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Dynamical consistency and covariance: Reply to Staniforth and White

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The derivation of consistent sets of approximated equations in geophysical fluid dynamics, which has been the subject of recent publications, is a challenging topic. We are therefore grateful to Staniforth and White (2015) for contributing to the clarification of some aspects of that derivation. We hope the following paragraphs will also help clarifying these issues.

This reply is based on the following two main points, which will be justified later on:

1. "dynamical consistency" should mean "consistency with the fundamental assumptions underlying Newtonian mechanics";
2. equations of motion in Newtonian mechanics may be written in any admissible coordinate system in which time is absolute.

As shown in Charron and Zadra (2014) and further discussed here, the definition of consistency previously proposed by White *et al.* (2005) may contradict, under certain conditions, these two points. This is why we have proposed what we believe to be a more fundamental definition of dynamical consistency.

Let us consider the principle of relativity. It states that the laws of physics are the same in all admissible coordinate systems. Even though the term "relativity" is often associated with Albert Einstein's theories, this principle actually applies to Newtonian mechanics as well. What could be referred to as the principle of *Newtonian relativity*[†] may be regarded as a generalisation of the principle of Galilean relativity to non-inertial, curvilinear coordinates. The equations of motion of fluids moving at velocities much slower than the speed of light must obey the principle of Newtonian relativity.

This principle should not be violated even when the equations are approximated, otherwise unphysical results — i.e. inconsistent to leading order of approximation with the fundamental assumptions underlying Newtonian mechanics — may be obtained from the approximated equations. These fundamental assumptions are the absolute nature of time, the existence of a law of inertia, and the equality between the rate of change, in an inertial frame, of a particle's momentum and the sum of physical forces acting on it.

Equations obeying a principle of relativity are always covariant under admissible coordinate transformations. The covariance of the equations may or may not be manifest. It is said to be manifest when the governing equations are written using tensors only. By construction, tensor equations have the same form in all admissible coordinate systems. When covariant equations are manipulated to form non-tensors (for example, by choosing non-tensor dependent variables), covariance is no longer manifest. In this case, the principle of relativity continues to be obeyed, but this is somewhat hidden by the form of the non-tensorial equations.

In Newtonian mechanics, covariance is defined with respect to synchronous coordinate transformations (eqs (24)–(27) in Charron *et al.* 2014). Because time is considered absolute, these are the most general coordinate transformations admissible in Newtonian mechanics[‡].

In sum, our point of view is that "obeying the principle of Newtonian relativity", "covariance under synchronous

[†]This expression is however not well established. We propose it here to distinguish this concept from Galilean relativity.

[‡]In fact, the most general transformation of time t in Newtonian mechanics is $\tilde{t} = At + B$, where $A > 0$ and B are real constants. The choice $A \neq 1$ is equivalent to a change in the unit of time, and $B \neq 0$ to a change in the origin of time. The choice $A < 0$ implies a reversal of time, and is incompatible with the existence of irreversible processes such as viscous effects and diffusion. One may choose $A = 1$ and $B = 0$ without loss of generality.

coordinate transformations", and "dynamical consistency" are equivalent concepts. This point of view is justified by the fact that any approximation (geometric and/or dynamical) of the governing equation

$$T^{\mu\nu}{}_{:\nu} = -\rho h^{\mu\nu}\Phi_{,\nu} \tag{1}$$

that remains covariant will automatically be consistent with the fundamental assumptions underlying Newtonian mechanics. Equivalently, any approximation that preserves the scalar property of the action functional for inviscid fluids

$$\mathcal{S} = -\int d^4x\sqrt{g}\,\rho\left(K + I + \Phi + v^0\right) \tag{2}$$

will always lead to approximated equations of motion compatible with Newton's fundamental assumptions (see Charron *et al.* 2014; Charron and Zadra 2014, 2015; Zadra and Charron 2015, for definitions and details).

In certain cases, a given coordinate system may be preferred for practical reasons. The equations of motion in this coordinate system must nevertheless remain covariant.

One of the consistency criteria proposed by White *et al.* (2005) is the existence of a Lagrange's form for the momentum equations. White *et al.* (2005) do not impose constraints on the form of the Lagrangian. It is however known that not all Lagrangians will lead to consistent equations of motion (see for example Dirac 1964). To generate covariant equations of motion, a Lagrangian must be a scalar under admissible coordinate transformations (see Zadra and Charron 2015). Therefore, we conclude that this criterion of White *et al.* (2005) is incomplete.

The need for a renewed definition of dynamical consistency emerged after an analysis of the non-traditional shallow-atmosphere approximation (in short, the quasi-shallow approximation, Tort and Dubos 2014; Staniforth 2015). The quasi-shallow equations are compatible with the consistency criteria of White *et al.* (2005). However, the Lagrangian resulting from the quasi-shallow approximation is not a scalar under synchronous coordinate transformations (Charron and Zadra 2014). Consequently, it does not lead to covariant equations of motion[§] consistent with the fundamental assumptions underlying Newtonian mechanics.

To illustrate how non-covariant equations are incompatible with Newton's fundamental assumptions, consider the following example: the law of inertia states that a test particle not acted on by any physical forces (gravitational, electromagnetic, resulting from pressure gradients and viscous effects, etc.) follows a path defined by a geodesic trajectory. In Euclidean geometry, this trajectory is a straight line when viewed by an observer at rest within another inertial frame. In curved geometry (i.e. when geometric approximations are applied, Charron and Zadra 2014), this trajectory is not necessarily a straight line[¶]. For example, it can be shown that it is generally a spiral under the traditional shallow-atmosphere approximation. When arbitrary, time-dependent, curvilinear coordinates are used, the equation describing geodesic trajectories is

$$\frac{Du^i}{Dt} = \frac{d^2x^i}{dt^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0, \tag{3}$$

where centrifugal and Coriolis terms are described by Γ^i_{00} and $2\Gamma^i_{0j}dx^j/dt$, respectively (for the definition of $\Gamma^i_{\mu\nu}$ and details on the notation, see Charron *et al.* 2014). In the context of the quasi-shallow approximation, a test particle not acted on by any physical forces does not follow any geodesic trajectory because the space-time geometry of the quasi-shallow approximation is ill-defined: the Coriolis terms appear to exist in a curved metric space that is different from the metric space of all the other terms of the momentum equations. Therefore, there are no defined $\Gamma^i_{\mu\nu}$ (see Charron and Zadra 2014, section 4, for details). Consequently, inertial motion in the context of the quasi-shallow approximation cannot be defined. This implies that even when a test particle is not acted on by any physical forces, it will not follow a path normally defined by its inertia. Indefinable, unphysical "forces" come into play. It is therefore difficult to justify the use of the terms "dynamically consistent" when referring to the quasi-shallow approximation, or to any set of non-covariant approximated equations of motion. This is precisely why we proposed a renewed definition of dynamical consistency based on the concept of covariance.

The two criteria by White *et al.* (2005) to define consistency (existence of a Lagrange's form for the momentum equations, and conservation properties for axial angular momentum, energy, and potential vorticity) are not necessarily independent from each other. Conservation laws usually follow from symmetries of the Lagrangian (more generally, of the action functional). In the case of geophysical fluids, if the gravitational potential has no space-time symmetries, angular momentum and energy are not conserved quantities. Still, if the Lagrangian is a scalar, the resulting equations of motion will be covariant and dynamical consistency will be obeyed even though these equations do not satisfy conservation principles for axial angular momentum and energy (Charron and Zadra 2015). White *et al.* (2005) have limited their definition of consistency to axially symmetric and time-independent gravitational potentials (which arguably cover most current applications in geophysical fluid dynamics).

As noted by Staniforth and White (2015), applying either the criteria proposed by White *et al.* (2005) or the covariance criterion proposed by Charron and Zadra (2014, 2015) may result, under specific conditions, in the same equations of motion under a given approximation. These conditions are:

1. the approximated action functional remains a scalar under synchronous coordinate transformations;
2. the gravitational potential and metric tensor exhibit the space-time symmetries needed for axial angular momentum and energy conservation.

The second condition becomes necessary only if one imposes the conservation criterion of White *et al.* (2005). Recall that the covariance criterion does not require such a constraint. In sum, compared with the criteria of White *et al.* (2005), the covariance criterion for defining consistency imposes a single constraint on the approximated action functional (i.e. the approximated $\mathcal{S}$

[§] Staniforth and White (2015) suggest that the quasi-shallow equations could be covariant based on a submitted manuscript by T. Dubos. Dr Dubos has kindly sent us a copy of that manuscript. We believe there are problems in his demonstration and that the quasi-shallow equations remain non-covariant as shown in Charron and Zadra (2014).

[¶] The law of inertia in curved geometry is an extension of Newton's law of inertia which, strictly speaking, exists in Euclidean space-time.

obtained starting from Eq. (2) must remain a scalar), but does not specify any conservation constraints (conservation equations follow from symmetries of the approximated Lagrangian).

In their comments, Staniforth and White (2015) have written: “Working in a geopotential coordinate system, CZ1 show that, for the two non-hydrostatic equations sets, covariance implies axial angular momentum, energy, and potential vorticity conservation principles.” There are two issues in this statement that require clarification. Firstly, the conditions leading to those conservation principles were studied by Charron and Zadra (2014) in arbitrary coordinates, not specifically in geopotential coordinates. Secondly and as specified earlier, it is the symmetries that imply the conservation laws, not the covariance. Symmetries and covariance are independent properties.

Finally, Staniforth and White (2015) have mentioned that they struggle to understand why one would wish to transform geopotential coordinates to other coordinate systems. There is at least one practical reason one would wish to do so — aside from the theoretical arguments for having transformable governing equations, as discussed earlier. Most numerical weather prediction and climate models employ terrain-following coordinates. These are not geopotential coordinates, and a transformation from geopotential to terrain-following coordinates is needed to write the governing equations of these models. Should one take the quasi-shallow equations and attempt to transform their geopotential coordinates to terrain-following coordinates, would they remain dynamically consistent according to the criteria of White *et al.* (2005)? We believe it is impossible to answer this question affirmatively because there is no systematic and unambiguous way of transforming these equations since they are not covariant.

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