}_3^3 = 0$ corresponds to: $\sum_{u \in h} \zeta_u^t = \sum_{l \in h} \zeta_l^t$



$$\zeta_{l,u}^t = \mp \frac{4}{3d}(u_1 + u_2 + u_3)$$

...checkerboard! **Bad!**



Gassmann, JCP (2011)

Proof of $\widetilde{u_1}^1 + \widetilde{u_2}^2 + \widetilde{u_3}^3 = 0$

$$\widetilde{u_1}^1 = \frac{1}{3}\left(\overline{u_1}^1 + 2\overline{\overline{u_1}}^{23}\right)$$

Helmholtz decomposition (periodic domain):

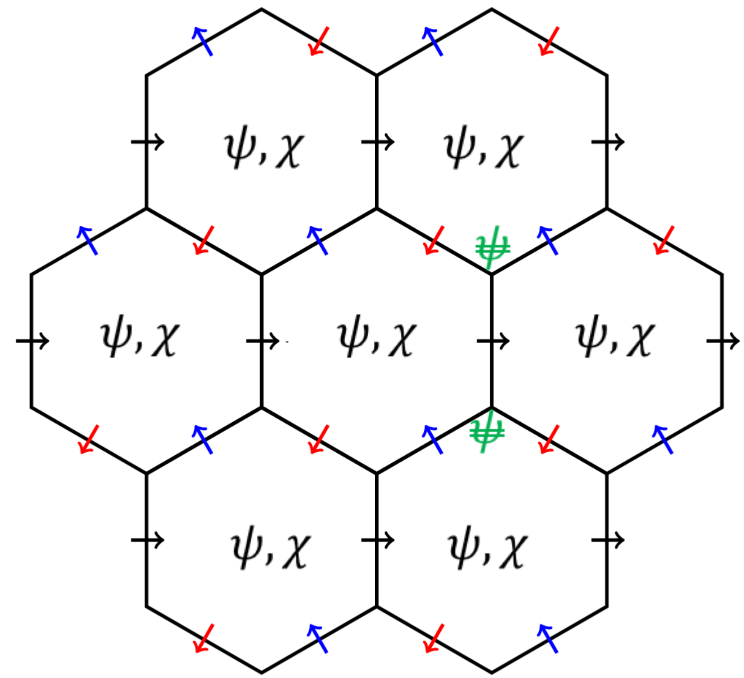
$$\boldsymbol{v} = \boldsymbol{k} \times \nabla\psi + \nabla\chi$$

$$\widetilde{u_1}^1 = -\frac{1}{\sqrt{3}}\left(\cancel{\widetilde{\delta_2\tilde{\psi}^3}^1} - \cancel{\widetilde{\delta_3\tilde{\psi}^2}^1}\right) + \widetilde{\delta_1\chi}^1$$

$$\widetilde{u_2}^2 = -\frac{1}{\sqrt{3}}\left(\cancel{\widetilde{\delta_3\tilde{\psi}^1}^2} - \cancel{\widetilde{\delta_1\tilde{\psi}^3}^2}\right) + \widetilde{\delta_2\chi}^2$$

$$\widetilde{u_3}^3 = -\frac{1}{\sqrt{3}}\left(\cancel{\widetilde{\delta_1\tilde{\psi}^2}^3} - \cancel{\widetilde{\delta_2\tilde{\psi}^1}^3}\right) + \widetilde{\delta_3\chi}^3$$

$$0 = 0 + 0$$



$$\widetilde{\delta_1\chi}^1 + \widetilde{\delta_2\chi}^2 + \widetilde{\delta_3\chi}^3 = 0$$

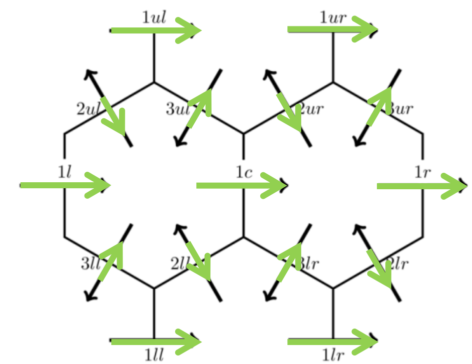
If the normal velocities are rotated by 90°, the hexagonal D-grid is the only consistent possibility for the numerics, not the triangular C-grid. Scalar variables at triangle midpoints lead to inconsistent schemes.

Shallow water velocity equation

$$\frac{\partial \mathbf{v}}{\partial t} = \boxed{-\nabla(k+gh)} - \frac{\zeta + \boxed{f}}{h}\mathbf{k} \times h\boxed{\mathbf{v}} \boxed{- \frac{1}{h}\nabla \cdot \mathbb{T}}$$

fix $\square$, vary $\dashrightarrow$

constant zonal fow



$$\partial_t(\widetilde{u_1}^1 + \widetilde{u_2}^2 + \widetilde{u_3}^3)|_m = 0$$

?

Linearized shallow water velocity component equations

$$\begin{aligned}
\partial_t u_1 &= -\delta_1(k+gh) + f(\widetilde{u_2}^3 - \widetilde{u_3}^2)/\sqrt{3} + (\overline{\zeta_3}^2(-U/2) - \overline{\zeta_2}^3(-U/2))/\sqrt{3} + K\delta_1 D - K(\delta_2\zeta_3 - \delta_3\zeta_2)/\sqrt{3} \\
\partial_t u_2 &= -\delta_2(k+gh) + f(\widetilde{u_3}^1 - \widetilde{u_1}^3)/\sqrt{3} + (\overline{\zeta_1}^3(-U/2) - \overline{\zeta_3}^1(\ U\ ))/\sqrt{3} + K\delta_2 D - K(\delta_3\zeta_1 - \delta_1\zeta_3)/\sqrt{3} \\
\partial_t u_3 &= -\delta_3(k+gh) + f(\widetilde{u_1}^2 - \widetilde{u_2}^1)/\sqrt{3} + (\overline{\zeta_2}^1(\ U\ ) - \overline{\zeta_1}^2(-U/2))/\sqrt{3} + K\delta_3 D - K(\delta_1\zeta_2 - \delta_2\zeta_1)/\sqrt{3}
\end{aligned}$$

proven

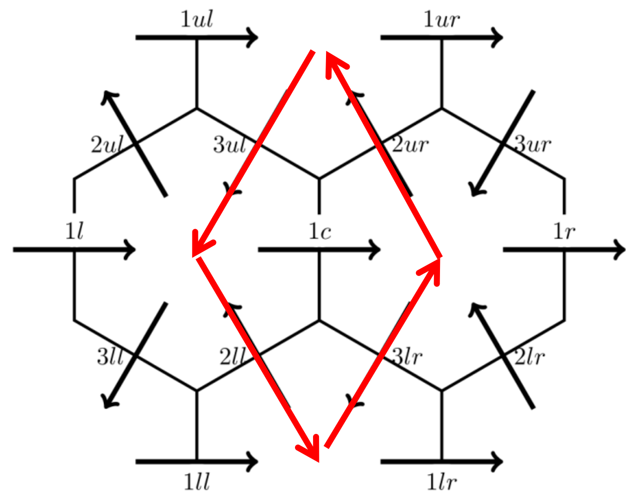
proven
Thuburn (2008)

to be proven
(next but one page)

to be proven
next page:
Laplacian of Helmholtz decomposition
if ζ defined on rhombi

This term has not yet been considered in the literature!

$$
\begin{aligned}
\zeta_1 &= 2(\delta_2 u_3 - \delta_3 u_2)/\sqrt{3} = \hat{\nabla}^2 \widetilde{\psi}^1 \\
&= 2(\delta_2(-(\delta_1\widetilde{\psi}^2 - \delta_2\widetilde{\psi}^1)/\sqrt{3} + \delta_3\chi) - \delta_3(-(\delta_3\widetilde{\psi}^1 - \delta_1\widetilde{\psi}^3)/\sqrt{3} + \delta_2\chi))/\sqrt{3} \\
&= 2/3(-\delta_{12}\widetilde{\psi}^2 + \delta_{22}\widetilde{\psi}^1 + \delta_{33}\widetilde{\psi}^1 - \delta_{13}\widetilde{\psi}^3) \\
&= 2/3(-\delta_{12}\widetilde{\psi}^2 - \delta_{13}\widetilde{\psi}^3 - \delta_{11}\widetilde{\psi}^1 + \delta_{11}\widetilde{\psi}^1 + \delta_{22}\widetilde{\psi}^1 + \delta_{33}\widetilde{\psi}^1) \\
&= 2/3(-\delta_1(\delta_2\widetilde{\psi}^2 + \delta_3\widetilde{\psi}^3 + \delta_1\widetilde{\psi}^1) + \delta_{11}\widetilde{\psi}^1 + \delta_{22}\widetilde{\psi}^1 + \delta_{33}\widetilde{\psi}^1) \\
&= 2/3(\delta_{11}\widetilde{\psi}^1 + \delta_{22}\widetilde{\psi}^1 + \delta_{33}\widetilde{\psi}^1)
\end{aligned}
$$

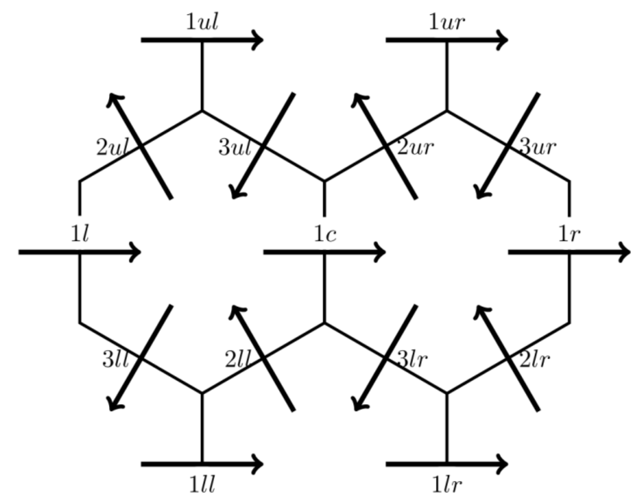


Diffusion term …is Laplacian of … Helmholtz decomposition

$$
\begin{aligned}
&+ K\delta_1 D - K(\delta_2\zeta_3 - \delta_3\zeta_2)/\sqrt{3} \\
&+ K\delta_2 D - K(\delta_3\zeta_1 - \delta_1\zeta_3)/\sqrt{3} \\
&+ K\delta_3 D - K(\delta_1\zeta_2 - \delta_2\zeta_1)/\sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= -\frac{1}{\sqrt{3}}\left(\delta_2\widetilde{\psi}^3 - \delta_3\widetilde{\psi}^2\right) + \delta_1\chi \\
u_2 &= -\frac{1}{\sqrt{3}}\left(\delta_3\widetilde{\psi}^1 - \delta_1\widetilde{\psi}^3\right) + \delta_2\chi \\
u_3 &= -\frac{1}{\sqrt{3}}\left(\delta_1\widetilde{\psi}^2 - \delta_2\widetilde{\psi}^1\right) + \delta_3\chi
\end{aligned}
$$

Consequences for the vorticity flux term



$$\boxed{\widetilde{\zeta_1}^3 = \widetilde{\hat{\nabla}^2\widetilde{\psi}^1}^3 = \widetilde{\hat{\nabla}^2\widetilde{\psi}^3}^1 = \widetilde{\zeta_3}^1}$$

Vorticity flux term

$$(\overline{\zeta_3}^2(-U/2) - \overline{\zeta_2}^3(-U/2))/\sqrt{3}$$

$$(\overline{\zeta_1}^3(-U/2) - \overline{\zeta_3}^1(\;\;U\;\;))/\sqrt{3}$$

$$(\overline{\zeta_2}^1(\;\;U\;\;) - \overline{\zeta_1}^2(-U/2))/\sqrt{3}$$

proven !

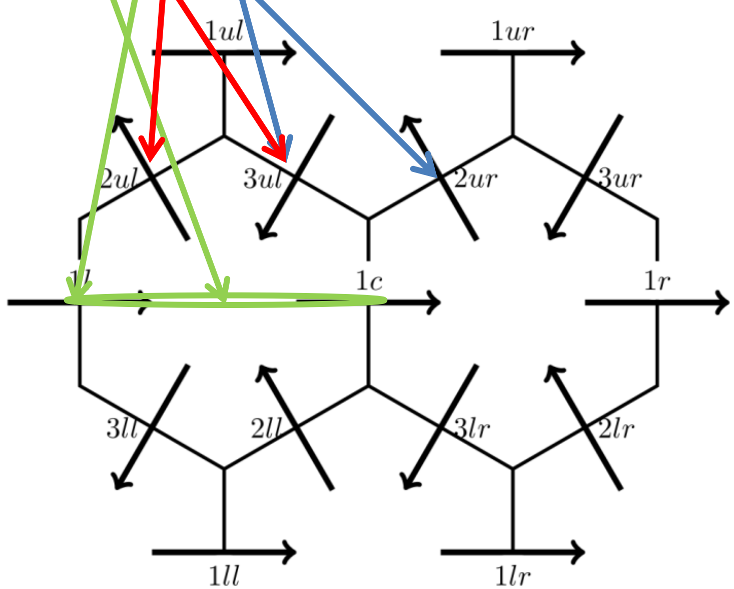
$$-U/2(\widetilde{\overline{\zeta_3}^2}^1 - \widetilde{\overline{\zeta_1}^2}^3) - U/2(\widetilde{\overline{\zeta_1}^3}^2 - \widetilde{\overline{\zeta_2}^3}^1) + U(\widetilde{\overline{\zeta_2}^1}^3 - \widetilde{\overline{\zeta_3}^1}^2) = 0$$

vorticity usage

Practical consequence for PV flux term

$$Q_e^{\perp} = \sum_{e' \in NB} w_{ee'} q_{ee'} \, h_{e'} v_{e'}^{n}$$

- reconstruction weights remain as in TRSK
- **PV usage is new**
- energy conservation remains as in TRSK, because swapping e and e‘ touches the same PV‘s



Nonlinear shallow water velocity equation

$$\frac{\partial \mathbf{v}}{\partial t} = \boxed{-\nabla(k+\cancel{g}h) - \frac{\zeta+\cancel{f}}{\cancel{h}}\mathbf{k}\times\cancel{h}\mathbf{v}} - \frac{1}{h}\nabla\cdot\mathbb{T}$$

nonlinear Coriolis term
vorticity flux term

Is the vector invariant form

$$\partial_t u_1 = -\frac{1}{3}\delta_1\left(\overline{u_1^2}^1 + \overline{u_2^2}^2 + \overline{u_3^2}^3\right) + \frac{1}{3\sqrt{3}}\left(\left(2\overline{\zeta_3\overline{u_2}^1}^2 + \widehat{\zeta_3 u_2}^{1\perp}\right) - \left(2\overline{\zeta_2\overline{u_3}^1}^3 + \widehat{\zeta_2 u_3}^{1\perp}\right)\right)$$

equal to the ‚pseudo-continuous' advective form?

$$\partial_t u_1|_{adv} = \frac{2}{3}\left(-u_1\partial_1 u_1 - u_2\partial_2 u_1 - u_3\partial_3 u_1\right) + \boxed{\frac{d_e^2}{18}\left(\partial_2(\partial_1 u_2)^2 + \partial_3(\partial_1 u_3)^2\right)}$$

No!

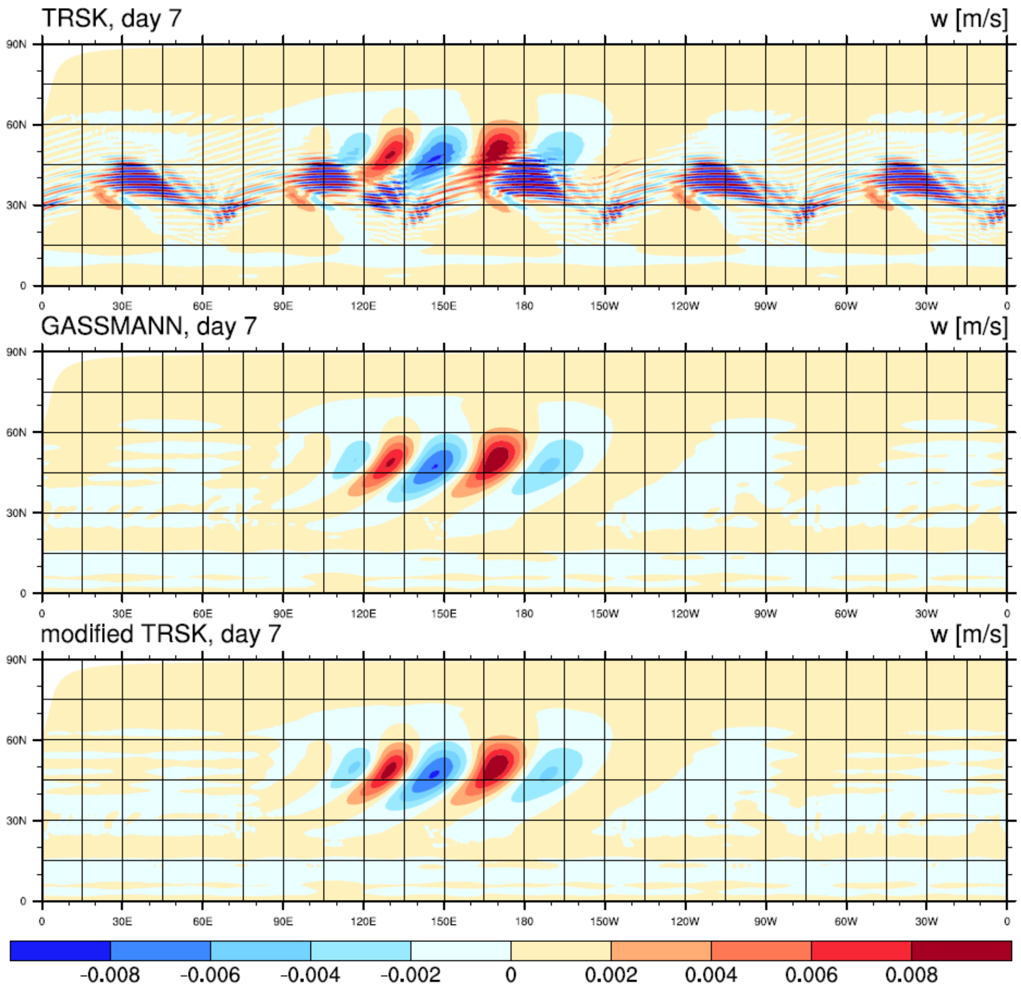
Hollingsworth et al. (1983) instability seems to be possible. **Bad!**

Linear dependency constraint in the linear limit prevents checkerboard. **Good!**

Baroclinic wave test : Day 7

Linear dependency fulfilled?

$$\partial_t(\widetilde{u_1}^1 + \widetilde{u_2}^2 + \widetilde{u_3}^3) = \cdots$$



$$\tfrac{3}{4}U(\tilde{\zeta_3}^3 - \tilde{\zeta_2}^2)$$

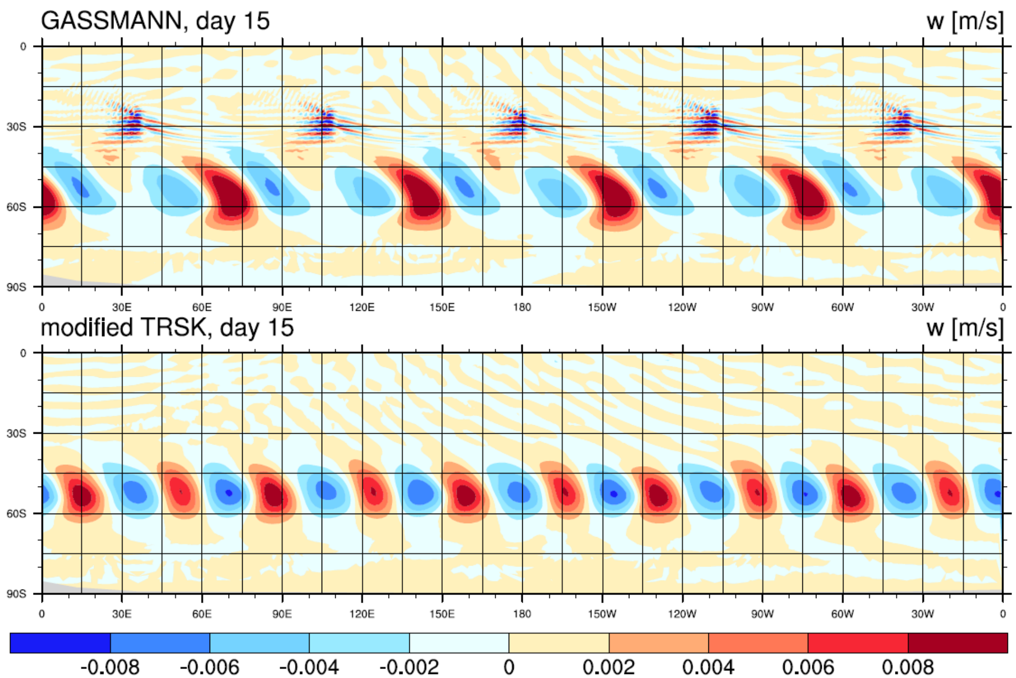
$$\tfrac{1}{2}U(\tilde{\zeta_3}^3 - \tilde{\zeta_2}^2 + \overline{\tilde{\zeta_3}^2}^1 - \overline{\tilde{\zeta_2}^3}^1)$$

$$0$$

Linear dependency fulfilled?

$$\partial_t(\widetilde{u_1}^1 + \widetilde{u_2}^2 + \widetilde{u_3}^3) = \cdots$$

Baroclinic wave test : Day 15



$$\frac{1}{2}U\left(\tilde{\zeta}_3^{\,3} - \tilde{\zeta}_2^{\,2} + \overline{\tilde{\zeta}_3^{\,2}}^{\,1} - \overline{\tilde{\zeta}_2^{\,3}}^{\,1}\right)$$

$$0$$

The linear dependency constraint prevents the instability.
An instability of the Hollingsworth kind seems not to be present.

Friction tensor

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla(k+gh) - \frac{\zeta + f}{h}\mathbf{k}\times h\mathbf{v} \boxed{-\frac{1}{h}\nabla\cdot\mathbb{T}}$$

$$\begin{pmatrix} h\partial_t u_1 \\ h\partial_t u_2 \\ h\partial_t u_3 \end{pmatrix}_{fric} = (\delta_1, \delta_2, \delta_3)\cdot \begin{pmatrix} \frac{4K_c}{9}[\delta_1 u_1 - \delta_2 u_2 + \delta_1 u_1 - \delta_3 u_3] & \frac{2K_3}{3}[\delta_2 u_1 + \delta_1 u_2] & \frac{2K_2}{3}[\delta_3 u_1 + \delta_1 u_3] \\ \frac{2K_3}{3}[\delta_2 u_1 + \delta_1 u_2] & \frac{4K_c}{9}[\delta_2 u_2 - \delta_3 u_3 + \delta_2 u_2 - \delta_1 u_1] & \frac{2K_1}{3}[\delta_2 u_3 + \delta_3 u_2] \\ \frac{2K_2}{3}[\delta_3 u_1 + \delta_1 u_3] & \frac{2K_1}{3}[\delta_2 u_3 + \delta_3 u_2] & \frac{4K_c}{9}[\delta_3 u_3 - \delta_1 u_1 + \delta_3 u_3 - \delta_2 u_2] \end{pmatrix}$$

physical constraints:

- traceless (volume-preserving),
- symmetric (invariant to addition of solid body rotation)
- entropy production is positive

mathematical constraint:

- linear dependency in linearized case

But, for a constant diffusion coefficient, the Laplacian is not obtained!

$$h\partial_t u_1|_{fric} = K[\frac{2}{3}(\delta_{11}u_1 + \delta_{22}u_1 + \delta_{33}u_1) + \frac{1}{3}\delta_1(\frac{2}{3}(\delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3))] = K[\hat{\nabla}^2 u_1 + \frac{1}{3}\delta_1 D]$$

Shear and strain defomations in the stress tensor

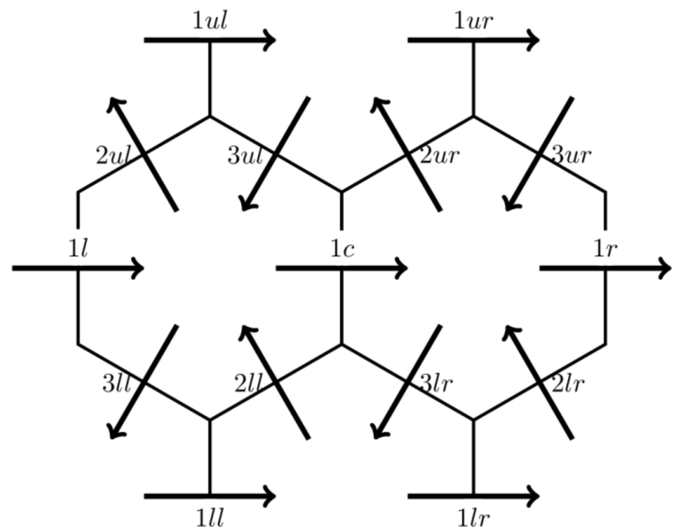
$$h\partial_t u_1|_{fric} = \delta_1(K_c E^1) + \delta_1^{\perp}\left(\frac{K_1 F_1^1 + K_2 F_2^1 + K_3 F_3^1}{3}\right)$$

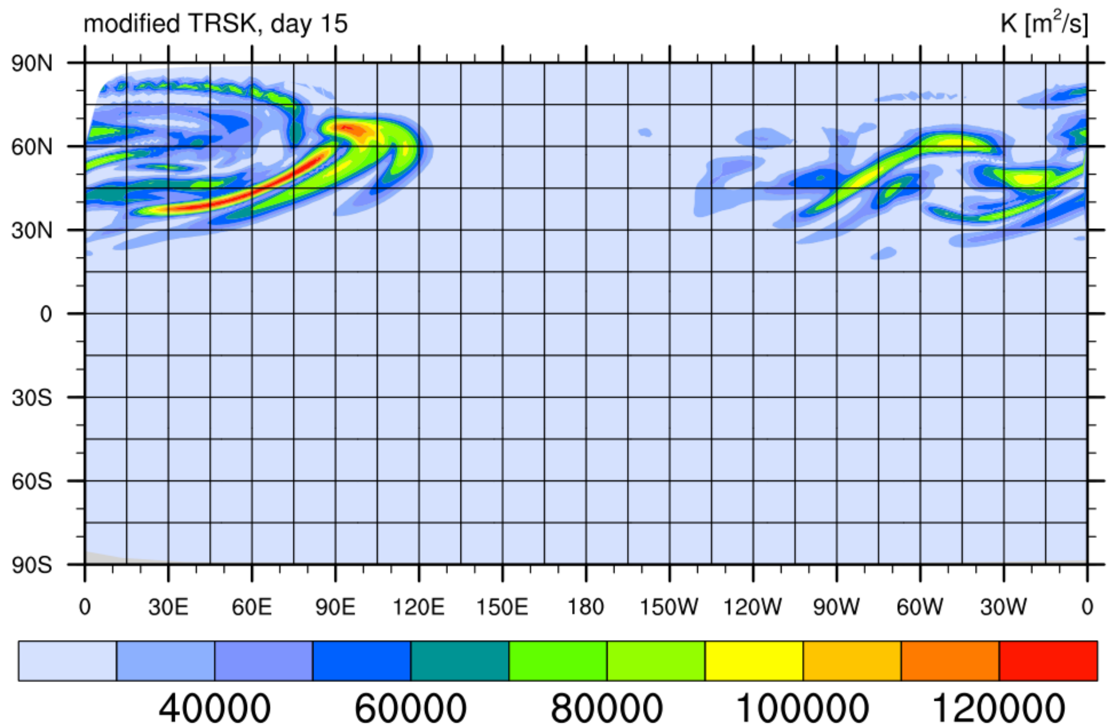
Deformations may not be computed by with Gauss or Stokes.
Derivations have to be estimated via least squares reconstructions for instance in the rhombus center in the coordinate system of the central edge:

$$\begin{aligned} E^1 = E_{c(e=1)} &= \frac{2}{3}(-D + 2\delta_x u_1|_c) \\ F_2^1 = F_{r(e=1)=2} &= \zeta_2 + 2\delta_y u_1|_2 \\ F_3^1 = F_{r(e=1)=3} &= \zeta_3 + 2\delta_y u_1|_3 \end{aligned}$$

$$\mathbf{u} = \left\{ \mathbf{N_e}\cdot\mathbf{e}_x \quad \mathbf{N_e}\cdot\mathbf{e}_y \quad \mathbf{N_e}\cdot\mathbf{e}_x\frac{dy_i}{dx_r} \quad \mathbf{N_e}\cdot\mathbf{e}_y\frac{dx_i}{dx_r} \right\} \cdot \begin{pmatrix} u_r \\ v_r \\ \widetilde{\partial_y u} \\ \widetilde{\partial_x v} \end{pmatrix}$$

$$\mathbf{u} = \{v_e^n\}.$$





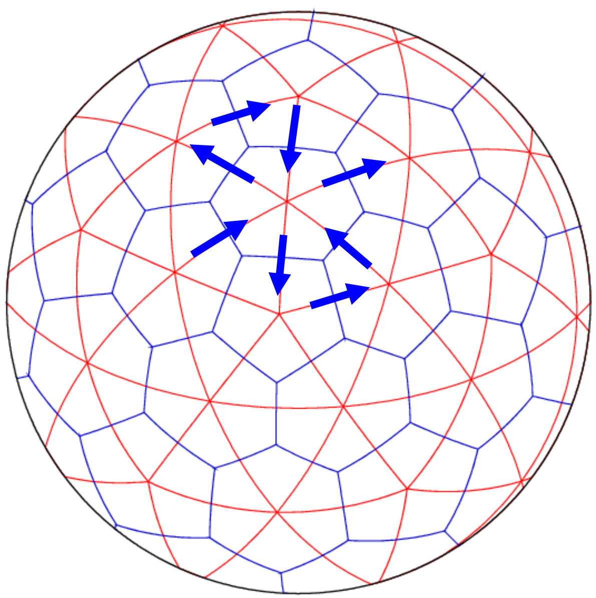
Diagnosed Smagorinsky diffusion coefficient

$K = c_s L^2 |S|$

$|S| = \mathrm{sqrt}(E^2+F^2)$

Future further development direction: dynamic Smagorinsky scheme

Summary:
Full equivalence to quadrilateral C-grid is achieved!



- regular trivariate grid introduces **linear dependency constraint**

- vector invariant form of momentum transport must use **special edge vorticities**

- **Hollingsworth instability does not occur**

- **friction tensor** leads to unexpected diffusion term: **Laplacian + grad D/3**

- **reconstruction of gradients of vector components** is needed for shear/strain deformation on a deformed mesh.