GEM4.4: A non-hydrostatic atmospheric model (Euler equations)

in terrain-following vertical coordinate ($\zeta = \zeta_S + \ln\eta$; $\zeta_S = \ln p_{ref}$)
of the **log-hydrostatic-pressure** type [$\ln\pi = \zeta + Bs$; $s = \ln(\pi_S / p_{ref})$]

discretized,
with simple differences and means,
horizontally on a Ying-Yan Arakawa-C grid,
vertically on a Charney-Phillips grid

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GEM4

Revision 4

Dynamics Documentation compatible with Model Version

GEM/4.8.0

(back to the LOG-tendencies formulation)

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PREFACE: GEM4, an evolving dynamical core code

Introduction of staggering

Reduction of noise in GEM3 was the main motivation for the original project of GEM4, consisting in the introduction of vertical staggering (Charney-Phillips grid). It was deemed the first and primary ingredient to achieve this goal. In effect, there are numerical modes which were theoretically diagnosed on the previous un-staggered grid which are absent from the new one in the hydrostatic case at least. As a first step therefore, only the grid was changed. Everything else, the equations, the independent as well as the dependent variables, were kept unchanged. Very *positive results* were obtained *with respect to noise*. But there remain problems in GEM, in particular an accuracy problem in the hydrostatic relation at upper levels when the true resolution (in terms of height) is insufficient.

Improving the accuracy of the hydrostatic relation using *logarithmic differencing* wherever appropriate was therefore the goal of a second step. The *results* from this modification of the code were very satisfying with *improved scores in the stratosphere*.

The log-hydrostatic-pressure coordinate

With this incentive, it was tempting to try and implement *a full log-hydrostatic-pressure coordinate*, ζ. A theoretical advantage of ζ is its linear relationship with ln*p*, [$\ln p = \ln(p/\pi) + \ln(\pi/\pi_*) + \ln \pi_* = q + Bs + \zeta$]. Along with the fact that $q = \ln(p/\pi)$ and $s = \ln(\pi_S / p_{ref})$ are already model variables, this greatly simplifies the linearization of model equations. Again the accuracy of the hydrostatic equation is improved since the finite differences not only are calculated logarithmically but also become defined at logarithmic mid-points. This third step though had *little impact* on model performance.

An important development: it was discovered that the initial staggered version of the semi-Lagrangian scheme, linear vertical interpolation of the departure positions for variables arriving on thermodynamic levels, resulted in significant loss of kinetic energy. *Cubic interpolation* is rather the thing to do.

Resolving numerical inconsistencies within the semi-Lagrangian scheme

A secondary motivation for the project was the resolution of accuracy and noise problems encountered in the simulation of non-hydrostatic mountain waves, specifically what we call Schär's case. Well, a completely satisfactory solution has been achieved, not via staggering though but again through modifications of the semi-Lagrangian scheme: tri-dimensional *cubic interpolation* of the departure positions replacing linear ones combined with *trapezoidal means* of the velocities instead of the mid-point rule. And if off-centering is present, off-centering in trajectory calculations as well as in advective calculations is necessary for consistency.

In addition to resolving numerical inconsistencies, the introduction of the trapezoidal rule and cubic interpolation in the trajectory calculations has improved model accuracy.

Reintroduction of logarithmic tendencies (***the sole reason for renewing the documentation***)

Tendencies inside non-linear terms can only be dealt with through non-linear iterations. Our latest attempt to eliminate them (see GEM4.3) has failed. The model becomes noisier near the surface, indeed less stable, in absence of off-centering. There is also a hint that accuracy is affected in presence of off-centering (the unfiltered mountain test).

Taking advantage of the Yin-Yang grid: improving efficiency

On a Yin-Yang grid, horizontal resolution is much more uniform than on a latitude-longitude grid. We are getting rid of pole problems. The need to calculate semi-Lagrangian trajectories along great circle or to implement implicit horizontal diffusion schemes disappears. These optimizations were implemented.

Variable T_* and modified epsilon, ε': improving stability

Improving stability, especially over steep topography, is perhaps the main remaining challenge for the dynamical core of GEM. The introduction of two new options: a variable T_* and a modified epsilon, $\varepsilon' = r\varepsilon$, are features which have shown their utility, although the problem is still present.

Lifting the last thermodynamic level: improving accuracy (!?)

As far as the dynamics was concerned, the last thermodynamic level was assumed to be at the earth's surface and therefore also the thermodynamic equation along with the temperature. In particular, the vertical motion was assumed to vanish. It turns out though that the thickness of the layer involved is typically far from negligible and the assumption of vanishing vertical motion in terrain-following coordinate, where the 'horizontal wind' vary substantially between the base and top of a mountain, leads to artificial cooling/warming due to lack of inflow/outflow of air in the layer. These are particularly evident when the model is run in adiabatic mode (no physics). A proper correction has required extensive modifications to the code. When the physics is included though, the impact is surprisingly small.

Dynamics-Physics interface: further improving accuracy

Improving accuracy along with improving efficiency and stability are the permanent general objectives of Numerical Weather Prediction. This document focuses on the Dynamics aspects. Two items involving the interface between Dynamics and Physics have recently been addressed though and are worth mentioning here:

a) Dynamics and Physics work in slightly different coordinate system: Is then the coordinate transformation done correctly?
b) There are water vapor and precipitation fluxes through the Earth's surface. How are we to take into account these fluxes?

Older versions of this document, GEM4.0, GEM4.1 GEM4.2 *and* GEM4.3 *remain available*.

1) The meteorological equations

- 4 independent variables: t, $\mathbf{r}=(\mathbf{r}_h, z)$
- 6 dependent variables: $\mathbf{V}=(\mathbf{V}_h, w)$, T, ρ, p
- 6 scalar equations:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{V} = 0$$

$$c_p \frac{dT}{dt} - RT \frac{d \ln p}{dt} = T \frac{ds}{dt} = Q + f$$

$$\rho = \frac{p}{RT}$$

- There are: 5 prognostic equations (momentum + mass + energy conservation),
  one diagnostic equation (perfect gas law).

N.B. The Coriolis force remains approximated in GEM4.4 (traditional meteorological approximations).

N.B. The atmospheric substance is assumed to contain, in addition to dry air, not only a variable quantity of water vapor but also condensed water and precipitations. The above equations are valid under the assumptions of *dynamic* (precipitations falling at terminal velocity) and *thermodynamic* (neglecting temperature differences between air and hydrometeors) *equilibrium.* Equations for the displacement and evolution of the hydrometeors are required to complete the system. Water vapor and precipitation fluxes through the earth's surface affecting the mass of the atmosphere is accounted for separately (see **Appendix 15**).

N.B. In the above equations the coefficients R, c_p and their ratio $\kappa = R/c_p$ are variable. A simplification occurs with the introduction of *virtual temperature* whereby RT is replaced by $R_d T_v$ with R_d now a constant) (see **Appendix 1**). The variation of κ in the thermodynamic equation is now taken into account.

N.B. The second law, $Tds / dt \geq Q$, requires that the frictional dissipation of kinetic energy $f \geq 0$.

2) The equations transformed to generalized η-coordinate

Note the necessary decomposition of vector equations into their horizontal/vertical components due to the different horizontal/vertical transformation rules.

3 transformation rules: $\nabla_z \equiv \nabla_\eta - \nabla_\eta z \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$; $\frac{\partial}{\partial z} \equiv \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$; $\frac{\partial}{\partial t_z} \equiv \frac{\partial}{\partial t_\eta} - \frac{\partial z}{\partial t_\eta}\frac{\partial}{\partial \eta}$

- 4 independent variables : t, $\mathbf{r}_h$, η
- 8 dependent variables: $\mathbf{V}_h$, w, T, ρ, p, $\dot{\eta}$, z
- 8 equations (6 prognostic and 2 diagnostic):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left[\nabla_\eta \ln p - \nabla_\eta z\left(\frac{\partial z}{\partial \eta}\right)^{-1}\frac{\partial \ln p}{\partial \eta}\right] = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\left(\frac{\partial z}{\partial \eta}\right)^{-1}\frac{\partial \ln p}{\partial \eta} + g = F_w$$

$$\frac{d}{dt}\ln\left(\rho\frac{\partial z}{\partial \eta}\right) + \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\eta}}{\partial \eta} = 0$$

$$\frac{dT}{dt} - \kappa T\frac{d\ln p}{dt} = \frac{Q+f}{c_p}$$

$$\frac{dz}{dt} - w = 0$$

$$\rho = \frac{p}{RT}$$

$$z \equiv z(\eta, \mathbf{r}_h, t)$$

- Were added then: 1 prognostic equation (***dz/dt=w***) for varying height in space and time,
1 diagnostic equation (yet to be specified) defining the coordinate η.

N. B. the continuity equation is the only one requiring more than simple manipulation:

$$w = \frac{dz}{dt} = \frac{\partial z}{\partial t} + \mathbf{V}_h \cdot \nabla_\eta z + \dot{\eta}\frac{\partial z}{\partial \eta}$$

$$\frac{\partial \eta}{\partial z}\frac{\partial w}{\partial \eta} = \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial t} + \mathbf{V}_h \cdot \nabla_\eta z + \dot{\eta}\frac{\partial z}{\partial \eta}\right) = \frac{\partial \eta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \eta}\cdot \nabla_\eta z + \frac{\partial \dot{\eta}}{\partial \eta} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \eta}\right)$$

hence

$$\nabla_z \cdot \mathbf{V}_h + \frac{\partial w}{\partial z} = \nabla_\eta \cdot \mathbf{V}_h - \frac{\partial \eta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \eta}\cdot \nabla_\eta z + \frac{\partial \eta}{\partial z}\frac{\partial w}{\partial \eta} = \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\eta}}{\partial \eta} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \eta}\right)$$

See **Appendix 2** for some details on transformation rules.

3) Eliminating density, ρ, introducing log-hydrostatic pressure, $\ln\pi$, replacing height, z, by the geopotential, ϕ, and adding μ (ratio of vertical acceleration to gravitational acceleration). *For convenience*, isolating the exclusively non-hydrostatic equations.

$$\frac{\partial \pi}{\partial z} = -g\rho\,; \qquad RT = -\frac{p}{\pi}\frac{\partial \phi}{\partial \ln \pi}; \quad \phi = gz\,; \qquad \mu = \frac{p}{\pi}\frac{\partial \ln p}{\partial \ln \pi} - 1$$

- 9 dependent variables: $\mathbf{V}_h, w, T, p, \dot{\eta}, \phi, \mu, \pi$

- 9 equations (added diagnostic equation for μ):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\eta \ln p + (1+\mu)\nabla_\eta \phi = \mathbf{F}_h$$

$$\frac{d}{dt}\ln\left(\pi \frac{\partial \ln \pi}{\partial \eta}\right) + \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\eta}}{\partial \eta} = 0$$

$$\frac{d \ln T}{dt} - \kappa \frac{d \ln p}{dt} = \frac{Q + f}{c_p T}$$

$$\frac{d\phi}{dt} - gw = 0$$

$$RT + \frac{p}{\pi}\frac{\partial \phi}{\partial \ln \pi} = 0$$

$$\ln \pi \equiv \ln \pi(\eta, \mathbf{r}_h, t)$$

...

$$\frac{dw}{dt} - g\mu = F_w$$

$$1 + \mu - \frac{p}{\pi}\frac{\partial \ln p}{\partial \ln \pi} = 0$$

N.B. Making the hydrostatic approximation allows for the elimination of (the last) two equations and the following variables: p and μ. The **...** will be used to separate the 'hydrostatic' from the 'non-hydrostatic' equations.

N.B. At this point, η is still a general coordinate of the hydrostatic-pressure type: in the next section we will specify both ζ and η.

N.B. For the rest of the presentation, the ***physical forcings,*** $\mathbf{F}_h$, F_w, Q and f, will be excluded and R will be treated as constant: the *Pure Dynamics* or so-called *No Physics* formulation (although κ is allowed to vary). It is worth noting that horizontal hyper-diffusion used in $\mathbf{F}_h$ neither conserves angular momentum nor respect the second law and f, due to all processes of kinetic energy dissipation, is not accounted for, viz. f=0.

4) The ζ-coordinate is lnπ-like

$$\zeta = \zeta_S + \ln\eta; \qquad \zeta_S = \ln p_{ref}; \quad p_{ref} = 10^5$$

$$\ln\pi = A(\zeta) + B(\zeta)s; \qquad s = \ln\pi_S - \zeta_S = \ln(\pi_S / p_{ref})$$

$$A = \zeta; \quad B = \lambda^r; \quad \lambda = \max\left[\frac{\zeta - \zeta_U}{\zeta_S - \zeta_U}, 0\right] \quad \begin{cases} \zeta_U \ge \zeta_T; \quad \zeta_T = \ln p_T \\ 0 \le r = r_{\max} - (r_{\max} - r_{\min})\lambda \le 50 \end{cases}$$

$$\ln\pi = \zeta + B(\zeta)s$$

transformation rules: $\nabla_\eta = \nabla_\zeta; \quad \frac{\partial}{\partial\eta} = \frac{1}{\eta}\frac{\partial}{\partial\zeta}; \quad \frac{\partial}{\partial t_\eta} = \frac{\partial}{\partial t_\zeta}$

- 9 dependent variables: $\mathbf{V}_h$, w, T, p, $\dot{\zeta}$, ϕ, μ, π

- 9 equations:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\mu)\nabla_\zeta\phi = 0$$

$$\frac{d}{dt}\ln\left(\pi\frac{\partial\ln\pi}{\partial\zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} = 0$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = 0$$

$$\frac{d\phi}{dt} - gw = 0$$

$$RT + \frac{p}{\pi}\frac{\partial\phi}{\partial\ln\pi} = 0$$

$$\ln\pi \equiv \zeta + Bs$$

...

$$\frac{dw}{dt} - g\mu = 0$$

$$1 + \mu - \frac{p}{\pi}\frac{\partial\ln p}{\partial\ln\pi} = 0$$

- **obviously,** at this point, the form of the equations in ζ and η coordinates is identical

N.B. $p_{top} / p_{ref} < \eta < 1$ is now but a label characterizing model ζ-levels. Entering the model, a set of η- values are required from which the ζ-values are obtained.

See **Appendix 3** for more information on the metric parameter *B*. ζ_U is typically chosen to correspond to ζ_1, the top momentum level.

5) Perturbation thermodynamic variables, T', ϕ', q, and simplifications

Introducing the logarithm of the non-hydrostatic pressure perturbation, $q=\ln(p/\pi)$, and eliminating the pressures p and π. Transforming the continuity equation in order to eliminate the logarithmic tendency. Further introducing a mean temperature *profile*, $T_*(\zeta)$, and a related geopotential profile, ϕ_*. Perturbation temperature, $T'=T-T_*$, and geopotential, $\phi'=\phi-\phi_*$, are then considered. Finally, modifying the notation for the vertical derivative to ∂_ζ.

$$\ln p = \ln \pi + q = \zeta + Bs + q$$
$$\partial_\zeta \phi_* = -RT_*$$

- **8 variables**: $\mathbf{V}_h, w, T, q, \left(\dot{\zeta}, s\right), \phi', \mu$

- **8 equations** [6 prognostic & 2 diagnostic]:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\mu)\nabla_\zeta \phi' = 0$$
$$\frac{d}{dt}\left[Bs + \ln(1+\partial_\zeta Bs)\right] + \nabla_\zeta \cdot \mathbf{V}_h + (\partial_\zeta + 1)\dot{\zeta} = 0$$
$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(Bs+q)\right] - \kappa\dot{\zeta} + \dot{\zeta}\,\partial_\zeta \ln T_* = 0$$
$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$
$$\frac{T}{T_*} + e^q \frac{\partial_\zeta \phi / RT_*}{1+\partial_\zeta Bs} = 0$$
$$\ldots$$
$$\frac{dw}{dt} - g\mu = 0$$
$$1 + \mu - e^q\left(1 + \frac{\partial_\zeta q}{1+\partial_\zeta Bs}\right) = 0$$

N.B. The variable s is 2-D only and $\dot{\zeta}$ vanishes at the surface. The combination ($\dot{\zeta}$, s) may therefore be considered to constitute a single 3-D variable.

N.B. In the thermodynamic equation, a vertical advection term for T_* has appeared. The term is to be calculated in an Eulerian fashion. Variable T_* is only an option (see **Appendix 13** for the motivation behind this option). When constant, T_*=240K is usually chosen. In the non-hydrostatic case, T_*=200K might be more appropriate.

6) Boundary Conditions

The model top (subscript $_T$) and bottom (subscript $_S$, for earth's surface when talking of the bottom of the atmosphere), are defined to be material surfaces. Therefore we have the following **top and bottom boundary conditions**:

$$\dot{\zeta}_T = \dot{\zeta}(\zeta_T) = 0; \qquad \dot{\zeta}_S = \dot{\zeta}(\zeta_S) = 0$$

In addition, the behavior of theses surfaces must be specified and this will lead to **an additional** condition in the non-hydrostatic case. The bottom surface is assumed to be terrain-following and not moving: $\partial\phi_S / \partial t = 0$. In terrain-following coordinates, this does not imply a vertical velocity that necessarily vanishes at the surface. In effect, $gw_S = [d\phi / dt]_S \neq 0$, generally. At the top, we consider a *flexible surface* whereby the top pressure:

$$p_T = \pi_T$$

is assumed to remain constant. This is automatic in the hydrostatic case since the top surface pressure cannot be anything other than a material hydrostatic pressure surface. In the non-hydrostatic case, to maintain a constant top pressure equal to the constant top hydrostatic pressure surface provides a **top boundary specification** for pressure. In terms of the non-hydrostatic pressure variable q, this becomes:

$$q_T = \ln(p_T / \pi_T) = 0$$

The top surface is then assumed free to move, constrained only by this *artificially imposed* pressure p_T (the atmosphere above only exerting its weight). In fact, this condition is strictly applied at the first momentum level: therefore we set $q_1 = 0$.

N.B. *Open top boundary conditions* are a possibility: see **Appendix 9**.

N.B. For the Limited Area Model (LAM) version, there are **lateral boundary conditions**: see **Appendix 10**.

N.B. *Time varying topography*, $\partial\phi_S / \partial t \neq 0$, is also an option: see **Appendix 11**. In effect, when adapting a given atmospheric state to a higher resolution topography inter(extra)polation is required. Artificially varying ϕ_S in time for a short period is an attractive alternative.

N.B. Initial conditions are time boundary conditions. At initial time, $\mathbf{V}_h, T$ and s are analyzed fields; $\dot{\zeta}, \phi'$ and w (in the hydrostatic case) are diagnosed: see **Appendix 16** for the calculation of $\dot{\zeta}$ and the estimation of w. In the non-hydrostatic case, w and q could be analyzed but usually w is estimated and q set to vanish; μ is diagnosed.

7) Vertical discretization

For vertical discretization, the following choice is made:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\mu}^{\zeta}\right)\nabla_{\zeta}\phi' = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1+\delta_{\zeta}\overline{B}^{\zeta}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \delta_{\zeta}\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(\breve{\overline{B}}^{\zeta}s + \breve{\overline{q}})\right] - \kappa\breve{\dot{\zeta}} + \breve{\dot{\zeta}}\,\frac{\delta_{\zeta}\overline{T_*}^{\zeta}}{T_*} = 0$$

$$\frac{d\breve{\overline{\phi'}}^{\zeta}}{dt} - RT_*\breve{\dot{\zeta}} - gw = 0$$

$$\frac{T}{T_*} - e^{\breve{\overline{q}}^{\zeta}}\,\frac{1-\delta_{\zeta}\phi'/RT_*}{1+\delta_{\zeta}Bs} = 0$$

$$\ldots$$

$$\frac{dw}{dt} - g\mu = 0$$

$$1+\mu - e^{\breve{\overline{q}}^{\zeta}}\left(1+\frac{\delta_{\zeta}q}{1+\delta_{\zeta}Bs}\right) = 0$$

In other words, the derivatives are replaced by simple finite differences represented by the operator δ_{ζ} and averaging operators represented by over bars (linear interpolation typically) are introduced wherever required. From the notation, it may be gathered that $\mathbf{V}_h, q, \phi'$ are defined on the same levels to be called *full* or **momentum** levels. They are staggered with respect to $w, T, \mu, \dot{\zeta}$, placed on *half* or **thermodynamic** levels. **Figures 1** and **2**, next pages, illustrate the *vertical grid,* for the variables on the first, for the equations on the second. Note that the last thermodynamic level is defined half way between the surface and the last momentum level, hence the need for a specific averaging of $\dot{\zeta}$ as well as ϕ' and q for that level (averaging represented by the *curly over bar*). *The full details on discretization* are disclosed in **Appendices 4, 5** and **6.**

N.B. The metric parameter B is provided at the top and bottom and on full levels. It is averaged for the half levels.

N.B. The absence of a thermodynamic level above the first momentum level implies that T and μ will be extrapolated to the momentum level.

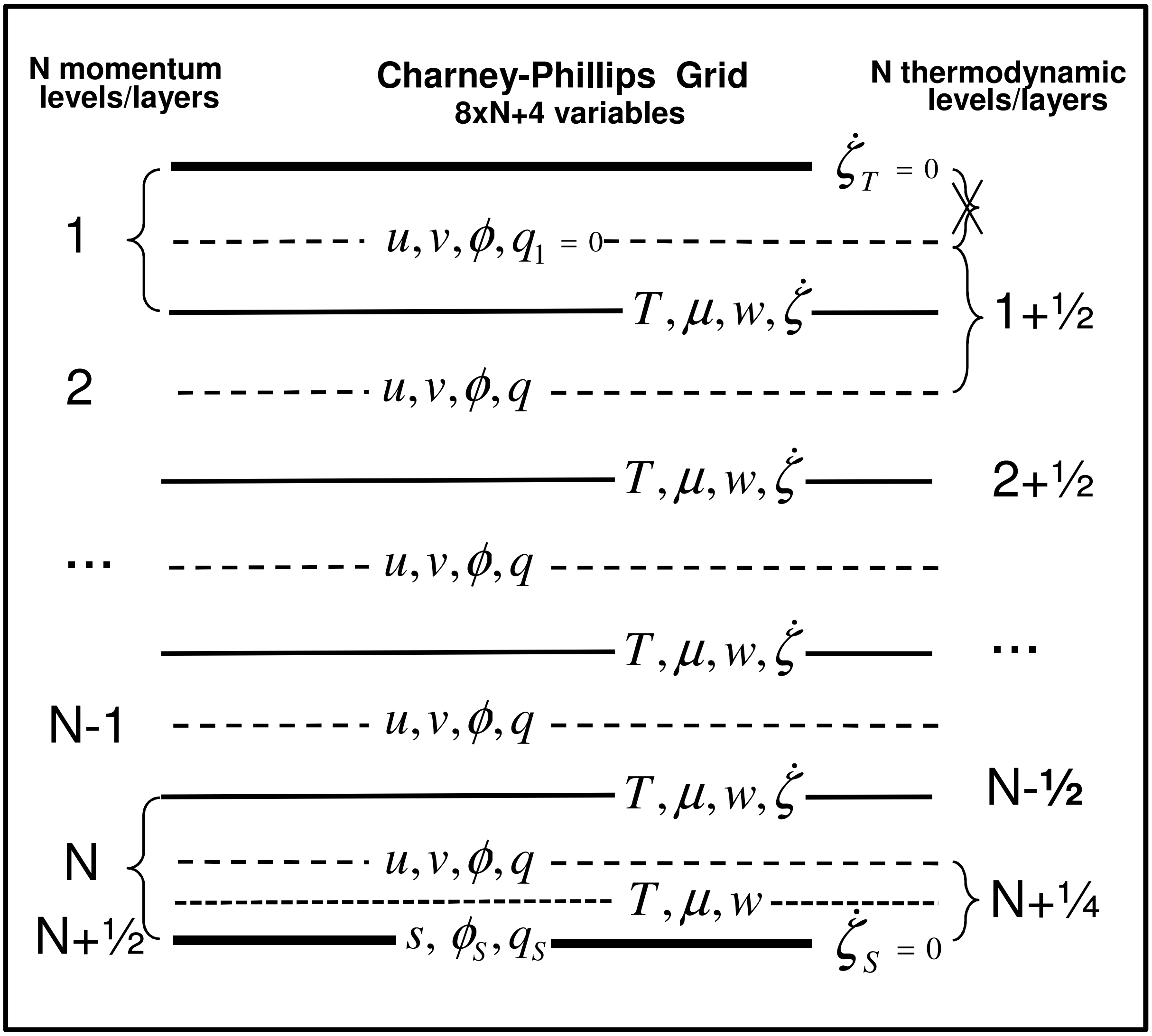


Figure 1. The Charney-Phillips grid, giving the position occupied by each variable in the vertical domain. The model is to be composed of N layers, inside of which (in the middle of which only if the layers are equal) are the momentum levels [1,2,…,N] where the wind components, *u* and *v*, the geopotential, ϕ, and *q* are positioned. Additionally, there are surface values for ϕ and *q* and note that *q* is set to vanish at the first momentum level rather than the top. These N layers are delimited by N-1 interfaces (the solid lines) corresponding to N-1 so-called thermodynamic levels [3/2,…,N-½] where the remaining variables, temperature, *T,* μ and the two vertical motion fields, *w*, $\dot{\zeta}$, are placed, exactly in the middle of the momentum levels. *T,* μ and *w* have an additional level [N+¼] positioned half way between the last momentum level and the surface while $\dot{\zeta}$ is placed directly at the surface [N+½]. Note that there is no thermodynamic level between the top and the first momentum level: an asymmetry therefore between the top and bottom. Of course, $\dot{\zeta}$ vanishes at both boundaries (see **Appendices 4, 5** and **6** for more details on the vertical discretization). In total, there are 8N+4 variables.

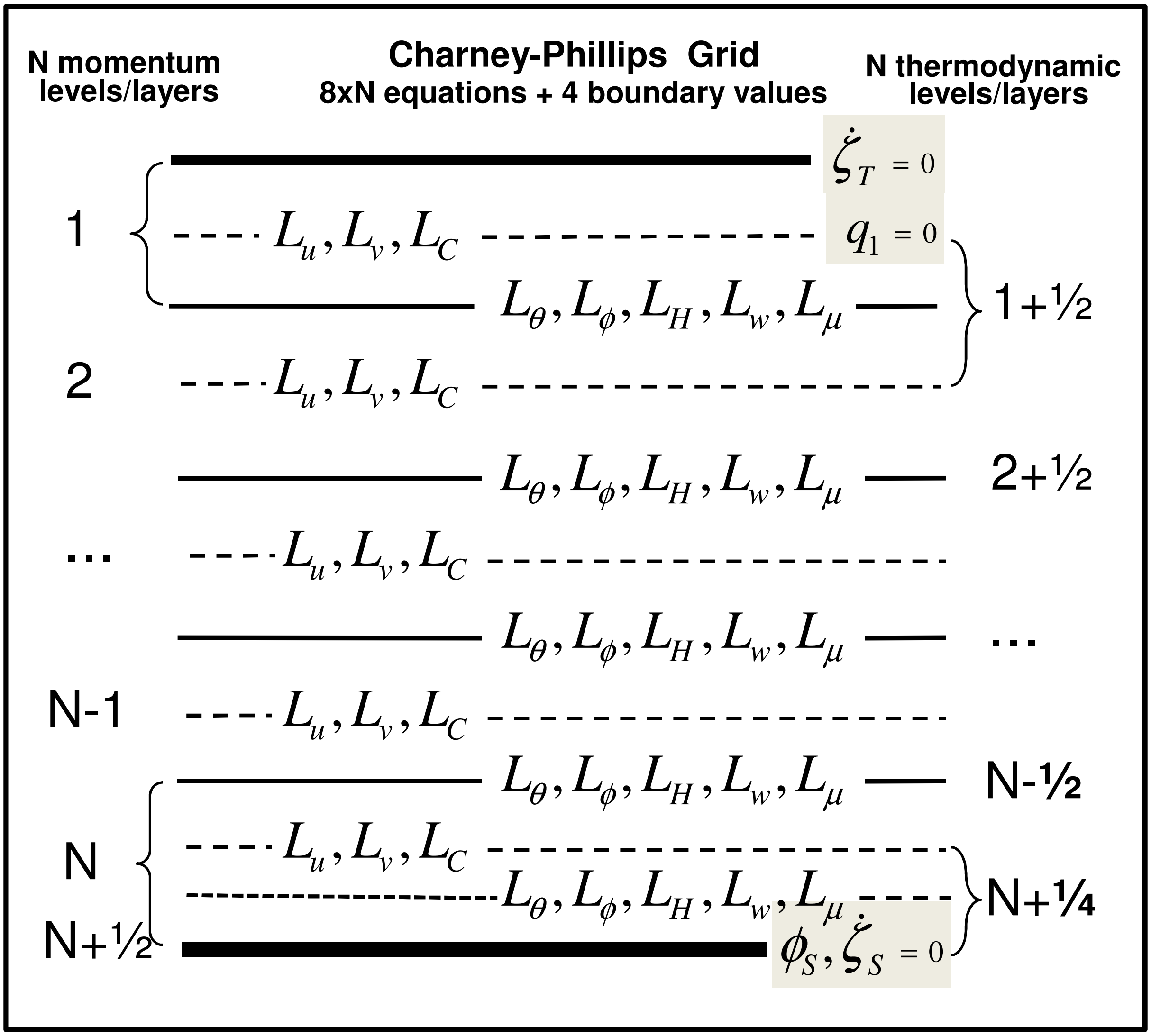


Figure 2. **The Charney-Phillips grid**, giving the position occupied by each of the ten equations (denoted by the letter *L* with appropriate subscript). Three equations, horizontal momentum, subscripts *u* and *v,* and continuity, subscript *C,* are placed on the momentum levels. The remaining five equations: thermodynamic, geopotential displacement, hydrostatic, vertical momentum and vertical acceleration ratio, respectively subscripts θ, ϕ, *H, w* and μ, are placed on the thermodynamic levels. Note that the last level, N+¼, is half way between the last momentum level and the surface. There are thus, in the vertical direction 8N equations for 8N+4 variables (Figure 1). Four extra relations are therefore necessary to complete the system: the boundary conditions: $\dot{\zeta}_S = \dot{\zeta}_T = 0$, the surface geopotential: $\phi_S = g z_S$ where z_S is the terrain, and the non-hydrostatic log-pressure perturbation at the first momentum level: $q_1 = 0$.

8) Horizontal discretization

Previously the horizontal discretization was only discussed in Appendix. This was somewhat justified since the horizontal momentum equations were treated as a single vector equation. In the model three modes remain available with respect to the horizontal: a Limited Area Mode (LAM) and two global modes: a Global Uniform mode (GU) and a so-called Global Yin-yang (GY) mode which may be considered composed of two interacting LAM modes. The GU mode is gradually being abandoned. In LAM and GY modes, the horizontal momentum equation is now decomposed into its components leading to the appearance of explicit metric terms. The presentation of the horizontal discretization is therefore more relevant. The horizontal grid is of the Arakawa-C type, with *u* staggered in the *X*-direction and *v* staggered in the *Y*-direction with respect to all other variables (*X* and *Y* along geometric longitude and latitude respectively). Hence, the following fully discretized equations:

$$\frac{du}{dt} - \left(f + \tfrac{\tan\theta}{a} u\right)\overline{v}^{XY} + R\overline{T}^{X\zeta}\delta_X\left(Bs + q\right) + (1 + \overline{\mu}^{X\zeta})\delta_X\phi' = 0$$

$$\frac{dv}{dt} + \left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T}^{Y\zeta}\delta_Y\left(Bs + q\right) + (1 + \overline{\mu}^{Y\zeta})\delta_Y\phi' = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \delta_\zeta \overline{B}^\zeta s\right)\right] + \delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^\zeta = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(\overline{\breve{B}}^\zeta s + \breve{\overline{q}})\right] - \kappa\breve{\dot{\zeta}} + \breve{\dot{\zeta}}\frac{\delta_\zeta\overline{T_*}^\zeta}{T_*} = 0$$

$$\frac{d\overline{\breve{\phi}'}^\zeta}{dt} - RT_*\breve{\dot{\zeta}} - gw = 0$$

$$\frac{T}{T_*} - e^{\overline{\breve{q}}^\zeta}\,\frac{1 - \delta_\zeta\phi'/RT_*}{1 + \delta_\zeta Bs} = 0$$

...

$$\frac{dw}{dt} - g\mu = 0$$

$$1 + \mu - e^{\overline{\breve{q}}^\zeta}\left(1 + \frac{\delta_\zeta q}{1 + \delta_\zeta Bs}\right) = 0$$

with $\delta_X = \frac{1}{a\cos\theta}\delta_\lambda$ and $\delta_Y = \frac{1}{a}\delta_\theta$. Horizontal averages are usually simple means. The winds in the Coriolis terms may however be interpolated cubically.

9) Semi-Lagrangian **Implicit** time discretization (n.b. not Semi-Implicit)

- Approximating the ***substantial derivatives*** and averaging the ***dynamical forcings***, each of the equations (index *i*) may be formally written and time discretized as follows:

$$\boxed{\frac{dF_i}{dt} + G_i = 0}$$

$$\frac{dF_i}{dt} \approx \frac{F_i^A - F_i^D}{\Delta t}; \quad G_i \approx b^A G_i^A + \left(1 - b^A\right) G_i^D; \qquad 0.5 \le b^A \le 0.6 \ \ \text{(off-centering)}$$

$$\frac{F_i^A - F_i^D}{\Delta t} + b^A G_i^A + \left(1 - b^A\right) G_i^D = 0$$

$A: \ (\mathbf{r}, t)$ Arrival

$D: \ (\mathbf{r} - \Delta\mathbf{r}, t - \Delta t)$ Departure

- Separating the time levels ($\tau = \Delta t b^A; \beta = \left(1 - b^A\right)/b^A$)

$$\frac{F_i^A}{\tau} + G_i^A = \frac{F_i^D}{\tau} - \beta G_i^D \equiv R_i$$

- Decomposing the left-hand side into linear and residual non-linear parts

$$\frac{F_i^A}{\tau} + G_i^A = L_i + N_i = R_i$$

$$L_i \equiv \left(\frac{F_i^A}{\tau} + G_i^A\right)_{lin}; \qquad N_i \equiv \frac{F_i^A}{\tau} + G_i^A - \left(\frac{F_i^A}{\tau} + G_i^A\right)_{lin}$$

- Defining the solution method (a Crank-Nicholson scheme): $L_i = R_i - N_i$

Iterating (*jter*: *departure loop, iter*: *non-linear loop*) :

Do jter=1,2
Do iter=1,2

$$\left(L_i\right)^{iter,jter} = \left(R_i\right)^{jter} - \left(N_i\right)^{iter-1,jter}; \qquad \left(N_i\right)^{0,1} = N_i\left(\mathbf{r}, t - \Delta t\right)$$

end do
end do

$$\left(R_i\right)^{jter} = R_i\left(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{jter}\right); \quad \Delta\mathbf{r}^{jter} = \Delta t\left[b_A \mathbf{v}^A + \left(1 - b_A \mathbf{v}^D\right)\right]$$

$\mathbf{v}^A = \mathbf{v}\left(t, \mathbf{r}\right)^{jter}$; $\mathbf{v}^D = \mathbf{v}(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{jter})$; $\mathbf{v}\left(t\right)^1 = \mathbf{v}(t - \Delta t)$; $\Delta\mathbf{r}^0$ from previous timestep.

N.B. The displacement $\Delta\mathbf{r}^{jter}$ is calculated by the *trapezoidal* rule with off-centering and using cubic interpolations.

N.B. The elliptic solver is called a minimum of four times per time step in this scheme. An alternative, a variant of **SETTLS** scheme used at ECMWF, was developed. It is described in **Appendix 12.** The scheme is more efficient, calling the solver only twice per time step, and marginally stable. Unfortunately, it seems, for the moment, less accurate than the present scheme.

10) The *F's* and the *G's*

$$F_u \equiv u \qquad G_u \equiv -\left(f + \tfrac{\tan\theta}{a} u\right)\overline{v}^{XY} + R\overline{T}^{X\zeta}\delta_X\left(Bs+q\right) + (1+\overline{\mu}^{X\zeta})\,\delta_X\phi'$$

$$F_v \equiv v \qquad G_v \equiv +\left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T}^{Y\zeta}\delta_Y\left(Bs+q\right) + (1+\overline{\mu}^{Y\zeta})\delta_Y\phi'$$

$$F_C \equiv Bs + \ln\left(1+\delta_\zeta\overline{B}^{\zeta}s\right) \qquad G_C \equiv \delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}$$

$$F_\theta \equiv \ln\left(\frac{T}{T_*}\right) - \kappa\left(\breve{\overline{B}}^{\zeta}s + \breve{\overline{q}}\right) \qquad G_\theta \equiv -\kappa\breve{\dot{\zeta}} + \breve{\dot{\zeta}}\,\frac{\delta_\zeta\overline{T_*}^{\zeta}}{T_*}$$

$$F_\phi \equiv \breve{\overline{\phi'}}^{\zeta} \qquad G_\phi \equiv -RT_*\breve{\dot{\zeta}} - gw$$

$$F_H \equiv 0 \qquad G_H \equiv \frac{T}{T_*} - e^{\breve{\overline{q}}^{\zeta}}\,\frac{1-\delta_\zeta\phi'/RT_*}{1+\delta_\zeta Bs} = 0$$

...

$$F_w \equiv w \qquad G_w \equiv -g\mu$$

$$F_\mu \equiv 0 \qquad G_\mu \equiv 1+\mu - e^{\breve{\overline{q}}^{\zeta}}\left[1+\frac{\delta_\zeta q}{1+\delta_\zeta Bs}\right] = 0$$

N.B. Since F_μ=F_H=0 and G_μ=G_H=0, then of course R_μ=R_H=0.

N.B. *The role of diagnostic equations* is to abbreviate other equations. If, in the 6 prognostic equations, we replace the symbols μ and T by their definitions, the diagnostic equations as well as the associated variables vanish.

11) The Previous time step on the Right-Hand Sides: R_i

$$R_i \equiv \frac{F_i^D}{\tau} - \beta G_i^D$$

(dropping the superscript *D*)

$$R_u = \frac{u}{\tau} \qquad - \beta\left[-\left(f + \tfrac{\tan\theta}{a} u\right)\overline{v}^{XY} + R\overline{T}^{X\zeta}\delta_X\left(Bs+q\right) + (1+\overline{\mu}^{X\zeta})\delta_X\phi'\right]$$

$$R_v = \frac{v}{\tau} \qquad - \beta\left[\left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T}^{Y\zeta}\delta_Y\left(Bs+q\right) + (1+\overline{\mu}^{Y\zeta})\,\delta_Y\phi'\right]$$

$$R_C = \frac{Bs + \ln\left(1+\delta_\zeta\overline{B}^{\zeta}s\right)}{\tau} \qquad - \beta\left[\delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}\right]$$

$$R_\theta = \frac{1}{\tau}\ln\left(\frac{T}{T_*}\right) - \kappa\frac{\breve{\overline{B}}^{\zeta}s + \breve{\overline{q}}}{\tau} \qquad - \beta\left[-\kappa\breve{\dot{\zeta}} + \breve{\dot{\zeta}}\frac{\delta_\zeta\overline{T_*}^{\zeta}}{T_*}\right]$$

$$R_\phi = \frac{\breve{\overline{\phi'}}^{\zeta}}{\tau} \qquad - \beta\left(-RT_*\breve{\dot{\zeta}} - gw\right)$$

...

$$R_w = \frac{w}{\tau_{nh}} \qquad - \beta_{nh}\left(-g\mu\right)$$

N.B. In the non-hydrostatic case, off-centering for the sound waves may be set to a larger value. Hence $b_{nh}^A \geq b^A$ and

$$\tau_{nh} = \Delta t b_{nh}^A; \qquad \beta_{nh} = \left(1 - b_{nh}^A\right)/\,b_{nh}^A$$

12) The Left-Hand Side terms: $L_i + N_i$

$$\boxed{L_i + N_i \equiv \frac{F_i^A}{\tau} + G_i^A}$$

Prognostic (dropping the superscript *A*):

$$L_u + N_u = \frac{u}{\tau} - \left(f + \tfrac{\tan\theta}{a}u\right)\overline{v}^{XY} + R\overline{T}^{X\zeta}\delta_X\left(Bs+q\right) + (1+\overline{\mu}^{X\zeta})\delta_X\phi'$$

$$L_v + N_v = \frac{v}{\tau} + \left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T}^{Y\zeta}\delta_Y\left(Bs+q\right) + (1+\overline{\mu}^{Y\zeta})\,\delta_Y\phi'$$

$$L_C + N_C = \frac{Bs + \ln\left(1+\delta_\zeta\overline{B}^\zeta s\right)}{\tau} + \delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^\zeta$$

$$L_\theta + N_\theta = \frac{1}{\tau}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\breve{\dot{\zeta}} + \frac{\breve{\overline{B}}^\zeta s + \breve{\overline{q}}}{\tau}\right] + \breve{\dot{\zeta}}\,\frac{\delta_\zeta\overline{T_*}^\zeta}{T_*}$$

$$L_\phi + N_\phi = \frac{\breve{\overline{\phi}}'^\zeta}{\tau} - RT_*\breve{\dot{\zeta}} - gw$$

...

$$L_w + N_w = \frac{w}{\tau_{nh}} - g\mu$$

Diagnostic:

$$L_H + N_H = \frac{T}{T_*} - e^{\breve{\overline{q}}^\zeta}\,\frac{1-\delta_\zeta\phi'/RT_*}{1+\delta_\zeta Bs} = 0$$

...

$$L_\mu + N_\mu = 1 + \mu - e^{\breve{\overline{q}}^\zeta}\left[1 + \frac{\delta_\zeta q}{1+\delta_\zeta Bs}\right] = 0$$

13) The linear Left-Hand Side terms: L_i

$$L_i \equiv \left[\frac{F_i^A}{\tau} + G_i^A \right]_{linear}$$

Linearizing (approximating the exponentials $\left[e^{\alpha} \approx 1+\alpha\right]$, the (returned) logarithms $\left[\ln(1+\alpha) \approx \alpha\right]$ and the products $\left[(1+\alpha)(1+\beta)^{\pm 1} \approx 1+\alpha \pm \beta\right]$) yields:

$$L_u = \frac{u}{\tau} + \delta_X [\phi' + R\overline{T_*}^{\zeta}(Bs + \boldsymbol{r}'q)]$$

$$L_v = \frac{v}{\tau} + \delta_Y [\phi' + R\overline{T_*}^{\zeta}(Bs + \boldsymbol{r}'q)]$$

$$L_C = \frac{\overline{B}^{\zeta\zeta} s + \delta_\zeta \overline{B}^{\zeta} s}{\tau} + \delta_X u + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta\, v) + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}$$

$$L_\theta = \frac{T'}{\tau T_*} - \kappa_d \left(\breve{\dot{\zeta}} + \frac{\breve{\overline{B}}^{\zeta} s + \boldsymbol{r}'\breve{\overline{q}}^{\zeta}}{\tau} \right)$$

$$L_\phi = \frac{\breve{\overline{\phi'}}^{\zeta}}{\tau} - RT_* \breve{\dot{\zeta}} - gw$$

$$L_H = \frac{T'}{T_*} + \frac{\delta_\zeta [\phi' + R\overline{T_*}^{\zeta}(Bs + \boldsymbol{r}'q)]}{RT_*} - \boldsymbol{r}'\left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right) \neq 0$$

...

$$L_w = \frac{w}{\tau_{nh}} - g\mu$$

$$L_\mu = \mu - \left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right) \neq 0$$

N.B. κ is reduced to κ_d in the linear system.

N.B. The Coriolis and metric terms and the vertical advection of T_* are linear terms but absent from the L's; they are treated as if they were non-linear terms.

N.B. T_* is first defined on thermodynamic levels and averaged for the momentum levels; it is also needed at the first momentum level and the surface.

N.B. $\boldsymbol{r} \le 1$ reduces the influence of non-hydrostatic pressure perturbation, q, in the linear system modifying the so-called epsilon term.

N.B. Definition: $\boldsymbol{r}' = \boldsymbol{r}\,\tau / \tau_{nh}$.

14) The non-linear Left-Hand side terms, N_i, are the left-over differences

$$N_i = \left[\frac{F_i^A}{\tau} + G_i^A \right] - \left[\frac{F_i^A}{\tau} + G_i^A \right]_{linear} = \left[\frac{F_i^A}{\tau} + G_i^A \right] - L_i$$

and therefore:

$$\begin{aligned}
N_u &= -\left(f + \tfrac{\tan\theta}{a} u\right)\overline{v}^{XY} + R\overline{T}^{X\zeta}\delta_X\left(Bs+q\right) + \overline{\mu}^{X\zeta}\delta_X\phi' - \delta_X R\overline{T_*}^{\zeta}\left(Bs + \boldsymbol{r}'q\right) \\
N_v &= +\left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T}^{Y\zeta}\delta_Y\left(Bs+q\right) + \overline{\mu}^{Y\zeta}\delta_Y\phi' - \delta_Y R\overline{T_*}^{\zeta}\left(Bs + \boldsymbol{r}'q\right) \\
N_C &= \frac{1}{\tau}\left[Bs - \overline{B}^{\zeta\zeta}s + \ln\left(1 + \delta_\zeta \overline{B}^{\zeta}s\right) - \delta_\zeta \overline{B}^{\zeta}s\right] \\
N_\theta &= \frac{1}{\tau}\left[\ln\left(\frac{T}{T_*}\right) - \frac{T'}{T_*}\right] - \kappa\frac{\breve{\overline{B}}^{\zeta}s + \breve{\overline{q}}}{\tau} + \kappa_d\frac{\breve{\overline{B}}^{\zeta}s + \boldsymbol{r}'\breve{\overline{q}}}{\tau} + \breve{\dot{\zeta}}\frac{\delta_\zeta\overline{T_*}^{\zeta}}{T_*} \\
N_\phi &= 0 \\
N_H &= -\left[\frac{T'}{T_*} + \frac{\delta_\zeta[\phi' + R\overline{T_*}^{\zeta}\left(Bs + \boldsymbol{r}'q\right)]}{RT_*} - \boldsymbol{r}'\left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right)\right] = -L_H \\
&\;\;\cdots \\
N_w &= 0 \\
N_\mu &= -\left[\mu - \left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right)\right] = -L_\mu
\end{aligned}$$

15) The elliptic problem: preliminary step

A first step consists of eliminating the two diagnostic equations from the left-hand sides, i.e. eliminate L_H, L_μ and N_H, N_μ from the solution system along with two variables: T', μ. This is convenient since the right-hand sides, R_μ, R_H, vanish. In a second step, we eliminate two other variables, $\dot{\zeta}$ and $\dot{q}$, from the linear system, using the kinematic relations, L_ξ, L_q. In this case, the non-linear parts, N_ξ, N_q, vanish but not the right-hand sides, R_ξ, R_q. Third, we introduce the auxiliary variables *P* and *X*:

$$P = \phi' + R\overline{T_*}^{\zeta}\left(Bs + \boldsymbol{r}'q\right)$$
$$X = \dot{\zeta} + \frac{\overline{B}^{\zeta} s + \boldsymbol{r}'\overline{q}^{\zeta}}{\tau}$$

We are left with the following 6 *basic* equations for the final form of the linear system involving now only the following 6 variables, u, v, P, X, w, q:

$$L_u = \frac{u}{\tau} + \delta_X P$$
$$L_v = \frac{v}{\tau} + \delta_Y P$$
$$L_C \equiv L_C' = \delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta X + \overline{X}^{\zeta} - \frac{\boldsymbol{r}'}{\tau}\left(\delta_\zeta \overline{q}^{\zeta} + \overline{q}^{\zeta\zeta}\right)$$
$$L_\theta - \frac{L_H}{\tau} \equiv L_\theta' = -\frac{\delta_\zeta P}{\tau R T_*} - \kappa_d \breve{X} + \frac{\boldsymbol{r}'}{\tau}\left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right)$$
$$L_\phi - \frac{\text{EXTRA}}{\tau} \equiv L_\phi' = \frac{\breve{\overline{P}}^{\zeta}}{\tau} - RT_*\breve{X} - gw$$

...

$$L_w + gL_\mu \equiv L_w' = \frac{w}{\tau_{nh}} - g\left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right)$$

where $\text{EXTRA} = RT_*(\breve{\overline{B}}^{\zeta} s + \boldsymbol{r}'\breve{\overline{q}}) - \breve{\overline{\chi}}^{\zeta}$, with $\chi = R\overline{T_*}^{\zeta}\left(Bs + \boldsymbol{r}'q\right)$, a term allowing *P* to appear in L'_ϕ which then takes the required form on the way to the elliptic problem. EXTRA will therefore be added to the non-linear system. In effect, since *L* = *R* - *N*, modifications to the *L*'s are to be carried on the *R*'s and the *N*'s.

N.B. *X* is not defined at the top, since *q* is not defined there. To validate the difference and average operations in the continuity equation we consider *truncated operators* at the top momentum level:

$$\left[\delta_\zeta X\right]_1 = X_{\frac{3}{2}} / \Delta\zeta_1; \quad \left[\overline{X}^{\zeta}\right]_1 = \varpi_1^+ X_{\frac{3}{2}}$$

The result on $\dot{\zeta}$ is un-changed since it vanishes at top. Once the operators are truncated, there is no need to carry it at the top and in fact we don't.

For the non-linear system, we obtain (noting that $N_\phi = 0$) :

$$N_u = -\left(f + \tfrac{\tan\theta}{a}u\right)\overline{v}^{XY} + R\overline{T'}^{X\zeta}\delta_X\left(Bs+q\right) + \overline{\mu}^{X\zeta}\delta_X\phi' + (1-r')R\overline{T_*}^{\zeta}\delta_X q$$

$$N_v = \left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T'}^{Y\zeta}\delta_Y\left(Bs+q\right) + \overline{\mu}^{Y\zeta}\delta_Y\phi' + (1-r')R\overline{T_*}^{\zeta}\delta_Y q$$

$$N_C \equiv N_C' = \frac{1}{\tau}\left[Bs - \overline{B}^{\zeta\zeta}s + \ln\left(1+\delta_\zeta\overline{B}^{\zeta}s\right) - \delta_\zeta\overline{B}^{\zeta}s\right]$$

$$N_\theta - \frac{1}{\tau}N_H \equiv N_\theta' = \frac{1}{\tau}\ln\left(\frac{T}{T_*}\right) + \frac{\delta_\zeta[\phi' + R\overline{T_*}^{\zeta}(Bs + \boldsymbol{r}'q)]}{\tau RT_*} - \boldsymbol{r}'\frac{\delta_\zeta q + \breve{\overline{q}}^{\zeta}}{\tau} - \kappa\frac{\breve{\overline{B}}^{\zeta}s + \breve{\overline{q}}}{\tau} + \kappa_d\frac{\breve{\overline{B}}^{\zeta}s + \boldsymbol{r}'\breve{\overline{q}}}{\tau} + \breve{\zeta}\frac{\delta_\zeta\overline{T_*}^{\zeta}}{T_*}$$

$$\frac{\text{EXTRA}}{\tau} \equiv N_\phi'$$

…

$$N_w + gN_\mu \equiv N_w' = -g\left(\mu - \delta_\zeta q - \breve{\overline{q}}^{\zeta}\right)$$

For the right-hand sides (noting that $R_H = R_\mu = 0$), we have nothing to calculate.

16) The linear elliptic problem: final step

At this point the system of six equations is formally identical to previous model versions, except for the presence of curly operators:

$$L_u = \frac{u}{\tau} + \delta_X P$$

$$L_v = \frac{v}{\tau} + \delta_Y P$$

$$L'_C = \delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta X + \overline{X}^\zeta - \frac{\boldsymbol{r}'}{\tau}\left(\delta_\zeta \overline{q}^\zeta + \overline{q}^{\zeta\zeta}\right)$$

$$L'_\theta = -\frac{\delta_\zeta P}{\tau R T_*} - \kappa_d \breve{X} + \frac{\boldsymbol{r}'}{\tau}\left(\delta_\zeta q + \breve{\overline{q}}^\zeta\right)$$

$$L'_\phi = \frac{\breve{\overline{P}}^\zeta}{\tau} - RT_*\breve{X} - gw$$

$$\ldots$$

$$L'_w = \frac{w}{\tau_{nh}} - g\left(\delta_\zeta q + \breve{\overline{q}}^\zeta\right)$$

The fact that T_* varies in the vertical will however produce variable coefficients Γ and ε'. The number of equations can then be reduced to three essentially as before:

$$\delta_X L_u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, L_v\right) - \frac{1}{\tau}\left(L'_C - \frac{\boldsymbol{r}}{g\tau_{nh}}\overrightarrow{\overline{L'_w}}^\zeta\right) \equiv L''_C = \nabla^2_\zeta P - \frac{1}{\tau}\left(\delta_\zeta X + \overline{X}^\zeta\right) + \frac{g\varepsilon'}{\tau R T_*}\vec{w}^\zeta$$

$$\frac{1}{\tau(\kappa_d + \varepsilon')}\left(L'_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}L'_w + \frac{\boldsymbol{r}}{g^2\tau^2_{nh}}L'_\phi\right) \equiv \breve{L}''_\theta = -\Gamma\left(\delta_\zeta P - \varepsilon'\breve{\overline{P}}^\zeta\right) - \frac{\breve{X}}{\tau}$$

$$\ldots$$

$$\frac{1}{\tau(\kappa_d + \varepsilon')}\left(L'_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}L'_w - \frac{\kappa_d}{RT_*}L'_\phi\right) \equiv L''_\phi = -\Gamma\left(\delta_\zeta P + \kappa_d\breve{\overline{P}}^\zeta\right) + \frac{gw}{\tau R T_*}$$

where $\nabla^2_\zeta P = \delta_{XX} P + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\,\delta_Y P\right)$, $\Gamma = \dfrac{1}{(\kappa_d + \varepsilon')\tau^2 R T_*}$ and $\varepsilon' = \boldsymbol{r}\varepsilon = \boldsymbol{r}\dfrac{RT_*}{g^2\tau^2_{nh}}$, hence the so-called *modified epsilon formulation*.

N.B. The following must hold to complete the elimination of *q*: $\overrightarrow{\overline{\delta_\zeta q}}^\zeta + \overrightarrow{\overline{\breve{\overline{q}}^\zeta}}^\zeta = \delta_\zeta \overline{q}^\zeta + \overline{q}^{\zeta\zeta}$. This is possible in virtue of the forced commutation rule between the difference and mean operators (see **Appendix 5**), also taking into account the truncation of operators at the first level and the fact that q_1=0. The averaging must be adapted for the last thermodynamic level: hence the introduction of an *arrowed* averaging operator (see **Appendix 4**).

The averages $\breve{X}^{\zeta}$, $\breve{\overline{P}}^{\zeta}$ and $\vec{w}^{\zeta}$ modify the final elimination, again only in relation to the last thermodynamic level and therefore only for the bottom row of the matrices involved in the elliptic problem. The next step brings us to L_P:

$$L_C'' - \hat{\delta}_{\zeta}\breve{L}_{\theta}'' - \overrightarrow{\overline{L}_{\theta}''}^{\zeta} - \overrightarrow{\varepsilon' L_{\phi}''}^{\zeta} \equiv L_P = \nabla_{\zeta}^2 P + \hat{\delta}_{\zeta}\Gamma\left(\delta_{\zeta}P - \varepsilon'\breve{\overline{P}}^{\zeta}\right) + \overrightarrow{\Gamma(\delta_{\zeta}P - \varepsilon'\breve{\overline{P}}^{\zeta})}^{\zeta} + \overrightarrow{\Gamma\varepsilon'(\delta_{\zeta}P + \kappa_d\breve{\overline{P}}^{\zeta})}^{\zeta}$$

Here we have introduced and applied the difference operator $\hat{\delta}_{\zeta}$ allowing the elimination of X since

$$\hat{\delta}_{\zeta}\breve{X} + \vec{\breve{X}} = \delta_{\zeta}X + \overline{X}^{\zeta}$$

Further introducing $\tilde{P}$ (differing from P again at the last level only) such that

$$\hat{\delta}_{\zeta}\Gamma\varepsilon'\breve{\overline{P}}^{\zeta} - \overrightarrow{\Gamma\varepsilon'\delta_{\zeta}P}^{\zeta} = \tilde{P}\hat{\delta}_{\zeta}\Gamma\varepsilon'$$

the final result is

$$L_P = \nabla_{\zeta}^2 P + \breve{\delta}_{\zeta}\Gamma\delta_{\zeta}P + \overrightarrow{\Gamma\delta_{\zeta}P}^{\zeta} - \tilde{P}\breve{\delta}_{\zeta}\Gamma\varepsilon' - (1-\kappa_d)\overrightarrow{\Gamma\varepsilon'\breve{\overline{P}}^{\zeta}}^{\zeta}$$

This is a generalization of the equation in previous model versions. With T_* constant, Γ and ε' become constant

$$L_P = \nabla_{\zeta}^2 P + \Gamma\left[\hat{\delta}_{\zeta}\delta_{\zeta}P + \overrightarrow{\delta_{\zeta}P}^{\zeta} - (1-\kappa_d)\varepsilon'\overrightarrow{\breve{\overline{P}}^{\zeta}}^{\zeta}\right]$$

The ε' formulation, with $\varepsilon' = \mathbf{r}\varepsilon$ and $\mathbf{r}<1$, is also a generalization as well as the introduction of the curly average operator $(\breve{\ })$, lifting the last thermodynamic level from the surface to the mid-point between the surface and last momentum level and forcing the introduction of the curly difference $\breve{\delta}$ and arrowed average $(\vec{\ })$, all operators which differ from the originals for the last levels only, all depending on a single weight $\varpi_*^+ < 1$. With $\mathbf{r} = \varpi_*^+ = 1$, the equation of document GEM4.2 (still a valid option) is recovered:

$$L_P = \nabla_{\zeta}^2 P + \Gamma\left[\delta_{\zeta}\delta_{\zeta}P + \overline{\delta_{\zeta}P}^{\zeta} - (1-\kappa_d)\varepsilon\overline{\overline{P}^{\zeta}}^{\zeta}\right]$$

The above equation for P corresponds to an elliptic problem and still needs a surface boundary condition. In effect, there are N+1unknowns remaining, [P_k $(k=1,N)$, plus $P_{N+\frac{1}{2}}$], but only N equations. However,

$$\frac{P_{N+\frac{1}{2}} - \phi_S}{\tau R T_{*N+\frac{1}{2}}} = \frac{s + \boldsymbol{r}' q_{N+\frac{1}{2}}}{\tau} = X_{N+\frac{1}{2}}$$

since $\dot{\zeta}_{N+\frac{1}{2}}=0$. On the other hand, eliminating $X_{N-\frac{1}{2}}$ from

$$\left[\breve{L}''_\theta+\Gamma(\delta_\zeta P-\varepsilon\breve{\overline{P}}^\zeta)\right]_{N+\frac{1}{4}}=-\frac{\breve{X}_{N+\frac{1}{4}}}{\tau}=-\frac{1}{\tau}\left(\varpi_*^+X_{N+\frac{1}{2}}+\varpi_*^-X_{N-\frac{1}{2}}\right)$$

$$\left[L''_\theta+\Gamma(\delta_\zeta P-\varepsilon\overline{P}^\zeta)\right]_{N-\frac{1}{2}}=-\frac{X_{N-\frac{1}{2}}}{\tau}$$

gives

$$\frac{1}{\varpi_*^+}\left[(L''_\theta)_{N+\frac{1}{4}}-\varpi_*^-(L''_\theta)_{N-\frac{1}{2}}\right]\equiv(L''_\theta)_{N+\frac{1}{2}}$$

where

$$(L''_\theta)_{N+\frac{1}{2}}=-\frac{1}{\varpi_*^+}\left[\Gamma(\delta_\zeta P-\varepsilon\breve{\overline{P}}^\zeta)\right]_{N+\frac{1}{4}}+\frac{\varpi_*^-}{\varpi_*^+}\left[\Gamma(\delta_\zeta P-\varepsilon\overline{P}^\zeta)\right]_{N-\frac{1}{2}}-\frac{1}{\tau}X_{N+\frac{1}{2}}$$

Hence, the combination

$$\left[L''_\theta-\frac{\phi_S}{\tau^2RT_*}\right]_{N+\frac{1}{2}}\equiv(L'''_\theta)_{N+\frac{1}{2}}$$

leads to a relation involving $P_{N+\frac{1}{2}}$ in terms of P_N and P_{N-1} only:

$$(L'''_\theta)_{N+\frac{1}{2}}=-\frac{1}{\varpi_*^+}\left[\Gamma(\delta_\zeta P-\varepsilon\breve{\overline{P}}^\zeta)\right]_{N+\frac{1}{4}}+\frac{\varpi_*^-}{\varpi_*^+}\left[\Gamma(\delta_\zeta P-\varepsilon\overline{P}^\zeta)\right]_{N-\frac{1}{2}}-\frac{P_{N+\frac{1}{2}}}{\tau^2RT_{*N+\frac{1}{2}}}$$

Rewritten as follows

$$P_{N+\frac{1}{2}}=\alpha_SP_N+\beta_SP_{N-1}-C_S(L'''_\theta)_{N+\frac{1}{2}}$$

it is combined with L_P at level N to get, on the left-hand side,

$$(L_P)_N+C''_S(L'''_\theta)_{N+\frac{1}{2}}\equiv(L'_P)_N$$

and, on the right-hand side, the same relation, *except that* $P_{N+\frac{1}{2}}$ *is replaced by* $\alpha_SP_N+\beta_SP_{N-1}$. See **Appendix 4** for the full details of these derivations including definitions of difference and mean operators as well as definitions of parameters α_S, β_S, C_S and C''_S.

N.B. Here the closed top boundary condition has been described. An open top boundary condition is considered and described in **Appendix 9**.

17) The elliptic problem: non-linear step

To find the solution to the non-linear problem we need to perform the following operations iteratively

$$(L_u)^{1+iter,\,jter} = (R_u)^{jter} - (N_u)^{iter,\,jter}$$
$$(L_v)^{1+iter,\,jter} = (R_v)^{jter} - (N_v)^{iter,\,jter}$$
$$(L''_C)^{1+iter,\,jter} = (R''_C)^{jter} - (N''_C)^{iter,\,jter}$$
$$(L''_\theta)^{1+iter,\,jter} = (R''_\theta)^{jter} - (N''_\theta)^{iter,\,jter}$$
$$(L''_\phi)^{1+iter,\,jter} = (R''_\phi)^{jter} - (N''_\phi)^{iter,\,jter}$$
$$\ldots$$
$$(L'_w)^{1+iter,\,jter} = (R_w)^{jter} - (N'_w)^{iter,\,jter}$$

$$(L_P)^{1+iter,\,jter} = (R_P)^{jter} - (N_P)^{iter,\,jter}$$

In order to obtain $R''_C, R''_\theta, R''_\phi, R_P$ and $N''_C, N''_\theta, N''_\phi, N_P$, the *R*'s and *N*'s are transformed like was done for the *L*'s, i.e. we compute the *R*'s:

$$\delta_X R_u + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta R_v) - \frac{1}{\tau}\left(R_C - \frac{r}{g\tau_{nh}}\overrightarrow{R_w}^{\zeta}\right) \equiv R''_C$$

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(R'_\theta + \frac{r}{g\tau_{nh}}R_w + \frac{r}{g^2\tau_{nh}^2}R'_\phi\right) \equiv \breve{R}''_\theta$$

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(R'_\theta + \frac{r}{g\tau_{nh}}R_w - \frac{\kappa_d}{RT_*}R'_\phi\right) \equiv R''_\phi$$

$$R''_C - \left(\widehat{\delta}_\zeta R''_\theta + \overline{\breve{R}''_\theta}^{\zeta} + \overrightarrow{\varepsilon' R''_\phi}^{\zeta}\right) \equiv R_P$$

$$\frac{1}{\varpi_*^+}[(R''_\theta)_{N+\frac{1}{4}} - \varpi_*^-(R''_\theta)_{N-\frac{1}{2}}] = (R''_\theta)_{N+\frac{1}{2}}$$

$$\left[R''_\theta - \frac{\phi_S}{\tau^2 RT_*}\right]_{N+\frac{1}{2}} = (R'''_\theta)_{N+\frac{1}{2}}$$

$$(R_P)_N + C''_S(R'''_\theta)_{N+\frac{1}{2}} = (R'_P)_N$$

Similarly for the *N*'s:

$$\delta_X N_u + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta N_v) - \frac{1}{\tau}\left(N_C - \frac{r}{g\tau_{nh}}\overrightarrow{N'_w}^{\zeta}\right) \equiv N''_C$$

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(N'_\theta + \frac{r}{g\tau_{nh}}N'_w + \frac{r}{g^2\tau_{nh}^2}N'_\phi\right) \equiv \breve{N}''_\theta$$

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(N'_\theta + \frac{r}{g\tau_{nh}}N'_w - \frac{\kappa_d}{RT_*}N'_\phi\right)_w \equiv N''_\phi$$

$$N''_C - \left(\widehat{\delta}_\zeta N''_\theta + \overline{\breve{N}''_\theta}^{\zeta} + \overrightarrow{\varepsilon' N''_\phi}^{\zeta}\right) \equiv N_P$$

$$\frac{1}{\varpi_*^+}[(N''_\theta)_{N+\frac{1}{4}} - \varpi_*^-(N''_\theta)_{N-\frac{1}{2}}] = (N''_\theta)_{N+\frac{1}{2}}$$

$$(N''_\theta)_{N+\frac{1}{2}} = (N'''_\theta)_{N+\frac{1}{2}}$$

$$(N_P)_N + C''_S(N'''_\theta)_{N+\frac{1}{2}} = (N'_P)_N$$

N.B. We have R_w but N'_w. Novelties are R'_θ and the fact that $N'_\phi \neq 0$.

18) The elliptic problem: back substitution

The following equations give in a straight forward manner the 8 prognostic variables $u, v, w, q, \ (s, \dot{\zeta}), \dot{\xi}, \dot{q}$ and ϕ':

$$w: \qquad w = \frac{\tau\, RT_*}{g}[\breve{R}''_\phi - \breve{N}''_\phi + \Gamma(\delta_\zeta P + \kappa_d \bar{\bar{\breve{P}}}^\zeta)]$$

$$q: \ \delta_\zeta q + \breve{\bar{q}}^\zeta = -\frac{1}{g}\left[R_w - N'_w - \frac{w}{\tau_{nh}}\right]; \qquad q_1 = 0$$

...

$$u: \qquad u = \tau[R_u - N_u - \delta_X P]$$

$$v: \qquad v = \tau[R_v - N_v - \delta_Y P]$$

$$s: \qquad s = \frac{P_S - \phi_S}{RT_{*S}} - \boldsymbol{r}' q_S$$

$$\dot{\zeta}: \qquad \dot{\zeta} = -\tau[R''_\theta - N''_\theta + \Gamma(\delta_\zeta P - \varepsilon \overline{P}^\zeta)] - \frac{\overline{B}^\zeta s - \boldsymbol{r}'\overline{q}^\zeta}{\tau}$$

$$\phi': \qquad \phi' = P - R\overline{T_*}^\zeta (Bs + \boldsymbol{r}' q)$$

Finally, we may compute μ and T diagnostically:

$$1 + \mu = e^{\bar{\breve{q}}^\zeta}\left[1 + \frac{\delta_\zeta q}{1 + \delta_\zeta Bs}\right]$$

$$\frac{T}{T_*} = e^{\bar{\breve{q}}^\zeta}\, \frac{1 - \delta_\zeta \phi' / RT_*}{1 + \delta_\zeta Bs}$$

THE END

Appendix 1. Virtual temperature

In presence of water vapor q_v and various types of hydrometeors q_i, the density of atmospheric substance is given by

$$\rho = \rho\left(q_d + q_v + \sum q_i\right)$$

where q_d is the dry air specific mass. The equation of state is given by

$$\begin{aligned} p &= \rho\left(R_d q_d + R_v q_v\right)T \\ &= \rho R_d\left(1 + \delta_v q_v - \sum q_i\right)T \end{aligned}$$

where $\delta_v = R_v / R_d - 1 \approx 0.6$ and we rewrite the equation of state as follows:

$$p = \rho R_d T_v$$

defining virtual temperature thus

$$T_v = T\left(1 + \delta_v q_v - \sum q_i\right)$$

Rewriting the equations to appear in terms of virtual temperature, the equations of **section 1** may then be replaced by the following:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + R_d T_v \nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{dT_v}{dt} - \kappa T_v \frac{d \ln p}{dt} = \frac{Q_v}{c_p} = \frac{R}{R_d}\frac{Q}{c_p} + T\left(\delta_v \frac{dq_v}{dt} - \sum \frac{dq_i}{dt}\right)$$

$$\frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{V} = 0$$

$$\rho - \frac{p}{R_d T_v} = 0$$

From the point of view of the pure dynamics $\left(\mathbf{F} = Q_v = 0\right)$, these equations are formally identical to those in **section 1** in which R would take the dry air constant value and temperature be replaced by virtual temperature. The advantage of this formulation is of course the fact that the parameter R no longer varies while all of the virtual effects, including *water vapor buoyancy* and *condensed water loading* effects, are implicitly taken into account. The replacement of κ by κ_d in the thermodynamic equation would however constitute an approximation and is avoided:

$$\begin{aligned} \kappa &= \frac{R}{c_p} = \frac{R_d q_d + R_v q_v}{c_{pd} q_d + c_{pv} q_v} = \frac{R_d}{c_{pd}} \frac{1 + \left(R_v / R_d - 1\right) q_v}{1 + \left(c_{pv} / c_{pd} - 1\right) q_v} \approx \kappa_d \left[1 + \left(\frac{R_v}{R_d} - \frac{c_{pv}}{c_{pd}}\right) q_v\right] \\ &\approx \kappa_d \left[1 + \left(\frac{461.51}{287.05} - \frac{1850}{1005}\right) q_v\right] \approx \kappa_d \left[1 + \left(1.608 - 1.841\right) q_v\right] \approx \kappa_d \left(1 - 0.233 q_v\right) \end{aligned}$$

(values taken from Atmospheric Thermodynamics, Iribarne & Godson)

Appendix 2. Coordinate transformation rules

Appendix 2a. Invariance of the total derivative

By the *chain rule* we first verify the invariance of the total derivative *df/dt* under a general coordinate transformation. In effect, if we consider $f(x,y,z,t)$, then:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{x,y,z} + \left(\frac{\partial f}{\partial x}\right)_{y,z,t} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{x,z,t} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{x,y,t} \frac{dz}{dt}$$

while for $f(x,y,\zeta,t)$, we naturally have:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{x,y,\zeta} + \left(\frac{\partial f}{\partial x}\right)_{y,\zeta,t} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{x,\zeta,t} \frac{dy}{dt} + \left(\frac{\partial f}{\partial \zeta}\right)_{x,y,t} \frac{d\zeta}{dt}$$

Here we only have changed the vertical coordinate from z to ζ with the result that the horizontal components of the velocity $(dx/dt, dy/dt) = (u,v) = \mathbf{V}_h$ remain unchanged. The vertical motion though has transformed from $dz/dt = w$ into $d\zeta / dt = \dot{\zeta}$. Shortening the notation, we also write the above relations respectively as follows:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_z + u\left(\frac{\partial f}{\partial x}\right)_z + v\left(\frac{\partial f}{\partial y}\right)_z + w\frac{\partial f}{\partial z} = \frac{\partial f}{\partial t} + \mathbf{V}_h \cdot \nabla_z f + w\frac{\partial f}{\partial z}$$

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_\zeta + u\left(\frac{\partial f}{\partial x}\right)_\zeta + v\left(\frac{\partial f}{\partial y}\right)_\zeta + \dot{\zeta}\frac{\partial f}{\partial \zeta} = \frac{\partial f}{\partial t} + \mathbf{V}_h \cdot \nabla_\zeta f + \dot{\zeta}\frac{\partial f}{\partial \zeta}$$

Thus we minimized the indices. We also introduced the vector notation for the 'horizontal' part of the advection operator. Note though that the new coordinate ζ is generally curvilinear and non-orthogonal and the scalar product must be interpreted with care (see appendix 2c)

Appendix 2b. Transformation rules for derivatives.

It is remarkable that not only can all these rules *be recovered* from the invariance of the total derivative but also that these derivative transformation rules suffice to transform the Euler equations. In effect, the three velocity components may be treated as three independent scalars ('pseudo-scalars'), the velocity vector not being transformed. We are left though with a 'hybrid' system since maintaining two vertical velocities w and $\dot{\eta}$ or $\dot{\zeta}$ and therefore needing an additional [prognostic when $(\partial z / \partial t)_\zeta \neq 0$), diagnostic otherwise] equation. A complete transformation to a time-varying non-orthogonal curvilinear coordinate, a complete elimination of w, is of course possible but then the notions of four-dimensional tensor calculus is very useful (see appendix 2d).

The transformation rules may be obtained by equating the above two relations. In effect, we must have

$$0=\left(\frac{\partial f}{\partial t}\right)_z-\left(\frac{\partial f}{\partial t}\right)_\zeta+u\left[\left(\frac{\partial f}{\partial x}\right)_z-\left(\frac{\partial f}{\partial x}\right)_\zeta\right]+v\left[\left(\frac{\partial f}{\partial y}\right)_z-\left(\frac{\partial f}{\partial y}\right)_\zeta\right]+w\frac{\partial f}{\partial z}-\dot{\zeta}\frac{\partial f}{\partial \zeta}$$

and since

$$w=\frac{dz}{dt}=\left(\frac{\partial z}{\partial t}\right)_\zeta+u\left(\frac{\partial z}{\partial x}\right)_\zeta+v\left(\frac{\partial z}{\partial y}\right)_\zeta+\dot{\zeta}\frac{\partial z}{\partial \zeta}$$

then

$$\begin{aligned}0=&\left[\left(\frac{\partial f}{\partial t}\right)_z-\left(\frac{\partial f}{\partial t}\right)_\zeta+\left(\frac{\partial z}{\partial t}\right)_\zeta\frac{\partial f}{\partial z}\right]+u\left[\left(\frac{\partial f}{\partial x}\right)_z-\left(\frac{\partial f}{\partial x}\right)_\zeta+\left(\frac{\partial z}{\partial x}\right)_\zeta\frac{\partial f}{\partial z}\right]\\&+v\left[\left(\frac{\partial f}{\partial y}\right)_z-\left(\frac{\partial f}{\partial y}\right)_\zeta+\left(\frac{\partial z}{\partial y}\right)_\zeta\frac{\partial f}{\partial z}\right]+\dot{\zeta}\left[\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \zeta}\right]\end{aligned}$$

Each bracket must vanish independently. Therefore the rules are:

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_z&=\left(\frac{\partial f}{\partial t}\right)_\zeta-\left(\frac{\partial z}{\partial t}\right)_\zeta\frac{\partial f}{\partial z}\\\left(\frac{\partial f}{\partial x}\right)_z&=\left(\frac{\partial f}{\partial x}\right)_\zeta-\left(\frac{\partial z}{\partial x}\right)_\zeta\frac{\partial f}{\partial z}\\\left(\frac{\partial f}{\partial y}\right)_z&=\left(\frac{\partial f}{\partial y}\right)_\zeta-\left(\frac{\partial z}{\partial y}\right)_\zeta\frac{\partial f}{\partial z}\\\frac{\partial f}{\partial z}&=\frac{\partial \zeta}{\partial z}\frac{\partial f}{\partial \zeta}\end{aligned}$$

Appendix 2c. Vectors in non-orthogonal curvilinear coordinates

In non-orthogonal curvilinear coordinates $\hat{\mathbf{x}}=\left(\hat{x}^1,\hat{x}^2,\hat{x}^3\right)$ (see Dutton, John A, *The Ceaseless Wind*, chapters 5 and 7), there appear two sets of basis vectors (usually not even of unit length) and two sets of vector components. Applying the chain rule, we obtain the following two expansions (summation convention):

$$\begin{aligned}d\mathbf{x}&=\frac{\partial \mathbf{x}}{\partial \hat{x}^j}d\hat{x}^j & \mathrm{d}\hat{x}^i&=\frac{\partial \hat{x}^i}{\partial x^j}dx^j=\left(\nabla\hat{x}^i\right)\cdot d\mathbf{x}\\&=\boldsymbol{\tau}_j d\hat{x}^j & &=\boldsymbol{\eta}^{\mathrm{i}}\cdot d\mathbf{x}\end{aligned}$$

where $\boldsymbol{\tau}_j$ is covariant: *tangent to the curve along which only* $\hat{x}^j$ *varies* and $\boldsymbol{\eta}^i$ is contra- variant: *normal to the surface* $\hat{x}^i$ = const. and we have the orthogonality relation

$$\boldsymbol{\tau}_j \boldsymbol{\eta}^i = \delta^i_j$$

Representing a vector **A** as

$$\mathbf{A} = A^k \boldsymbol{\tau}_k = A_k \boldsymbol{\eta}^k$$

we may recover the components [A_k (A^k): covariant (contravariant) components] using the above orthogonality relation:

$$A^i = \mathbf{A} \cdot \boldsymbol{\eta}^i = A^j \boldsymbol{\tau}_j \cdot \boldsymbol{\eta}^i$$

$$A_j = \mathbf{A} \cdot \boldsymbol{\tau}_j = A_i \boldsymbol{\eta}^i \cdot \boldsymbol{\tau}_j$$

The scalar product is

$$\mathbf{A} \cdot \mathbf{B} = A^k B_k = A_k B^k$$

Therefore in generalized vertical coordinate $\hat{\mathbf{x}} = (x, y, \zeta)$ the basis vectors become [the original orthogonal Cartesian coordinate being $\mathbf{x} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$]

$$\boldsymbol{\eta}^1 = \nabla x = \mathbf{i} \qquad \boldsymbol{\tau}_1 = \frac{\partial \mathbf{x}}{\partial x} = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}$$

$$\boldsymbol{\eta}^2 = \nabla y = \mathbf{j} \qquad \boldsymbol{\tau}_2 = \frac{\partial \mathbf{x}}{\partial y} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}$$

$$\boldsymbol{\eta}^3 = \nabla \zeta \qquad \boldsymbol{\tau}_3 = \frac{\partial \mathbf{x}}{\partial \zeta} = \frac{\partial z}{\partial \zeta}\mathbf{k}$$

The contravariant components of the velocity vector $u^i = \mathbf{V} \cdot \boldsymbol{\eta}^i$ are found to be

$$u,\ v, \mathbf{V} \cdot \nabla \zeta = \dot{\zeta}$$

while the covariant components of the gradient $\partial f / \partial \hat{x}^j = \nabla f \cdot \boldsymbol{\tau}_i$ are found to be

$$\left(\frac{\partial f}{\partial x}\right)_\zeta, \left(\frac{\partial f}{\partial y}\right)_\zeta, \frac{\partial f}{\partial \zeta}$$

and the vector product $\mathbf{V} \cdot \nabla f$ may be computed as follows:

$$\mathbf{V} \cdot \nabla f = u^i \boldsymbol{\tau}_i \cdot \boldsymbol{\eta}^j \frac{\partial f}{\partial \hat{x}^j}$$

$$= \left[u\left(\mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}\right) + v\left(\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}\right) + \dot{\zeta}\frac{\partial z}{\partial \zeta}\mathbf{k} \right] \cdot \left[\left(\frac{\partial f}{\partial x}\right)_\zeta \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_\zeta \mathbf{j} + \frac{\partial f}{\partial \zeta}\nabla \zeta \right]$$

$$= u\left[\left(\frac{\partial f}{\partial x}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\mathbf{i}\cdot\nabla\zeta + \frac{\partial z}{\partial x}\mathbf{k}\cdot\nabla\zeta\right)\right] + v\left[\left(\frac{\partial f}{\partial y}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\mathbf{j}\cdot\nabla\zeta + \frac{\partial z}{\partial y}\mathbf{k}\cdot\nabla\zeta\right)\right] + \dot{\zeta}\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial \zeta}\mathbf{k}\cdot\nabla\zeta$$

$$= u\left[\left(\frac{\partial f}{\partial x}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial \zeta}{\partial z}\right)\right] + v\left[\left(\frac{\partial f}{\partial y}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial \zeta}{\partial z}\right)\right] + \dot{\zeta}\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}$$

$$= u\left(\frac{\partial f}{\partial x}\right)_\zeta + v\left(\frac{\partial f}{\partial y}\right)_\zeta + \dot{\zeta}\frac{\partial f}{\partial \zeta}$$

since

$$\left(\frac{\partial \zeta}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_z \frac{\partial \zeta}{\partial z} = \left(\frac{\partial \zeta}{\partial x}\right)_\zeta = 0$$

$$\left(\frac{\partial \zeta}{\partial y}\right)_z + \left(\frac{\partial z}{\partial y}\right)_z \frac{\partial \zeta}{\partial z} = \left(\frac{\partial \zeta}{\partial y}\right)_\zeta = 0$$

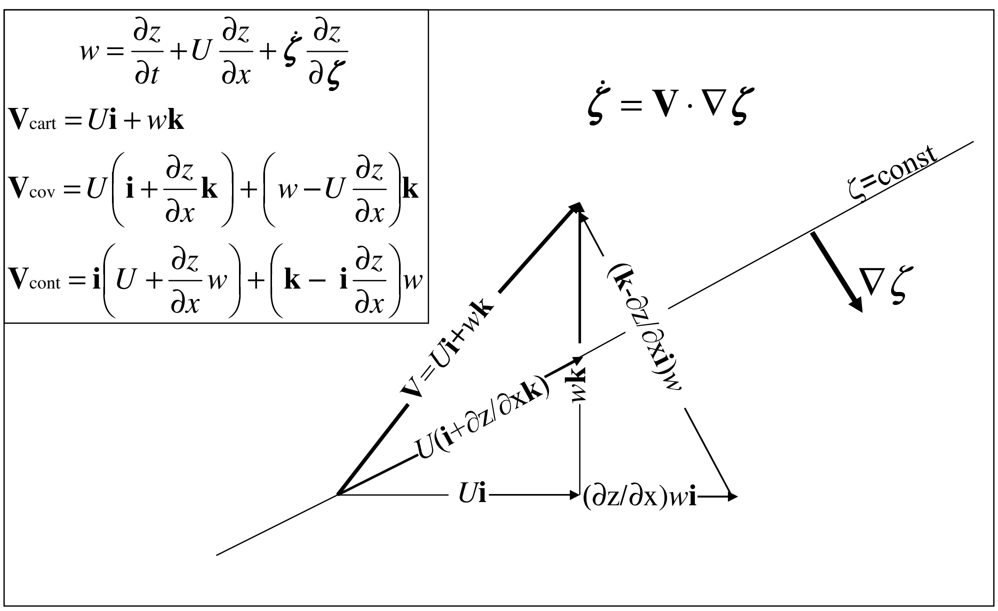


Figure 3. Representation of the wind vector in both orthogonal z-coordinate and oblique ζ-coordinate

Appendix 2d. Complete elimination of *w*.

Neglecting the Coriolis force and physical forcings, the four equations of motion in η-coordinate (*see page 5*) may be written:

$$\frac{d\mathbf{V}_h}{dt}+\frac{1}{\rho}\left(\nabla_\eta p-\nabla_\eta z\left(\frac{\partial z}{\partial \eta}\right)^{-1}\frac{\partial p}{\partial \eta}\right)=0 \qquad \text{(A2.1)}$$

$$\frac{dw}{dt}+\frac{1}{\rho}\left(\frac{\partial z}{\partial \eta}\right)^{-1}\frac{\partial p}{\partial \eta}+g=0 \qquad \text{(A2.2)}$$

$$\frac{dz}{dt}=w \qquad \text{(A2.3)}$$

with

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}_h\cdot\nabla_\eta+\dot{\eta}\frac{\partial}{\partial \eta} \qquad \text{(A2.4)}$$

Inserting (A2.3) in (A2.2) using (A2.1) and (A2.4), we obtain (Einstein summation convention):

$$\frac{d\dot{\eta}}{dt}+\Gamma^3_{\alpha\beta}u^\alpha u^\beta+\frac{1}{\rho}h^{3\alpha}\frac{\partial p}{\partial x^\alpha}+\left(\frac{\partial z}{\partial \eta}\right)^{-1}g=0$$

with $x^\alpha=(t,x,y,\eta)$ and $u^\alpha=(1,u,v,\dot{\eta})$, and where

$$\Gamma^3_{\alpha\beta}=\left(\frac{\partial z}{\partial \eta}\right)^{-1}\frac{\partial^2 z}{\partial x^\alpha \partial x^\beta}$$

is a Christoffel symbol and where

$$h^{30}=0;\; h^{31}=-\left(\frac{\partial z}{\partial \eta}\right)^{-1}\frac{\partial z}{\partial x};\; h^{32}=-\left(\frac{\partial z}{\partial \eta}\right)^{-1}\frac{\partial z}{\partial y};\; h^{33}=\left(\frac{\partial z}{\partial \eta}\right)^{-2}\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$

is a special tensor related to the metric tensor as follows:

$$h^{\mu\nu}=g^{\mu\nu}-g^{\mu 0}g^{0\nu}$$

(see Charron et al. 2013 in QJRMS for all the beautiful details).

Appendix 3. The metric parameter *B*.

In ζ-*coordinate*, the hydrostatic pressure is given by

$\ln\pi = A(\zeta) + B(\zeta)s;$	$s = \ln\pi_S - \zeta_S = \ln(\pi_S / p_{ref});\quad p_{ref} = 1000\,\text{hPa}$
$A = \zeta$	$B = \lambda^r$
	$\lambda = \dfrac{\zeta - \zeta_U}{\zeta_S - \zeta_U} \geq 0;\quad \zeta_U \geq \zeta_T;\quad r = r_{\max} - (r_{\max} - r_{\min})\lambda$
$\ln\pi = \zeta + B[\ln\pi_S - \zeta_S]$	

Here *B* is the unique relevant parameter as p_{ref} is not allowed to change. We note the logarithmic character of the relation and the presence of a variable exponent *r*. We have $0 \leq B \leq 1$ and a positive derivative:

$$\frac{\partial \ln B}{\partial\lambda} = \frac{\partial r\ln\lambda}{\partial\lambda} = \frac{1}{\lambda}[r - \Delta r\lambda\ln\lambda] \geq 0; \qquad \Delta r = r_{\max} - r_{\min}$$

Monotonicity requires that

$$\frac{\partial\ln\pi}{\partial\zeta} = 1 + \frac{\partial B}{\partial\lambda}[\ln\pi_S - \zeta_S]\frac{\partial\lambda}{\partial\zeta} > 0$$

$$\frac{\ln\pi_S}{\zeta_S} > K; \qquad K = 1 - \left(\frac{\partial B}{\partial\lambda}\right)_{\max}^{-1}\left(1 - \frac{\zeta_T}{\zeta_S}\right)$$

When *r* is constant (*Δr*=0), $(\partial B/\partial\lambda)_{\max} = r$ at the model surface where λ=1. *K*=1-1/*r* $(1 - \zeta_T/\zeta_S)$ and the monotonicity requirement is $r < \ln(p_{ref}/p_T)/\ln(p_{ref}/\pi_S)$. For $\pi_{S\,high} \approx p_{ref}/2$ and p_{top}=10 Pa, this implies $r < 4\ln 10/\ln 2 \approx 13.2$ and for p_{top}=10 hPa, $r < 2\ln 10/\ln 2 \approx 6.6$. Larger admitted exponents do not necessarily mean better coordinate straightening though and we must keep worrying about the ratio of model layer thicknesses. Considering

$$\left(\frac{\partial\ln\pi}{\partial\zeta}\right)_{\min} = 1 - \left(\frac{\partial B}{\partial\lambda}\right)_{\max}\frac{\ln(p_{ref}/\pi_S)}{\ln(p_{ref}/p_{top})}$$

we get, for constant *r* with p_{top}=10 Pa,

$$thfs = 1 - r\frac{\ln 2}{\ln(p_{ref}/p_{top})} \approx (1 - .075r)$$

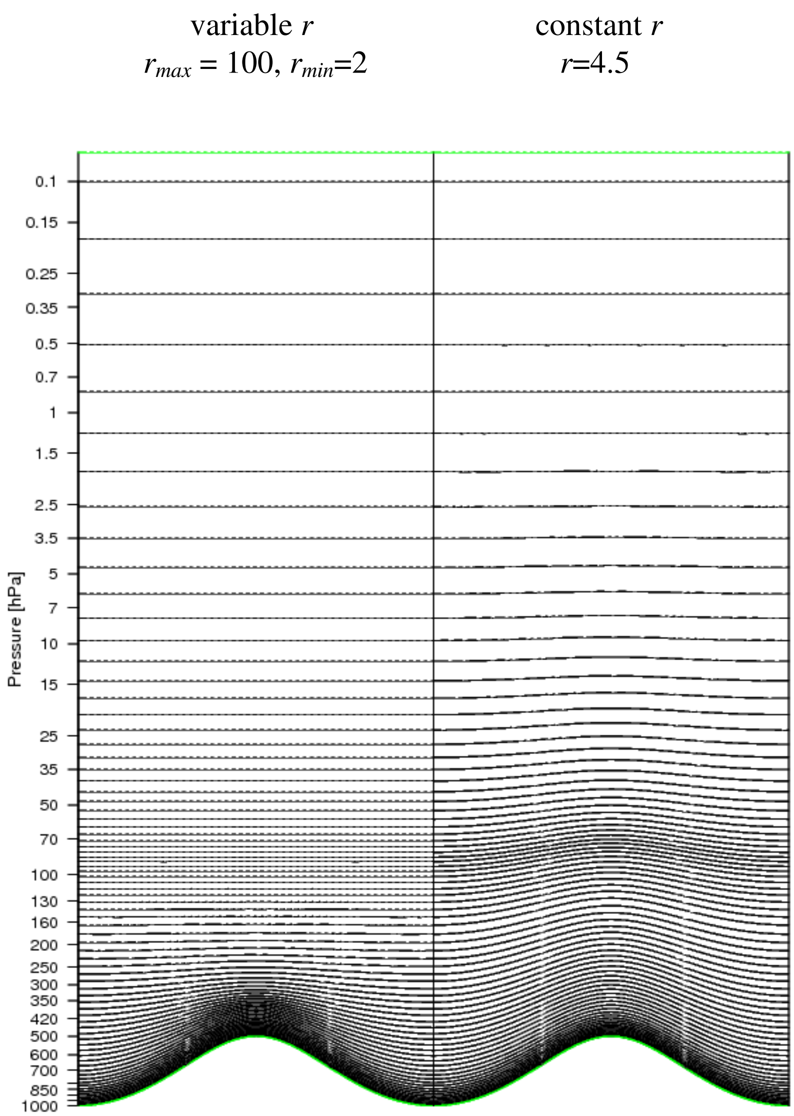


Figure 4. Variable *r* compared to constant *r*=4.5.

Hence, for *r*=1, $thfs \approx .925$ already. The value *thfs*=0.4 is reached for *r*=8 and *thfs*=0.57 is reached for *r*= 5.7 meanwhile $\pi(0.2) \approx 172$ hPa with *r*=8 and $\pi(0.2) \approx 159$ hPa with *r*=5.7, slightly better but no doubt insufficient rectification. Hence the need to keep *r* close to 0 near the surface while faster coordinate rectification requires increasingly larger values of *r* aloft and this is what we may attempt to achieve with the present formulation.

In **Figures 4** and **5,** we compare variable r (r_{max}=100, r_{min}=2) to constant r=4.5, clearly showing the rectifying possibilities inherent in variable r. The basic idea here is to essentially eliminate topography induced coordinate variation above 200 hPa.

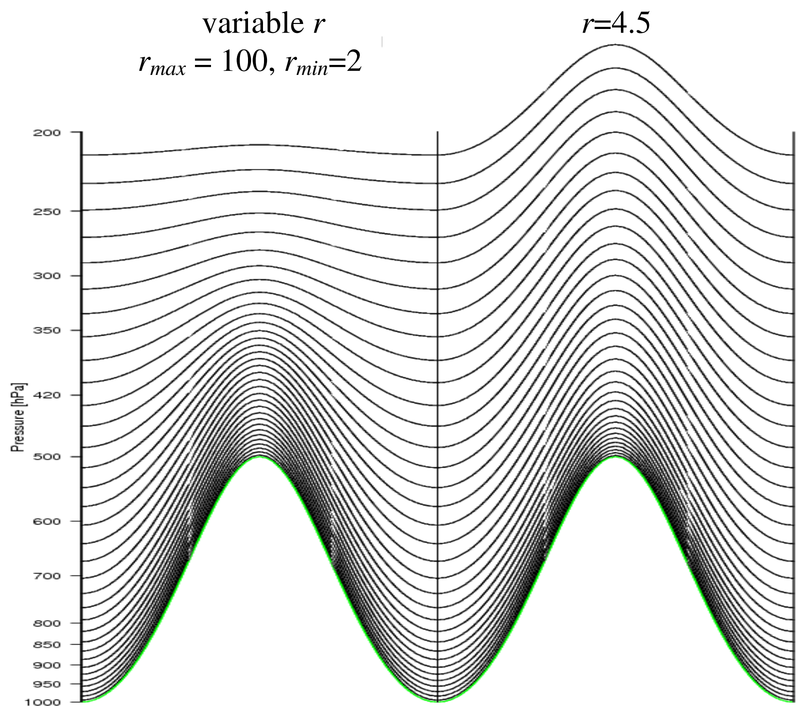


Figure 5. Variable r (r_{max}=100, r_{min}=2) compared to constant r=4.5 **below 200 hPa**.

In practice though, we have found difficult to implement models with r_{max} much greater than 15. The current (2015) GDPS uses r_{min}=3, r_{max}=15. An alternative would be to use a SLEVE-like coordinate (Schär et al., MWR, 2002).

Appendix 4. Detailed spatial discretization: *linear terms and matrices of the elliptic problem*

a) Initial terms and basic *difference* and *mean* operators

In **section 7**, we described the vertical discretization succinctly. In **sections 15 to 18**, we examined the elliptic problem formally. We now go back and examine the discrete linear system leading to the elliptic problem in full details. In particular, we clarify the definitions of *difference* and *averaging* operators.

As mentioned earlier, the *difference* operators replacing the derivatives are made as simple as possible. In effect, if ψ is a variable defined on full-levels and χ a variable defined on half-levels, then

$$\left(\delta_\zeta \psi\right)_{k+\frac{\mathrm{f}_k}{2}} = \frac{\psi_{k+\mathrm{f}_k} - \psi_k}{\Delta\zeta_{k+\frac{\mathrm{f}_k}{2}}} \quad ; \quad \left(\delta_\zeta \chi\right)_k = \frac{\chi_{k+\frac{1}{2}} - t_k \chi_{k-\frac{1}{2}}}{\Delta\zeta_k} \qquad (k = 1, N)$$
$$\Delta\zeta_{k+\frac{\mathrm{f}_k}{2}} = \zeta_{k+\mathrm{f}_k} - \zeta_k \qquad \Delta\zeta_k = \zeta_{k+\frac{1}{2}} - \zeta_{k-\frac{1}{2}}$$

with *specification* of the momentum levels, $\zeta_k \ (1 \le k \le N)$, plus the surface, $\zeta_S \equiv \zeta_{N+\frac{1}{2}}$, and *calculation* of the thermodynamic levels, $\zeta_{k+\frac{\mathrm{f}_k}{2}} = \left(\zeta_{k+\mathrm{f}_k} + \zeta_k\right)/2 \ (1 \le k \le N)$ plus the top $\zeta_T \equiv \zeta_{\frac{1}{2}} = 2\zeta_1 - \zeta_{\frac{3}{2}}$. Note the fraction at thermodynamic level *N*, [$\mathrm{f}_k = 1 \ (1 \le k \le N-1)$, $\mathrm{f}_N = \frac{1}{2}$], and the truncated difference at momentum level one [$t_1 = 0$, $t_k = 1 \ (2 \le k \le N)$]. Considering the linear equations and focusing first on *difference* operators, we place $\left(L_C\right)_k$, with u_k, v_k and $\dot{\zeta}_{k+\frac{1}{2}}$, leading to $\left(L_u\right)_k, \left(L_v\right)_k$ and $P_k = \phi'_k + RT_{*k} B_k s$. The hydrostatic case imposes $\left(L_H\right)_{k+\frac{\mathrm{f}_k}{2}}$ with $T'_{k+\frac{\mathrm{f}_k}{2}}$ and then $\left(L_\theta\right)_{k+\frac{\mathrm{f}_k}{2}}$. Note the need of an extra level for *P*, $P_{N+\frac{1}{2}} = \phi'_{N+\frac{1}{2}} + RT_{*N+\frac{1}{2}} s$, where $\phi'_{N+\frac{1}{2}} = \phi_S$, the surface geopotential, is a known quantity and *s* is a model variable. Hence, in the hydrostatic case, for *k*=1, *N*:

$$\left(L_u\right)_k = \left(\frac{u}{\tau} + \delta_X P\right)_k \qquad \left(L_v\right)_k = \left(\frac{v}{\tau} + \delta_Y P\right)_k$$

$$\left(L_C\right)_k = \left[\frac{\delta_\zeta \overline{B}^\zeta s + \overline{\overline{B}^\zeta}^\zeta s}{\tau} + \delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^\zeta\right]_k$$

$$\left(L_\theta\right)_{k+\frac{\mathrm{f}_k}{2}} = \left[\frac{T'}{\tau T_*} - \kappa_d\left(\breve{\dot{\zeta}} + \frac{\breve{\overline{B}}^\zeta s}{\tau}\right)\right]_{k+\frac{\mathrm{f}_k}{2}}$$

$$\left(L_\phi\right)_{k+\frac{\mathrm{f}_k}{2}} = \left(\frac{\breve{\overline{\phi'}}^\zeta}{\tau} - RT_*\breve{\dot{\zeta}} - gw\right)_{k+\frac{\mathrm{f}_k}{2}}$$

$$\left(L_H\right)_{k+\frac{\mathrm{f}_k}{2}} = \left(\frac{T'}{T_*} + \frac{\delta_\zeta P}{RT_*}\right)_{k+\frac{\mathrm{f}_k}{2}}$$

This is typical of the discretization on the well-established Charney-Phillips grid (**Figure 1**, page 12) for the hydrostatic case. The absence (in the present version of GEM) or presence (in previous versions) of a thermodynamic level near the top, above the first momentum level, and the presence (in this model) or absence (in other models) of a thermodynamic level, $N+\frac{f_k}{2}$, near the surface, below the last momentum level, remain model specific features.

For the less well-established non-hydrostatic case, the placement of q at momentum position k is determinant. It is suggested by its presence in both the full definition of P and in the full equation $\left(L_H\right)_{k+\frac{f_k}{2}}$. Note however the need for a value at N+½, the surface. With q_k, the positions of the remaining equations and non-hydrostatic variables are pretty well determined: $\left(L_\mu\right)_{k+\frac{f_k}{2}}$, $\mu_{k+\frac{f_k}{2}}$, then $\left(L_w\right)_{k+\frac{f_k}{2}}$, $w_{k+\frac{f_k}{2}}$ and $\left(L_\phi\right)_{k+\frac{f_k}{2}}$. Hence, for k=1, N:

$$\left(L_u\right)_k = \left(\frac{u}{\tau} + \delta_X P\right)_k$$

$$\left(L_v\right)_k = \left(\frac{v}{\tau} + \delta_Y P\right)_k$$

$$\left(L_C\right)_k = \left[\frac{\delta_\zeta \overline{B}^\zeta s + \overline{\overline{B}^\zeta}^\zeta s}{\tau} + \delta_X u + \frac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v\right) + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^\zeta\right]_k$$

$$\left(L_\theta\right)_{k+\frac{f_k}{2}} = \left[\frac{T'}{\tau T_*} - \kappa_d\left(\breve{\dot{\zeta}} + \frac{\breve{\overline{B}}^\zeta s + \breve{\overline{q}}^\zeta}{\tau}\right)\right]_{k+\frac{f_k}{2}}$$

$$\left(L_\phi\right)_{k+\frac{f_k}{2}} = \left(\frac{\breve{\overline{\phi'}}^\zeta}{\tau} - RT_*\breve{\dot{\zeta}} - gw\right)_{k+\frac{f_k}{2}}$$

$$\left(L_H\right)_{k+\frac{f_k}{2}} = \left[\frac{T'}{T_*} + \frac{\delta_\zeta P}{RT_*} - \boldsymbol{r}'\left(\delta_\zeta q + \breve{\overline{q}}^\zeta\right)\right]_{k+\frac{f_k}{2}}$$

...

$$\left(L_w\right)_{k+\frac{f_k}{2}} = \left(\frac{w}{\tau_{nh}} - g\mu\right)_{k+\frac{f_k}{2}}$$

$$\left(L_\mu\right)_{k+\frac{f_k}{2}} = \left[\mu - \left(\delta_\zeta q + \breve{\overline{q}}^\zeta\right)\right]_{k+\frac{f_k}{2}}$$

For the sake of maintaining second order accuracy, the use of more than one operator on the same terms is to be avoided. In fact, on the above linear terms, this occurs only on terms involving the parameter B, not a model variable, and it could have been avoided by explicitly calculating B at half-levels. For the vertical *averaging* operators, we formally write:

$$\left(\overline{\psi}^{\zeta}\right)_{k+\frac{1}{2}} = \varpi^{+}_{k+\frac{1}{2}} \psi_{k+1} + \varpi^{-}_{k+\frac{1}{2}} \psi_{k} \qquad \varpi^{-}_{k+\frac{1}{2}} = 1 - \varpi^{+}_{k+\frac{1}{2}} \quad ; \quad \left(\overline{\chi}^{\zeta}\right)_{k} = \varpi^{+}_{k} \chi_{k+\frac{1}{2}} + t_k \varpi^{-}_{k} \chi_{k-\frac{1}{2}} \qquad \varpi^{-}_{k} = 1 - \varpi^{+}_{k} \quad (k = 1, N)$$

Note again the truncation of the second operator at momentum level one. As can be seen, only one variable, $\dot{\zeta}$, could have been affected by the truncation of both difference and average. However, since $\dot{\zeta}_{\frac{1}{2}} \equiv \dot{\zeta}_T = 0$, the truncation has no impact. The truncation will take care, in the following development, of the absence of other variables above the first momentum level, while being of no consequence on $\dot{\zeta}$. The first averaging operator, averaging variables from momentum or full levels toward thermodynamic or half-levels, follows the rule of calculation for the half-levels, i.e.

$$\varpi^{+}_{k+\frac{1}{2}} = \frac{1}{2} \ (k = 1, N-1); \ \ \varpi^{+}_{N+\frac{1}{2}} = 1$$

This ensures optimal (second-order) accuracy for the hydrostatic equation L_H in particular. For the level *N*+½, no averaging is necessary for *B* or *q*, hence the value one. For the second averaging operator, averaging variables from thermodynamic levels toward momentum levels, averaging operator commuting with difference operators is adopted (see **Appendix 5**):

$$\varpi^{+}_{k} = \frac{\Delta\zeta_{k+\frac{1}{2}}}{2\Delta\zeta_{k}} \ (k = 1, N-1) \ ; \quad \varpi^{+}_{N} = \varpi^{+}_{*} \frac{\Delta\zeta_{N+\frac{1}{4}}}{\Delta\zeta_{N}}$$

The weight ϖ^{+}_{*} refers to a *special averaging* operator for level *N*+¼ which we have noted and defined as follows:

$$\breve{\chi}_{k+\frac{1}{2}} = \chi_{k+\frac{1}{2}} \quad (k = 1, N-1) \qquad \breve{\chi}_{N+\frac{1}{4}} = \varpi^{+}_{*} \chi_{N+\frac{1}{2}} + \varpi^{-}_{*} \chi_{N-\frac{1}{2}} \quad ; \quad \left(\breve{\overline{\psi}}^{\zeta}\right)_{k+\frac{t_k}{2}} = \left(\overline{\psi}^{\zeta}\right)_{k+\frac{1}{2}} \quad (k = 1, N-1) \qquad \left(\breve{\overline{\psi}}^{\zeta}\right)_{N+\frac{1}{4}} = \varpi^{+}_{*} \psi_{N+\frac{1}{2}} + \varpi^{-}_{*} \left(\overline{\psi}^{\zeta}\right)_{N-\frac{1}{2}}$$

and which serves to interpolate linearly the variables required at level *N*+¼ but not defined there. Hence the weights:

$$\varpi^{-}_{*} = \Delta\zeta_{N+\frac{1}{4}} / 2\Delta\zeta_{N}; \quad \varpi^{+}_{*} = 1 - \varpi^{-}_{*} \, .$$

This averaging is performed on $\dot{\zeta}$ in $\left(L_\theta\right)_{N+\frac{1}{4}}$, $\left(L_\phi\right)_{N+\frac{1}{4}}$, on $\overline{\phi'}^{\zeta}$ in $\left(L_\phi\right)_{N+\frac{1}{4}}$ and on $\overline{q}^{\zeta}$ in $\left(L_\theta\right)_{N+\frac{1}{4}}$, $\left(L_H\right)_{N+\frac{1}{4}}$ and $\left(L_\mu\right)_{N+\frac{1}{4}}$. Very explicitly, we have [exceptional values of *k* are listed separately, $\varpi^{\pm}_{k+\frac{1}{2}}$ being replaced by their values, either (½ , ½) or (1,0)] the linear terms (**section 13**):

$$\left(L_u\right)_k = \frac{u_k}{\tau} + \delta_X P_k \qquad \left(k = 1, N\right)$$

$$\left(L_v\right)_k = \frac{v_k}{\tau} + \delta_Y P_k \qquad \left(k = 1, N\right)$$

$$\left(L_C\right)_k = \delta_X u_k + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v_k\right) + \left(\tfrac{1}{\Delta\zeta_k} + \varpi_k^+\right)\left[\dot{\zeta}_{k+\frac{1}{2}} + \frac{B_{k+1} + B_k}{2}\frac{s}{\tau}\right] - \left(\tfrac{1}{\Delta\zeta_k} - \varpi_k^-\right)\left[\dot{\zeta}_{k-\frac{1}{2}} + \frac{B_k + B_{k-1}}{2}\frac{s}{\tau}\right]\left(k = 2, N-1\right)$$

$$\left(L_\theta\right)_{k+\frac{1}{2}} = \frac{T'_{k+\frac{1}{2}}}{\tau T_{*k+\frac{1}{2}}} - \kappa_d\left[\dot{\zeta}_{k+\frac{1}{2}} + \frac{1}{\tau}\frac{B_{k+1}s + \boldsymbol{r}' q_{k+1} + B_k s + \boldsymbol{r}' q_k}{2}\right] \qquad \left(k = 1, N-1\right)$$

$$\left(L_\phi\right)_{k+\frac{1}{2}} = \frac{1}{\tau}\frac{\phi'_{k+1} + \phi'_k}{2} - RT_{*k+\frac{1}{2}}\dot{\zeta}_{k+\frac{1}{2}} - g w_{k+\frac{1}{2}} \qquad \left(k = 1, N-1\right)$$

$$\left(L_H\right)_{k+\frac{1}{2}} = \frac{T'_{k+\frac{1}{2}}}{T_{*k+\frac{1}{2}}} + \frac{P_{k+1} - P_k}{RT_{*k+\frac{1}{2}}\Delta\zeta_{k+\frac{1}{2}}} - \boldsymbol{r}'\left(\frac{q_{k+1} - q_k}{\Delta\zeta_{k+\frac{1}{2}}} + \frac{q_{k+1} + q_k}{2}\right) \qquad \left(k = 1, N-1\right)$$

...

$$\left(L_w\right)_{k+\frac{1}{2}} = \frac{w_{k+\frac{1}{2}}}{\tau_{nh}} - g\mu_{k+\frac{1}{2}} \qquad \left(k = 1, N-1\right)$$

$$\left(L_\mu\right)_{k+\frac{1}{2}} = \mu_{k+\frac{1}{2}} - \left(\frac{q_{k+1} - q_k}{\Delta\zeta_{k+\frac{1}{2}}} + \frac{q_{k+1} + q_k}{2}\right) \qquad \left(k = 1, N-1\right)$$

and for the exceptional values, k=1 for L_C:

$$\left(L_C\right)_1 = \delta_X u_1 + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v_1\right) + \left(\tfrac{1}{\Delta\zeta_1} + \varpi_1^+\right)\left[\dot{\zeta}_{\frac{3}{2}} + \frac{B_2 + B_1}{2}\frac{s}{\tau}\right]$$

and k=N for all equations except L_u and L_v:

$$\left(L_C\right)_N = \delta_X u_N + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, V_N\right) + \left(\tfrac{1}{\Delta\zeta_N} + \varpi_N^+\right)\left[\dot{\zeta}_{N+\frac{1}{2}} + \frac{s}{\tau}\right] - \left(\tfrac{1}{\Delta\zeta_N} - \varpi_N^-\right)\left[\dot{\zeta}_{N-\frac{1}{2}} + \frac{B_N + B_{N-1}}{2}\frac{s}{\tau}\right]$$

$$\left(L_\theta\right)_{N+\frac{1}{4}} = \frac{T'_{N+\frac{1}{4}}}{\tau T_{*N+\frac{1}{4}}} - \kappa_d\left[\varpi_*^+\left(\dot{\zeta}_{N+\frac{1}{2}} + \frac{s + \boldsymbol{r}' q_{N+\frac{1}{2}}}{\tau}\right) + \varpi_*^-\left(\dot{\zeta}_{N-\frac{1}{2}} + \frac{1}{\tau}\frac{B_N s + \boldsymbol{r}' q_N + B_{N-1} s + \boldsymbol{r}' q_{N-1}}{2}\right)\right]$$

$$\left(L_\phi\right)_{N+\frac{1}{4}} = \frac{1}{\tau}\left(\varpi_*^+ \phi'_{N+\frac{1}{2}} + \varpi_*^- \frac{\phi'_N + \phi'_{N-1}}{2}\right) - RT_{*N+\frac{1}{4}}\left(\varpi_*^+ \dot{\zeta}_{N+\frac{1}{2}} + \varpi_*^- \dot{\zeta}_{N-\frac{1}{2}}\right) - g w_{N+\frac{1}{4}}$$

$$\left(L_H\right)_{N+\frac{1}{4}} = \frac{T'_{N+\frac{1}{4}}}{T_{*N+\frac{1}{4}}} + \frac{P_{N+\frac{1}{2}} - P_N}{RT_{*N+\frac{1}{4}}\Delta\zeta_{N+\frac{1}{4}}} - \boldsymbol{r}'\left(\frac{q_{N+\frac{1}{2}} - q_N}{\Delta\zeta_{N+\frac{1}{4}}} + \varpi_*^+ q_{N+\frac{1}{2}} + \varpi_*^- \frac{q_N + q_{N-1}}{2}\right)$$

...

$$\left(L_w\right)_{N+\frac{1}{4}} = \frac{w_{N+\frac{1}{4}}}{\tau_{nh}} - g\mu_{N+\frac{1}{4}}$$

$$\left(L_\mu\right)_{N+\frac{1}{4}} = \mu_{N+\frac{1}{4}} - \left(\frac{q_{N+\frac{1}{2}} - q_N}{\Delta\zeta_{N+\frac{1}{4}}} + \varpi_*^+ q_{N+\frac{1}{2}} + \varpi_*^- \frac{q_N + q_{N-1}}{2}\right)$$

b) The primed terms

After the preliminary step (**section 15**), having introduced

$$X_{k+\frac{1}{2}} = \dot{\zeta}_{k+\frac{1}{2}} + \frac{1}{\tau}\frac{B_{k+1}s + \boldsymbol{r}' q_{k+1} + B_k s + \boldsymbol{r}' q_k}{2}$$

and

$$P_k = \phi'_k + RT_{*k}\left(B_k s + q_k\right)$$

and eliminated two equations and variables *T'* and μ, we end up with:

$$\left(L'_C\right)_k = \delta_X u_k + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, V_k\right) + \left(\tfrac{1}{\Delta\zeta_k} + \varpi_k^+\right)\left(X_{k+\frac{1}{2}} - \frac{\boldsymbol{r}'}{\tau}\frac{q_{k+1} + q_k}{2}\right) - \left(\tfrac{1}{\Delta\zeta_k} - \varpi_k^-\right)\left(X_{k-\frac{1}{2}} - \frac{\boldsymbol{r}'}{\tau}\frac{q_k + q_{k-1}}{2}\right)$$

and

$$\left(L_\theta - \frac{L_H}{\tau}\right)_{k+\frac{1}{2}} = \left(L'_\theta\right)_{k+\frac{1}{2}} = -\frac{1}{\tau RT_{*k+\frac{1}{2}}}\frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \kappa_d X_{k+\frac{1}{2}} + \frac{\boldsymbol{r}'}{\tau}\left(\frac{q_{k+1} - q_k}{\Delta\zeta_{k+\frac{1}{2}}} + \frac{q_{k+1} + q_k}{2}\right)$$

$$\left[L_\phi - \frac{EXTRA}{\tau}\right]_{k+\frac{1}{2}} = \left(L'_\phi\right)_{k+\frac{1}{2}} = \frac{1}{\tau}\frac{P_{k+1} + P_k}{2} - RT_{*k+\frac{1}{2}} X_{k+\frac{1}{2}} - g w_{k+\frac{1}{2}}$$

...

$$\left[L_w + gL_\mu\right]_{k+\frac{1}{2}} = \left(L'_w\right)_{k+\frac{1}{2}} = \frac{w_{k+\frac{1}{2}}}{\tau_{nh}} - g\left(\frac{q_{k+1} - q_k}{\Delta\zeta_{k+\frac{1}{2}}} + \frac{q_{k+1} + q_k}{2}\right)$$

in addition to $\left(L_u\right)_k, \left(L_v\right)_k$. For the exceptional levels, we have

$$\left(L_C\right)_1 = \delta_X u_1 + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v_1\right) + \left(\tfrac{1}{\Delta\zeta_1} + \varpi_1^+\right)\left(X_{\frac{3}{2}} - \frac{\mathbf{r}'}{\tau}\frac{q_2 + q_1}{2}\right)$$

$$\left(L_C\right)_N = \delta_X u_N + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, v_N\right) + \left(\tfrac{1}{\Delta\zeta_N} + \varpi_N^+\right)\left(X_{N+\frac{1}{2}} - \frac{\mathbf{r}'}{\tau} q_{N+\frac{1}{2}}\right) - \left(\tfrac{1}{\Delta\zeta_N} - \varpi_N^-\right)\left(X_{N-\frac{1}{2}} - \frac{\mathbf{r}'}{\tau}\frac{q_N + q_{N-1}}{2}\right)$$

$$\left(L'_\theta\right)_{N+\frac{1}{4}} = -\frac{1}{\tau R T_{*N+4}}\frac{P_{N+\frac{1}{2}} - P_N}{\Delta\zeta_{N+\frac{1}{4}}} - \kappa_d\left(\varpi_*^+ X_{N+\frac{1}{2}} + \varpi_*^- X_{N-\frac{1}{2}}\right) + \frac{\mathbf{r}'}{\tau}\left(\frac{q_{N+\frac{1}{2}} - q_N}{\Delta\zeta_{N+\frac{1}{4}}} + +\varpi_*^+ q_{N+\frac{1}{2}} + \varpi_*^- \frac{q_N + q_{N-1}}{2}\right)$$

$$\left(L'_\phi\right)_{N+\frac{1}{4}} = \frac{1}{\tau}\left(\varpi_*^+ P_{N+\frac{1}{2}} + \varpi_*^- \frac{P_N + P_{N-1}}{2}\right) - R T_{*N+\frac{1}{4}}\left(\varpi_*^+ X_{N+\frac{1}{2}} + \varpi_*^- X_{N-\frac{1}{2}}\right) - g w_{N+\frac{1}{4}}$$

...

$$\left(L'_w\right)_{N+\frac{1}{4}} = \frac{w_{N+\frac{1}{4}}}{\tau_{nh}} - g\left(\frac{q_{N+\frac{1}{2}} - q_N}{\Delta\zeta_{N+\frac{1}{4}}} + +\varpi_*^+ q_{N+\frac{1}{2}} + \varpi_*^- \frac{q_N + q_{N-1}}{2}\right)$$

N.B.

$$\left(L'_\phi\right)_{k+\frac{1}{2}} = \left(L_\phi\right)_{k+\frac{1}{2}} - \frac{1}{\tau}\mathrm{EXTRA}_{k+\frac{1}{2}}$$

$$\left(L'_\phi\right)_{N+\frac{1}{4}} = \left(L_\phi\right)_{N+\frac{1}{4}} - \frac{1}{\tau}\mathrm{EXTRA}_{N+\frac{1}{4}}$$

where

$$\mathrm{EXTRA}_{k+\frac{1}{2}} = R T_{*k+\frac{1}{2}}\frac{\left(B_{k+1}s + \mathbf{r}' q_{k+1}\right) + \left(B_k s + \mathbf{r}' q_k\right)}{2} - \frac{R T_{*k+1}\left(B_{k+1}s + \mathbf{r}' q_{k+1}\right) + R T_{*k}\left(B_k s + \mathbf{r}' q_k\right)}{2}$$

$$EXTRA_{N+\frac{1}{4}} = R T_{*N+\frac{1}{4}}\left[\varpi_*^+\left(s + \mathbf{r}' q_{N+\frac{1}{2}}\right) + \varpi_*^- \frac{\left(B_N s + \mathbf{r}' q_N\right) + \left(B_{N-1}s + \mathbf{r}' q_{N-1}\right)}{2}\right]$$
$$- \varpi_*^+ R T_{*N+\frac{1}{2}}\left(s + \mathbf{r}' q_{N+\frac{1}{2}}\right) - \varpi_*^- \frac{R T_{*N}\left(B_N s + \mathbf{r}' q_N\right) + R T_{*N-1}\left(B_{N-1}s + \mathbf{r}' q_{N-1}\right)}{2}$$

and if T_* is constant, $\mathrm{EXTRA}_{k+\frac{\mathrm{f}_k}{2}} = 0$.

c) The double primed terms

Next we proceed towards $(L''_C), (L''_\phi)$ and (L''_θ)

$$(L''_C)_k = \left[\delta_X L_u + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta\, L_v) - \frac{L'_C}{\tau}\right]_k + \frac{r}{g\tau\tau_{nh}}\left[\varpi_k^+ (L'_w)_{k+\frac{1}{2}} + \varpi_k^- (L'_w)_{k-\frac{1}{2}}\right]$$

$$= \nabla_\zeta^2 P_k - \left(\tfrac{1}{\Delta\zeta_k} + \varpi_k^+\right)\frac{X_{k+\frac{1}{2}}}{\tau} + \left(\tfrac{1}{\Delta\zeta_k} - \varpi_k^-\right)\frac{X_{k-\frac{1}{2}}}{\tau} + \frac{r}{g\tau\tau_{nh}^2}\left(\varpi_k^+ w_{k+\frac{1}{2}} + \varpi_k^- w_{k-\frac{1}{2}}\right) + \frac{r}{\tau_{nh}^2} res_k$$

$$(L''_\theta)_{k+\frac{1}{2}} = \frac{1}{\tau\left(\kappa_d + \varepsilon'_{k+\frac{1}{2}}\right)}\left[(L'_\theta)_{k+\frac{1}{2}} + \frac{r}{g\tau_{nh}}(L'_w)_{k+\frac{1}{2}} + \frac{r}{g^2\tau_{nh}^2}(L'_\phi)_{k+\frac{1}{2}}\right]$$

$$= -\Gamma_{k+\frac{1}{2}}\left(\frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \varepsilon'_{k+\frac{1}{2}}\frac{P_{k+1} + P_k}{2}\right) - \frac{X_{k+\frac{1}{2}}}{\tau}$$

$$(L''_\phi)_{k+\frac{1}{2}} = \frac{1}{\tau\left(\kappa_d + \varepsilon'_{k+\frac{1}{2}}\right)}\left[(L'_\theta)_{k+\frac{1}{2}} + \frac{r}{g\tau_{nh}}(L'_w)_{k+\frac{1}{2}} - \frac{\kappa_d}{RT_{*k+\frac{1}{2}}}(L'_\phi)_{k+\frac{1}{2}}\right]$$

$$= -\Gamma_{k+\frac{1}{2}}\left(\frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} + \kappa_d\frac{P_{k+1} + P_k}{2}\right) + \frac{g w_{k+\frac{1}{2}}}{\tau RT_{*k+\frac{1}{2}}}$$

where $\Gamma_{k+\frac{1}{2}} = \dfrac{1}{\left(\kappa_d + \varepsilon'_{k+\frac{1}{2}}\right)\tau^2 RT_{*k+\frac{1}{2}}}$ and $\varepsilon'_{k+\frac{1}{2}} = r\varepsilon = r\dfrac{RT_{*k+\frac{1}{2}}}{g^2\tau_{nh}^2}$. Here the commutation rule applies:

$$res_k = \frac{q_{k+1} - q_{k-1}}{2\Delta\zeta_k} - \varpi_k^+\frac{q_{k+1} - q_k}{\Delta\zeta_{k+\frac{1}{2}}} - \varpi_k^-\frac{q_k - q_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}} = 0$$

since $\varpi_k^\pm = \dfrac{\Delta\zeta_{k\pm\frac{1}{2}}}{2\Delta\zeta_k}$. For the exceptional values, we have at the top (k=1)

$$(L''_C)_1 = \left[\delta_X L_u + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta\, L_v) - \frac{L'_C}{\tau}\right]_1 + \frac{r}{g\tau\tau_{nh}}\varpi_1^+ (L'_w)_{\frac{3}{2}}$$

$$= \delta_{XX} P_1 + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta\,\delta_Y P_1) - \left(\tfrac{1}{\Delta\zeta_1} + \varpi_1^+\right)\frac{X_{\frac{3}{2}}}{\tau} + \frac{r}{g\tau\tau_{nh}^2}\varpi_1^+ w_{\frac{3}{2}} + \frac{r}{\tau_{nh}^2} res_1$$

where

$$res_1 = \frac{1}{\Delta\zeta_1}\frac{q_2 + q_1}{2} - \varpi_1^+\frac{q_2 - q_1}{\Delta\zeta_{\frac{3}{2}}} = \frac{q_1}{\Delta\zeta_1} = 0$$

since $\varpi_1^+ = \dfrac{\Delta\zeta_{\frac{3}{2}}}{2\Delta\zeta_1} = \dfrac{1}{2}$ and $q_1 = 0$. Another possibility would consist in assuming that $q_{\frac{1}{2}} = 0$ instead. Then q_1 could be obtained by linearly interpolating between ζ_2 and $\zeta_{\frac{1}{2}}$, viz.

$q_1 = q_2 / 3$, in which case $res_1 = \left(1/\Delta\zeta_1 - \varpi_1^+ / \Delta\zeta_{\frac{3}{2}}\right) 2q_2 / 3$ would vanish only provided we modify $\varpi_1^+ = \Delta\zeta_{\frac{3}{2}} / \Delta\zeta_1 = 1$: an interesting possibility (to be investigated).

At the bottom (*k*=*N*), we have

$$\left(L_C''\right)_N = \left[\delta_X L_u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, L_v\right) - \frac{L_C'}{\tau}\right]_N + \frac{\boldsymbol{r}}{g\tau\tau_{nh}}\left[\vec{\boldsymbol{\omega}}_N^+ \left(L'_w\right)_{N+\frac{1}{4}} + \vec{\boldsymbol{\omega}}_N^- \left(L'_w\right)_{N-\frac{1}{2}}\right]$$

$$= \delta_{XX} P_N + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, \delta_Y P_N\right) - \left(\tfrac{1}{\Delta\zeta_N} + \varpi_N^+\right)\frac{X_{N+\frac{1}{2}}}{\tau} + \left(\tfrac{1}{\Delta\zeta_N} - \varpi_N^-\right)\frac{X_{N-\frac{1}{2}}}{\tau} + \frac{\boldsymbol{r}}{g\tau\tau_{nh}^2}\left(\vec{\boldsymbol{\omega}}_N^+ w_{N+\frac{1}{4}} + \vec{\boldsymbol{\omega}}_N^- w_{N-\frac{1}{2}}\right) + \frac{\boldsymbol{r}}{\tau_{nh}^2} res_N$$

where

$$res_N = \left(\tfrac{1}{\Delta\zeta_N} + \varpi_N^+\right) q_{N+\frac{1}{2}} - \left(\tfrac{1}{\Delta\zeta_N} - \varpi_N^-\right)\frac{q_N + q_{N-1}}{2}$$
$$- \vec{\boldsymbol{\omega}}_N^+ \left(\frac{q_{N+\frac{1}{2}} - q_N}{\Delta\zeta_{N+\frac{1}{4}}} + \varpi_*^+ q_{N+\frac{1}{2}} + \varpi_*^- \frac{q_N + q_{N-1}}{2}\right) - \vec{\boldsymbol{\omega}}_N^- \left(\frac{q_N - q_{N-1}}{\Delta\zeta_{N-\frac{1}{2}}} + \frac{q_N + q_{N-1}}{2}\right)$$

and $res_N = 0$ provided $\vec{\boldsymbol{\omega}}_N^+ = \dfrac{\Delta\zeta_{N+\frac{1}{4}}}{\Delta\zeta_N} = \dfrac{\varpi_N^+}{\varpi_*^+}$. It is interesting to note that the value of $\vec{\boldsymbol{\omega}}_N^+$ corresponds to the ancient value of ϖ_N^+, which is the one that must be modified by ϖ_*^+. (N.B. With uniform resolution, $\vec{\boldsymbol{\omega}}_N^+$ =1/2, ϖ_*^+ =3/4, ϖ_N^+ =3/8). Therefore

$$\left(L_C''\right)_N = \left[\delta_X L_u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, L_v\right) - \frac{L_C'}{\tau}\right]_N + \frac{\boldsymbol{r}}{g\tau\tau_{nh}}\left[\vec{\boldsymbol{\omega}}_N^+ \left(L'_w\right)_{N+\frac{1}{4}} + \vec{\boldsymbol{\omega}}_N^- \left(L'_w\right)_{N-\frac{1}{2}}\right]$$
$$= \delta_{XX} P_N + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, \delta_Y P_N\right)$$
$$- \left(\tfrac{1}{\Delta\zeta_N} + \varpi_N^+\right)\frac{X_{N+\frac{1}{2}}}{\tau} + \left(\tfrac{1}{\Delta\zeta_N} - \varpi_N^-\right)\frac{X_{N-\frac{1}{2}}}{\tau} + \frac{\boldsymbol{r}}{g\tau\tau_{nh}^2}\left(\vec{\boldsymbol{\omega}}_N^+ w_{N+\frac{1}{4}} + \vec{\boldsymbol{\omega}}_N^- w_{N-\frac{1}{2}}\right)$$

For the other two terms, we have

$$\left(L''_\theta\right)_{N+\frac{1}{4}} = \frac{1}{\tau\left(\kappa_d + \varepsilon'_{N+\frac{1}{4}}\right)}\left[\left(L'_\theta\right)_{N+\frac{1}{4}} + \frac{\boldsymbol{r}}{g\tau_{nh}}\left(L'_w\right)_{N+\frac{1}{4}} + \frac{\boldsymbol{r}}{g^2\tau_{nh}^2}\left(L'_\phi\right)_{N+\frac{1}{4}}\right]$$

$$= -\Gamma_{N+\frac{1}{4}}\left[\frac{P_{N+\frac{1}{2}} - P_N}{\Delta\zeta_{N+\frac{1}{4}}} + \varepsilon'_{N+\frac{1}{4}}\left(\varpi_*^+ P_{N+\frac{1}{2}} + \varpi_*^- \frac{P_N + P_{N-1}}{2}\right)\right] - \frac{1}{\tau}\left(\varpi_*^+ X_{N+\frac{1}{2}} + \varpi_*^- X_{N-\frac{1}{2}}\right)$$

$$\left(L''_\phi\right)_{N+\frac{1}{4}} = \frac{1}{\tau\left(\kappa_d + \varepsilon'_{N+\frac{1}{4}}\right)}\left[\left(L'_\theta\right)_{N+\frac{1}{4}} + \frac{\boldsymbol{r}}{g\tau_{nh}}\left(L'_w\right)_{N+\frac{1}{4}} - \frac{\kappa_d}{RT_{*k+\frac{1}{2}}}\left(L'_\phi\right)_{N+\frac{1}{4}}\right]$$

$$= -\Gamma_{N+\frac{1}{4}}\left[\frac{P_{N+\frac{1}{2}} - P_N}{\Delta\zeta_{N+\frac{1}{4}}} + \kappa_d\left(\varpi_*^+ P_{N+\frac{1}{2}} + \varpi_*^- \frac{P_N + P_{N-1}}{2}\right)\right] + \frac{g}{\tau RT_{*N+\frac{1}{4}}} w_{N+\frac{1}{4}}$$

Symbolically, the averaging operator corresponding to the weights

$$\vec{\boldsymbol{\omega}}_k^+ = \varpi_k^+ \ \ (k = 1, N-1)\,; \quad \vec{\boldsymbol{\omega}}_N^+ = \varpi_N^+ / \varpi_*^+$$
$$\vec{\boldsymbol{\omega}}_k^- = 1 - \vec{\boldsymbol{\omega}}_k^+$$

has been represented by $\overrightarrow{(\ )}^{\zeta}$. We may therefore write

$$L''_C = \delta_X L_u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\, L_v\right) - \frac{L'_C}{\tau} + \frac{\boldsymbol{r}}{g\tau\tau_{nh}}\overrightarrow{L'_w}^{\zeta}$$

$$= \nabla_\zeta^2 P - \left(\delta_\zeta X + \overline{X}^{\zeta}\right) + \frac{\boldsymbol{r}}{g\tau\tau_{nh}^2}\vec{w}^{\zeta}$$

$$\breve{L}''_\theta = -\Gamma\left[\delta_\zeta P - \varepsilon\breve{\overline{P}}\right] - \frac{\breve{X}}{\tau}$$

$$L''_\phi = -\Gamma\left[\delta_\zeta P + \kappa_d\breve{\overline{P}}\right] + \frac{gw}{\tau RT_*}$$

d) The final eliminations

From $\left(L''_\theta\right)_{N+\frac{1}{4}}$, we obtain $\left(L''_\theta\right)_{N+\frac{1}{2}}$:

$$\left(L''_\theta\right)_{N+\frac{1}{4}} - \varpi_*^-\left(L''_\theta\right)_{N-\frac{1}{2}} \equiv \varpi_*^+\left(L''_\theta\right)_{N+\frac{1}{2}} = -\Gamma_{N+\frac{1}{4}}\left[\frac{P_{N+\frac{1}{2}} - P_N}{\Delta\zeta_{N+\frac{1}{4}}} - \varepsilon'_{N+\frac{1}{4}}\left(\varpi_*^+ P_{N+\frac{1}{2}} + \varpi_*^- \frac{P_N + P_{N-1}}{2}\right)\right]$$
$$+ \varpi_*^-\Gamma_{N-\frac{1}{2}}\left(\frac{P_N - P_{N-1}}{\Delta\zeta_{N-\frac{1}{2}}} - \varepsilon'_{N-\frac{1}{2}}\frac{P_N + P_{N-1}}{2}\right) - \varpi_*^+\frac{X_{N+\frac{1}{2}}}{\tau}$$

before proceeding to the final eliminations,

$$
\begin{aligned}
\left(L_P\right)_1 &= \left(L'_C\right)_1 - \left(\tfrac{1}{\Delta\zeta_1}+\varpi_1^+\right)\left(L''_\theta\right)_{\frac{3}{2}} - \vec{\boldsymbol{\omega}}_1^+\left(\varepsilon'\, L''_\phi\right)_{\frac{3}{2}} \\
\left(L_P\right)_k &= \left(L'_C\right)_k - \left(\tfrac{1}{\Delta\zeta_k}+\varpi_k^+\right)\left(L''_\theta\right)_{k+\frac{1}{2}} + \left(\tfrac{1}{\Delta\zeta_k}-\varpi_k^-\right)\left(L''_\theta\right)_{k-\frac{1}{2}} - \vec{\boldsymbol{\omega}}_k^+\left(\varepsilon'\, L''_\phi\right)_{k+\frac{1}{2}} - \vec{\boldsymbol{\omega}}_k^-\left(\varepsilon'\, L''_\phi\right)_{k-\frac{1}{2}} \\
\left(L_P\right)_N &= \left(L'_C\right)_N - \left(\tfrac{1}{\Delta\zeta_N}+\varpi_N^+\right)\left(L''_\theta\right)_{N+\frac{1}{2}} + \left(\tfrac{1}{\Delta\zeta_N}-\varpi_N^-\right)\left(L''_\theta\right)_{N-\frac{1}{2}} - \vec{\boldsymbol{\omega}}_N^+\left(\varepsilon' L''_\phi\right)_{N+\frac{1}{4}} - \vec{\boldsymbol{\omega}}_N^-\left(\varepsilon' L''_\phi\right)_{N-\frac{1}{2}}
\end{aligned}
$$

getting respectively

$$
\left(L_P\right)_1 = \nabla_\zeta^2 P_1 + \left(\frac{1}{\Delta\zeta_1}+\varpi_1^+\right)\Gamma_{\frac{3}{2}}\frac{P_2-P_1}{\Delta\zeta_{\frac{3}{2}}} - \left(\Gamma\varepsilon'\right)_{\frac{3}{2}}\frac{P_1}{\Delta\zeta_1} - \left(1-\kappa_d\right)\varpi_1^+\left(\Gamma\varepsilon'\right)_{\frac{3}{2}}\frac{P_2+P_1}{2}
$$

$$
\begin{aligned}
\left(L_P\right)_k = {} & \nabla_\zeta^2 P_k + \left(\frac{1}{\Delta\zeta_k}+\varpi_k^+\right)\Gamma_{k+\frac{1}{2}}\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \left(\frac{1}{\Delta\zeta_k}-\varpi_k^-\right)\Gamma_{k-\frac{1}{2}}\frac{P_k-P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}} \\
& - \frac{P_k}{\Delta\zeta_k}\left[\left(\Gamma\varepsilon'\right)_{k+\frac{1}{2}} - \left(\Gamma\varepsilon'\right)_{k-\frac{1}{2}}\right] - \left(1-\kappa_d\right)\left[\varpi_k^+\left(\Gamma\varepsilon'\right)_{k+\frac{1}{2}}\frac{P_{k+1}+P_k}{2} + \varpi_k^-\left(\Gamma\varepsilon'\right)_{k-\frac{1}{2}}\frac{P_k+P_{k-1}}{2}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\left(L_P\right)_N = {} & \nabla_\zeta^2 P_N + \left(\frac{1}{\Delta\zeta_N\varpi_*^+}+\vec{\varpi}_N^+\right)\Gamma_{N+\frac{1}{4}}\frac{P_{N+\frac{1}{2}}-P_N}{\Delta\zeta_{N+\frac{1}{4}}} - \left(\frac{1}{\Delta\zeta_N\varpi_*^+}-\vec{\varpi}_N^-\right)\Gamma_{N-\frac{1}{2}}\frac{P_N-P_{N-1}}{\Delta\zeta_{N-\frac{1}{2}}} \\
& - \frac{\left(\Gamma\varepsilon'\right)_{N+\frac{1}{4}} - \left(\Gamma\varepsilon'\right)_{N-\frac{1}{2}}}{\varpi_*^+\Delta\zeta_N}\left[\left(1-\frac{\varpi_*^-}{2}\right)P_N + \frac{\varpi_*^-}{2}P_{N-1}\right] \\
& - \left(1-\kappa_d\right)\left[\vec{\varpi}_N^+\left(\Gamma\varepsilon'\right)_{N+\frac{1}{4}}\left(\varpi_*^+ P_{N+\frac{1}{2}} + \varpi_*^-\frac{P_N+P_{N-1}}{2}\right) + \vec{\varpi}_N^-\left(\Gamma\varepsilon'\right)_{N-\frac{1}{2}}\frac{P_N+P_{N-1}}{2}\right]
\end{aligned}
$$

Symbolically then, defining the special difference operator

$$
\left(\hat{\delta}_\zeta\right)_k = \left(\delta_\zeta\right)_k \quad (k=1,N-1)\ ; \qquad \left(\hat{\delta}_\zeta\right)_N = \frac{\left(\delta_\zeta\right)_N}{\varpi_*^+}
$$

and the special average

$$
\left(\tilde{P}\right)_k = P_k \quad (k=1,N-1)\ ; \qquad \left(\tilde{P}\right)_N = \left(1-\frac{\varpi_*^-}{2}\right)P_N + \frac{\varpi_*^-}{2}P_{N-1}
$$

we may write the elliptic problem as follows

$$L_P = L''_C - \delta_\zeta L''_\theta - \overline{L'''_\theta}^{\zeta} - \overrightarrow{\varepsilon' L''_\phi}^{\zeta}$$

$$= L''_C - \widehat{\delta}_\zeta \breve{L}''_\theta - \overrightarrow{\breve{L}'''_\theta}^{\zeta} - \overrightarrow{\varepsilon' L''_\phi}^{\zeta}$$

$$L_P = \nabla^2_\zeta P + \widehat{\delta}_\zeta \Gamma \delta_\zeta P + \overrightarrow{\Gamma \delta_\zeta P}^{\zeta} - \widetilde{P}\widehat{\delta}_\zeta \Gamma \varepsilon' - (1-\kappa_d)\overrightarrow{\Gamma \varepsilon' \breve{\overline{P}}^{\zeta}}^{\zeta}$$

Here note the equivalence $\delta_\zeta L''_\theta + \overline{L'''_\theta}^{\zeta} \equiv \widehat{\delta}_\zeta \breve{L}''_\theta + \overrightarrow{\breve{L}'''_\theta}^{\zeta}$ or again $\delta_\zeta X + \overline{X}^{\zeta} \equiv \widehat{\delta}_\zeta \breve{X} + \overrightarrow{\breve{X}}^{\zeta}$. In effect, we verify

$$\frac{X_{N+\frac{1}{2}} - X_{N-\frac{1}{2}}}{\Delta\zeta_N} + \varpi^+_N X_{N+\frac{1}{2}} + \varpi^-_N X_{N-\frac{1}{2}} = \left(\frac{1}{\varpi^+_* \Delta\zeta_N} + \vec{\boldsymbol{\omega}}^+_N\right)\left(\varpi^+_* X_{N+\frac{1}{2}} + \varpi^-_* X_{N-\frac{1}{2}}\right) - \left(\frac{1}{\varpi^+_* \Delta\zeta_N} - \vec{\boldsymbol{\omega}}^-_N\right) X_{N-\frac{1}{2}}$$

There are *N* equations with *N*+1 unknowns [$P_k \quad (k=1,N)$ plus $P_{N+\frac{1}{2}}$]. We use the boundary condition, $\dot{\zeta}_{N+\frac{1}{2}} = 0$, to eliminate $P_{N+\frac{1}{2}}$. In effect,

$$\varpi^+_* (L''_\theta)_{N+\frac{1}{2}} = -\Gamma_{N+\frac{1}{4}}\left[\frac{P_{N+\frac{1}{2}} - P_N}{\Delta\zeta_{N+\frac{1}{4}}} - \varepsilon'_{N+\frac{1}{4}}\left(\varpi^+_* P_{N+\frac{1}{2}} + \varpi^-_* \frac{P_N + P_{N-1}}{2}\right)\right]$$
$$+ \varpi^-_* \Gamma_{N-\frac{1}{2}}\left(\frac{P_N - P_{N-1}}{\Delta\zeta_{N-\frac{1}{2}}} - \varepsilon'_{N-\frac{1}{2}} \frac{P_N + P_{N-1}}{2}\right) - \frac{\varpi^+_* X_{N+\frac{1}{2}}}{\tau}$$

But, since $\dot{\zeta}_{N+\frac{1}{2}} = 0$, then

$$X_{N+\frac{1}{2}} = \frac{s + \boldsymbol{r}' q_{N+\frac{1}{2}}}{\tau} = \frac{P_{N+\frac{1}{2}} - \phi_S}{\tau R T_{*N+\frac{1}{2}}}$$

Hence

$$(L'''_\theta)_{N+\frac{1}{2}} = (L''_\theta)_{N+\frac{1}{2}} - \frac{\phi_S}{\tau^2 R T_{*N+\frac{1}{2}}}$$
$$= -\frac{\Gamma_{N+\frac{1}{4}}}{\varpi^+_*}\left[\frac{P_{N+\frac{1}{2}} - P_N}{\Delta\zeta_{N+\frac{1}{4}}} - \varepsilon'_{N+\frac{1}{4}}\left(\varpi^+_* P_{N+\frac{1}{2}} + \varpi^-_* \frac{P_N + P_{N-1}}{2}\right)\right]$$
$$+ \frac{\varpi^-_*}{\varpi^+_*}\Gamma_{N-\frac{1}{2}}\left(\frac{P_N - P_{N-1}}{\Delta\zeta_{N-\frac{1}{2}}} - \varepsilon'_{N-\frac{1}{2}} \frac{P_N + P_{N-1}}{2}\right) - \frac{P_{N+\frac{1}{2}}}{\tau^2 R T_{*N+\frac{1}{2}}}$$

Re-ordering, we write

$$P_{N+\frac{1}{2}} = \alpha_S P_N + \beta_S P_{N-1} - C_S \left(L_\theta'''\right)_{N+\frac{1}{2}}$$

with

$$C_S = \frac{1}{\Gamma_{N+\frac{1}{4}}\left\{1/\Delta\zeta_{N+\frac{1}{4}} + \left[\left(\kappa_d + \varepsilon'_{N+\frac{1}{4}}\right)T_{*N+\frac{1}{4}}/T_{*N+\frac{1}{2}} - \varepsilon'_{N+\frac{1}{4}}\right]\varpi_*^+\right\}}$$

$$\alpha_S = C_S\left\{\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}} + \varpi_*^-\left[\frac{(\Gamma\varepsilon')_{N+\frac{1}{4}} - (\Gamma\varepsilon')_{N-\frac{1}{2}}}{2} + \frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}}\right]\right\}$$

$$\beta_S = C_S\varpi_*^-\left[\frac{(\Gamma\varepsilon')_{N+\frac{1}{4}} - (\Gamma\varepsilon')_{N-\frac{1}{2}}}{2} - \frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}}\right]$$

and thus finally

$$\left(L_P'\right)_N = \left(L_P\right)_N + C_S''\left(L_\theta'''\right)_{N+\frac{1}{2}}$$

$$\begin{aligned}
\left(L_P'\right)_N = {} & \delta_{XX}P_N + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\,\delta_Y P_N\right) \\
& + \left(\frac{1}{\Delta\zeta_N\varpi_*^+} + \vec{\boldsymbol{\omega}}_N^+\right)\Gamma_{N+\frac{1}{4}}\frac{\left(\alpha_S P_N + \beta_S P_{N-1}\right) - P_N}{\Delta\zeta_{N+\frac{1}{4}}} - \left(\frac{1}{\Delta\zeta_N\varpi_*^+} - \vec{\boldsymbol{\omega}}_N^-\right)\Gamma_{N-\frac{1}{2}}\frac{P_N - P_{N-1}}{\Delta\zeta_{N-\frac{1}{2}}} \\
& - \frac{(\Gamma\varepsilon')_{N+\frac{1}{4}} - (\Gamma\varepsilon')_{N-\frac{1}{2}}}{\varpi_*^+\Delta\zeta_N}\left[\left(1 - \frac{\varpi_*^-}{2}\right)P_N + \frac{\varpi_*^-}{2}P_{N-1}\right] \\
& - \left(1-\kappa_d\right)\left\{\vec{\boldsymbol{\omega}}_N^+(\Gamma\varepsilon')_{N+\frac{1}{4}}\left[\varpi_*^+\left(\alpha_S P_N + \beta_S P_{N-1}\right) + \varpi_*^-\frac{P_N + P_{N-1}}{2}\right] + \vec{\boldsymbol{\omega}}_N^-(\Gamma\varepsilon')_{N-\frac{1}{2}}\frac{P_N + P_{N-1}}{2}\right\}
\end{aligned}$$

with

$$C_S'' = \left[\frac{1}{\varpi_*^+}\left(\frac{1}{\Delta\zeta_N} + \varpi_N^+\right)\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}} - \left(1-\kappa_d\right)\varpi_N^+(\Gamma\varepsilon')_{N+\frac{1}{4}}\right]C_S$$

Hence L_P' gives rise to the same expression

$$L_P' = \nabla_\zeta^2 P + \hat{\delta}_\zeta\Gamma\delta_\zeta P + \overrightarrow{\Gamma\delta_\zeta P}^\zeta - \tilde{P}\hat{\delta}_\zeta\Gamma\varepsilon' - \left(1-\kappa_d\right)\overrightarrow{\Gamma\varepsilon'\breve{\overline{P}}^\zeta}^\zeta$$

with $P_{N+\frac{1}{2}}$ replaced by $\alpha_S P_N + \beta_S P_{N-1}$.

This vertical matrix problem may be decomposed into a set of tri-diagonal matrices written as follows:

$$L_P' = \nabla_\zeta^2 P + \left[\mathbf{D}\Gamma\mathbf{D} + \mathbf{M}\Gamma\mathbf{D} - \mathbf{D}(\Gamma\varepsilon') - (1-\kappa_d)\mathbf{M}(\Gamma\varepsilon')\mathbf{M}\right]P\,,$$

the elements of which are:

$$\begin{aligned}
\left(L_P\right)_{k_0=1} &= \delta_{XX} P_{k_0} + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\,\delta_Y P_{k_0}\right) \\
&+ \frac{1}{\Delta\zeta_{k_0}} \frac{\Gamma_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0+\frac{1}{2}}}\left(P_{k_0+1} - P_{k_0}\right) \\
&+ \varpi_{k_0}^{+} \frac{\Gamma_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0+\frac{1}{2}}}\left(P_{k_0+1} - P_{k_0}\right) \\
&- \frac{\left(\Gamma\varepsilon'\right)_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0}} P_{k_0} \\
&- \left(1-\kappa_d\right)\frac{\varpi_{k_0}^{+}}{2}\left(\Gamma\varepsilon'\right)_{k_0+\frac{1}{2}}\left(P_{k_0+1} + P_{k_0}\right)
\end{aligned}$$

$$\begin{aligned}
\left(L_P\right)_{k} &= \delta_{XX} P_{k} + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\,\delta_Y P_{k}\right) \\
&+ \frac{1}{\Delta\zeta_{k}} \frac{\Gamma_{k+\frac{1}{2}}}{\Delta\zeta_{k+\frac{1}{2}}}\left(P_{k+1} - P_{k}\right) - \frac{1}{\Delta\zeta_{k}} \frac{\Gamma_{k-\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}}\left(P_{k} - P_{k-1}\right) \\
&+ \varpi_{k}^{+} \frac{\Gamma_{k+\frac{1}{2}}}{\Delta\zeta_{k+\frac{1}{2}}}\left(P_{k+1} - P_{k}\right) + \varpi_{k}^{-} \frac{\Gamma_{k-\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}}\left(P_{k} - P_{k-1}\right) \\
&- \frac{\left(\Gamma\varepsilon'\right)_{k+\frac{1}{2}} - \left(\Gamma\varepsilon'\right)_{k-\frac{1}{2}}}{\Delta\zeta_{k}} P_{k} \\
&- \left(1-\kappa_d\right)\left[\frac{\varpi_{k}^{+}}{2}\left(\Gamma\varepsilon'\right)_{k+\frac{1}{2}}\left(P_{k+1} + P_{k}\right) + \frac{\varpi_{k}^{-}}{2}\left(\Gamma\varepsilon'\right)_{k-\frac{1}{2}}\left(P_{k} + P_{k-1}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\left(L_P'\right)_{N} &= \delta_{XX} P_{N} + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\,\delta_Y P_{N}\right) \\
&- \frac{1}{\Delta\zeta_N \varpi_*^{+}}\left[\frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}} + \left(1-\alpha_S\right)\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}}\right] P_N + \frac{1}{\Delta\zeta_N \varpi_*^{+}}\left[\frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}} + \beta_S \frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}}\right] P_{N-1} \\
&+ \left[\vec{\boldsymbol{\omega}}_N^{-} \frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}} - \vec{\boldsymbol{\omega}}_N^{+}\left(1-\alpha_S\right)\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}}\right] P_N - \left[\vec{\boldsymbol{\omega}}_N^{-} \frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}} - \vec{\boldsymbol{\omega}}_N^{+}\beta_S \frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}}\right] P_{N-1} \\
&- \left(1 - \frac{\varpi_*^{-}}{2}\right)\frac{\left(\Gamma\varepsilon'\right)_{N+\frac{1}{4}} - \left(\Gamma\varepsilon'\right)_{N-\frac{1}{2}}}{\varpi_*^{+}\Delta\zeta_N} P_N - \frac{\varpi_*^{-}}{2}\frac{\left(\Gamma\varepsilon'\right)_{N+\frac{1}{4}} - \left(\Gamma\varepsilon'\right)_{N-\frac{1}{2}}}{\varpi_*^{+}\Delta\zeta_N} P_{N-1} \\
&- \left(1-\kappa_d\right)\left\{\frac{\vec{\boldsymbol{\omega}}_N^{-}}{2}\left(\Gamma\varepsilon'\right)_{N-\frac{1}{2}} + \vec{\boldsymbol{\omega}}_N^{+}\left(\frac{\varpi_*^{-}}{2} + \varpi_*^{+}\alpha_S\right)\left(\Gamma\varepsilon'\right)_{N+\frac{1}{4}}\right\} P_N \\
&- \left(1-\kappa_d\right)\left\{\frac{\vec{\boldsymbol{\omega}}_N^{-}}{2}\left(\Gamma\varepsilon'\right)_{N-\frac{1}{2}} + \vec{\boldsymbol{\omega}}_N^{+}\left(\frac{\varpi_*^{-}}{2} + \varpi_*^{+}\beta_S\right)\left(\Gamma\varepsilon'\right)_{N+\frac{1}{4}}\right\} P_{N-1}
\end{aligned}$$

Explicitly, the matrices then are:

$$\mathbf{D\Gamma D}=\begin{bmatrix} -\frac{1}{\Delta\zeta_{k_0}}\left((1-\alpha_T)\frac{\Gamma_{k_0-\frac{1}{2}}}{\Delta\zeta_{k_0-\frac{1}{2}}}+\frac{\Gamma_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0+\frac{1}{2}}}\right) & \frac{1}{\Delta\zeta_{k_0}}\frac{\Gamma_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0+\frac{1}{2}}} & 0 \\ \frac{1}{\Delta\zeta_{k}}\frac{\Gamma_{k-\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}} & -\frac{1}{\Delta\zeta_{k}}\left(\frac{\Gamma_{k-\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}}+\frac{\Gamma_{k+\frac{1}{2}}}{\Delta\zeta_{k+\frac{1}{2}}}\right) & \frac{1}{\Delta\zeta_{k}}\frac{\Gamma_{k+\frac{1}{2}}}{\Delta\zeta_{k+\frac{1}{2}}} \\ 0 & \frac{1}{\varpi_*^+\Delta\zeta_N}\left(\frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}}+\beta_S\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}}\right) & -\frac{1}{\varpi_*^+\Delta\zeta_N}\left(\frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}}+(1-\alpha_S)\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}}\right) \end{bmatrix}$$

$$\mathbf{M\Gamma D}=\begin{bmatrix} (1-\alpha_T)\varpi_{k_0}^-\frac{\Gamma_{k_0-\frac{1}{2}}}{\Delta\zeta_{k_0-\frac{1}{2}}}-\varpi_{k_0}^+\frac{\Gamma_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0+\frac{1}{2}}} & \varpi_{k_0}^+\frac{\Gamma_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0+\frac{1}{2}}} & 0 \\ -\varpi_k^-\frac{\Gamma_{k-\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}} & \varpi_k^-\frac{\Gamma_{k-\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}}-\varpi_k^+\frac{\Gamma_{k+\frac{1}{2}}}{\Delta\zeta_{k+\frac{1}{2}}} & \varpi_k^+\frac{\Gamma_{k+\frac{1}{2}}}{\Delta\zeta_{k+\frac{1}{2}}} \\ 0 & -\vec{\boldsymbol{\omega}}_N^-\frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}}+\vec{\boldsymbol{\omega}}_N^+\beta_S\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}} & \vec{\boldsymbol{\omega}}_N^-\frac{\Gamma_{N-\frac{1}{2}}}{\Delta\zeta_{N-\frac{1}{2}}}-\vec{\boldsymbol{\omega}}_N^+(1-\alpha_S)\frac{\Gamma_{N+\frac{1}{4}}}{\Delta\zeta_{N+\frac{1}{4}}} \end{bmatrix}$$

$$\mathbf{D}\Gamma\varepsilon'=\begin{bmatrix} \frac{(\Gamma\varepsilon')_{k_0+\frac{1}{2}}-(\Gamma\varepsilon')_{k_0-\frac{1}{2}}}{\Delta\zeta_{k_0}} & & \\ & \frac{(\Gamma\varepsilon')_{k+\frac{1}{2}}-(\Gamma\varepsilon')_{k-\frac{1}{2}}}{\Delta\zeta_{k}} & \\ & \frac{\varpi_*^-}{2}\frac{(\Gamma\varepsilon')_{N+\frac{1}{4}}-(\Gamma\varepsilon')_{N-\frac{1}{2}}}{\Delta\zeta_N\varpi_*^+} & \left(1-\frac{\varpi_*^-}{2}\right)\frac{(\Gamma\varepsilon')_{N+\frac{1}{4}}-(\Gamma\varepsilon')_{N-\frac{1}{2}}}{\Delta\zeta_N\varpi_*^+} \end{bmatrix}$$

$$\mathbf{M}\Gamma\varepsilon'\mathbf{M}=\begin{bmatrix} \varpi_{k_0}^-(\Gamma\varepsilon')_{k_0-\frac{1}{2}}+\frac{\varpi_{k_0}^+}{2}(\Gamma\varepsilon')_{k_0+\frac{1}{2}} & \frac{\varpi_{k_0}^+}{2}(\Gamma\varepsilon')_{k_0+\frac{1}{2}} & 0 \\ \frac{\varpi_k^-}{2}(\Gamma\varepsilon')_{k-\frac{1}{2}} & \frac{\varpi_k^-}{2}(\Gamma\varepsilon')_{k-\frac{1}{2}}+\frac{\varpi_k^+}{2}(\Gamma\varepsilon')_{k+\frac{1}{2}} & \frac{\varpi_k^+}{2}(\Gamma\varepsilon')_{k+\frac{1}{2}} \\ 0 & \frac{\vec{\boldsymbol{\omega}}_N^-}{2}(\Gamma\varepsilon')_{N-\frac{1}{2}}+\vec{\boldsymbol{\omega}}_N^+\left(\frac{\varpi_*^-}{2}+\varpi_*^+\beta_S\right)(\Gamma\varepsilon')_{N+\frac{1}{4}} & \frac{\vec{\boldsymbol{\omega}}_N^-}{2}(\Gamma\varepsilon')_{N-\frac{1}{2}}+\vec{\boldsymbol{\omega}}_N^+\left(\frac{\varpi_*^-}{2}+\varpi_*^+\alpha_S\right)(\Gamma\varepsilon')_{N+\frac{1}{4}} \end{bmatrix}$$

N.B. The operators are not truncated and the indices start at k_0 rather than one with the presence of α_T , indicating that the top boundary condition is different from the one discussed above. In fact it corresponds to the open top boundary condition (see **Appendix 9** for its description). The truncated operators are easily recovered though: simply considering $\Gamma_{\frac{1}{2}}=0$ if $k_0=1$.

Appendix 5. How were chosen the averaging operators and note about commutation

Let us consider two variables, ψ and χ, defined on separate staggered grids as follows:

$$\psi_k = \psi(\zeta_k) \quad ; \quad \chi_{k+\frac{1}{2}} = \chi(\zeta_{k+\frac{1}{2}})$$

indicating that ψ is defined on full levels while χ is defined on half-levels. Only the independent variable $\boldsymbol{\zeta}$ could and was defined on both types of levels and thus take the two types of indices. The metric parameter could also sometimes be defined on both types of level, hence two different symbols (*B* on full and B on half levels). To obtain the variables *G* and *H* on their alternative grids, averaging operators $\boldsymbol{\alpha}$ and *a* such that:

$$(\alpha\chi)_k = \alpha_k \chi_{k+\frac{1}{2}} + (1-\alpha_k)\chi_{k-\frac{1}{2}} \quad ; \quad (a\psi)_{k+\frac{1}{2}} = a_{k+\frac{1}{2}}\psi_{k+1} + \left(1-a_{k+\frac{1}{2}}\right)\psi_k$$

are introduced. In the following discussion, difference operators will be needed and we define them:

$$(\delta\chi)_k = \frac{\chi_{k+\frac{1}{2}} - \chi_{k-\frac{1}{2}}}{\zeta_{k+\frac{1}{2}} - \zeta_{k-\frac{1}{2}}} = \frac{\chi_{k+\frac{1}{2}} - \chi_{k-\frac{1}{2}}}{\Delta\zeta_k} \quad ; \quad (\delta\psi)_{k+\frac{1}{2}} = \frac{\psi_{k+1} - \psi_k}{\zeta_{k+1} - \zeta_k} = \frac{\psi_{k+1} - \psi_k}{\Delta\zeta_{k+\frac{1}{2}}}$$

Now, let us consider the following relations discussed in **section 17**

$$\delta_\zeta \Gamma\varepsilon' \overline{P}^\zeta - \overline{\Gamma\varepsilon'\delta_\zeta P}^\zeta = \delta_\zeta \Gamma\varepsilon' P$$

$$\overline{\delta_\zeta q}^\zeta + \overline{\overline{q}^\zeta}^\zeta = \delta_\zeta \overline{q}^\zeta + \overline{q}^{\zeta\zeta}$$

If Γε is constant, and if we do not consider the frontiers, the double averages cancel and both relations simplify to the same commuting relation

$$\overline{\delta_\zeta \psi}^\zeta - \delta_\zeta \overline{\psi}^\zeta = (\alpha\delta - \delta a)\psi = 0$$

Let us impose this condition to the above operators and examine the consequences. We get

$$(\alpha\delta P)_k = \alpha_k (\delta P)_{k+\frac{1}{2}} + (1-\alpha_k)(\delta P)_{k-\frac{1}{2}} = \alpha_k \frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} + (1-\alpha_k)\frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}$$

$$(\delta a P)_k = \frac{(aP)_{k+\frac{1}{2}} - (aP)_{k-\frac{1}{2}}}{\Delta\zeta_k} = \frac{a_{k+\frac{1}{2}} P_{k+1} + \left(1-a_{k+\frac{1}{2}}\right)P_k - a_{k-\frac{1}{2}} P_k - \left(1-a_{k-\frac{1}{2}}\right)P_{k-1}}{\Delta\zeta_k}$$

Implying that

$$\frac{\Delta\zeta_k}{\Delta\zeta_{k+\frac{1}{2}}}=\frac{a_{k+\frac{1}{2}}}{\alpha_k};\qquad \frac{\Delta\zeta_k}{\Delta\zeta_{k-\frac{1}{2}}}=\frac{1-a_{k-\frac{1}{2}}}{1-\alpha_k}$$

and either

$$\text{(a)}\qquad \frac{\Delta\zeta_{k+1}}{\Delta\zeta_k}=\frac{\alpha_k}{1-\alpha_{k+1}}\frac{1-a_{k+\frac{1}{2}}}{a_{k+\frac{1}{2}}}\quad\text{or}\qquad \text{(b)}\qquad \frac{\Delta\zeta_{k+\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}}=\frac{\alpha_k}{1-\alpha_k}\frac{1-a_{k-\frac{1}{2}}}{a_{k+\frac{1}{2}}}$$

If the relation between the half and full levels is given, for example if, as we have chosen:

$$\zeta_{k+\frac{1}{2}}=\frac{\zeta_{k+1}+\zeta_k}{2}$$

then we most likely want

$$a_{k+\frac{1}{2}}=\frac{1}{2}$$

From (b) we get

$$\alpha_k=\frac{\Delta\zeta_{k+\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}+\Delta\zeta_{k+\frac{1}{2}}}=\frac{\Delta\zeta_{k+\frac{1}{2}}}{2\Delta\zeta_k}$$

and thus

$$(\alpha G)_k=\frac{\Delta\zeta_{k+\frac{1}{2}}G_{k+\frac{1}{2}}+\Delta\zeta_{k-\frac{1}{2}}G_{k-\frac{1}{2}}}{2\Delta\zeta_k}\quad;\quad (aH)_{k+\frac{1}{2}}=\frac{H_{k+1}+H_k}{2}$$

Instead of choosing $a_{k+\frac{1}{2}}$ off-hand as we have done, we might have imposed another condition such as the symmetry of matrix M formed by the product of the matrix obtained from the double averaging operator αa and the diagonal matrix with elements $\Delta\zeta_k$, i.e. if we had imposed that the tri-diagonal matrix M whose elements are

$$\begin{aligned}(\Delta\zeta\alpha aP)_k&=\Delta\zeta_k\alpha_k\left(a_{k+\frac{1}{2}}P_{k+1}+\left(1-a_{k+\frac{1}{2}}\right)P_k\right)+(1-\alpha_k)\Delta\zeta_k\left(a_{k-\frac{1}{2}}P_k+\left(1-a_{k-\frac{1}{2}}\right)P_{k-1}\right)\\&=M_{k+1,k}P_{k+1}+M_{k,k}P_k+M_{k-1,k}P_{k-1}\end{aligned}$$

be symmetric, i.e. setting $M_{k+1,k}=M_{k,k+1}$, i.e. $\Delta\zeta_k\alpha_k a_{k+\frac{1}{2}}=(1-\alpha_{k+1})\Delta\zeta_{k+1}\left(1-a_{k+\frac{1}{2}}\right)$, i.e.

$$\text{(c)}\qquad \frac{\Delta\zeta_{k+1}}{\Delta\zeta_k}=\frac{\alpha_k}{1-\alpha_{k+1}}\frac{a_{k+\frac{1}{2}}}{1-a_{k+\frac{1}{2}}}$$

Then, combining (c) with (a), we would have again found

$$a_{k+\frac{1}{2}}=\frac{1}{2}$$

Appendix 6. The Dynamic Core Code and vertical discretization: *A brief description*

The dynamic core code is essentially organized as follows:

set_zeta, set_dync, set_opr, set_oprz : *compute* constants *and* parameters *of the vertical discretization*

***Timestep** Loop*

tstpdyn: *performs a* dynamical time step *calling* **rhs, adv, pre, nli, sol, bac**

- **rhs:** compute the 6 basic Right-Hand-Side terms: $R_u, R_v, R_C, R_\theta, R_\phi, R_w$ **(section 12)**

$$R_u = \frac{u}{\tau} \qquad -\beta\left[-\left(f + \tfrac{\tan\theta}{a}u\right)\overline{v}^{XY} + R\overline{T}^{X\zeta}\delta_X(Bs+q) + (1+\overline{\mu}^{X\zeta})\delta_X\phi'\right]$$

$$R_v = \frac{v}{\tau} \qquad -\beta\left[\left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T}^{Y\zeta}\delta_Y(Bs+q) + (1+\overline{\mu}^{Y\zeta})\,\delta_Y\phi'\right]$$

$$R_C = \frac{Bs + \ln\left(1+\delta_\zeta\overline{B}^\zeta s\right)}{\tau} \qquad -\beta\left[\delta_X u + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta\, v) + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^\zeta\right]$$

$$R_\theta = \frac{1}{\tau}\ln\left(\frac{T}{T_*}\right) - \kappa\frac{\breve{\overline{B}}^\zeta s + \breve{\overline{q}}}{\tau} \qquad -\beta\left[-\kappa\breve{\dot{\zeta}} + \breve{\dot{\zeta}}\,\frac{\delta_\zeta\overline{T_*}^\zeta}{T_*}\right]$$

$$R_\phi = \frac{\breve{\overline{\phi'}}^\zeta}{\tau} \qquad -\beta\left(-RT_*\breve{\dot{\zeta}} - gw\right)$$

...

$$R_w = \frac{w}{\tau_{nh}} \qquad -\beta_{nh}(-g\mu)$$

***Departure** Outer-Loop*

- **adv:** **adv_pos**: Compute the next estimate of the departure points.
  adv_int: Evaluate Right-Hand-Side terms at departure points.

- **pre:** compute Right-Hand-Side terms (**section 17**)

- - first combine R_u, R_v, R_w, R_C into R_C and $R_{\theta,} R_w, R_\phi$ into R''_θ and into R''_ϕ

$$\delta_X R_u + \tfrac{1}{\cos\theta}\delta_Y(\cos\theta R_v) - \frac{1}{\tau}\left(R_C - \frac{\boldsymbol{r}}{g\tau_{nh}}\overrightarrow{R_w}^{\zeta}\right) \equiv R''_C$$

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(R_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}R_w + \frac{\boldsymbol{r}}{g^2\tau_{nh}^2}R_\phi\right) \equiv \breve{R}''_\theta$$

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(R_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}R_w - \frac{\kappa}{RT_*}R_\phi\right) \equiv R''_\phi$$

- - second combine $R''_C, R''_\theta, R''_\phi$ into R_P:

$$R''_C - \left(\hat{\delta}_\zeta R''_\theta + \overline{\breve{R}''_\theta}^{\zeta} + \overrightarrow{\varepsilon' R''_\phi}^{\zeta}\right) \equiv R_P$$

- - third make extra combination at the last level

$$\frac{1}{\varpi_*^+}[(R''_\theta)_{N+\frac{1}{4}} - \varpi_*^-(R''_\theta)_{N-\frac{1}{2}}] = (R''_\theta)_{N+\frac{1}{2}}$$

$$\left[R''_\theta - \frac{\phi_S}{\tau^2 RT_*}\right]_{N+\frac{1}{2}} = (R'''_\theta)_{N+\frac{1}{2}}$$

$$(R_P)_N + C''_S(R'''_\theta)_{N+\frac{1}{2}} = (R'_P)_N$$

The final version of the Right-Hand sides are: R_u, R_v , R''_θ , R''_ϕ , R_w and R'_P

Non-linear *Inner_Loop*

- **nli:** compute non-linear Left-Hand-Side terms: (**sections 15 & 17**)

- first compute $N_u, N_v, N_C, N'_\theta, N'_w, N'_\phi$

$$N_u = -\left(f + \tfrac{\tan\theta}{a} u\right)\overline{v}^{XY} + R\overline{T}^{X\zeta}\delta_X(Bs+q) + \overline{\mu}^{X\zeta}\delta_X\phi' - \delta_X R\overline{T_*}^{\zeta}(Bs + \boldsymbol{r}'q)$$

$$N_v = \left(f + \tfrac{\tan\theta}{a}\overline{u}^{XY}\right)\overline{u}^{XY} + R\overline{T}^{Y\zeta}\delta_Y(Bs+q) + \overline{\mu}^{Y\zeta}\delta_Y\phi' - \delta_Y R\overline{T_*}^{\zeta}(Bs + \boldsymbol{r}'q)$$

$$N_C \equiv \frac{1}{\tau}\left[Bs - \overline{B}^{\zeta\zeta}s + \ln\left(1 + \delta_\zeta \overline{B}^{\zeta}s\right) - \delta_\zeta\overline{B}^{\zeta}s\right]$$

$$N_\theta - \frac{1}{\tau}N_H \equiv N'_\theta = \frac{1}{\tau}\ln\left(\frac{T}{T_*}\right) + \frac{\delta_\zeta[\phi' + R\overline{T_*}^{\zeta}(Bs + \boldsymbol{r}'q)]}{\tau RT_*} - \boldsymbol{r}'\frac{\delta_\zeta q + \breve{\overline{q}}^{\zeta}}{\tau}$$
$$- \kappa\frac{\breve{\overline{B}}^{\zeta}s + \breve{\overline{q}}}{\tau} + \kappa_d\frac{\breve{\overline{B}}^{\zeta}s + \boldsymbol{r}'\breve{\overline{q}}}{\tau} + \breve{\zeta}\frac{\delta_\zeta\overline{T_*}^{\zeta}}{T_*}$$

$$\frac{\text{EXTRA}}{\tau} \equiv N'_\phi$$

...

$$N_w + gN_\mu \equiv N'_w = -g\left(\mu - \delta_\zeta q - \breve{\overline{q}}^{\zeta}\right)$$

- second combine N_u, N_v, N'_w, N_C into N'_C and N'_θ, N'_w, N'_ϕ into N''_θ and N''_ϕ

$$\delta_X N_u + \frac{1}{\cos\theta}\delta_Y(\cos\theta N_v) - \frac{1}{\tau}\left(N_C - \frac{\boldsymbol{r}}{g\tau_{nh}}\overrightarrow{N'_w}^{\zeta}\right) \equiv N''_C$$

$$\frac{1}{\tau(\kappa_d + \varepsilon')}\left(N'_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}N'_w + \frac{\boldsymbol{r}}{g^2\tau_{nh}^2}N'_\phi\right) \equiv \breve{N}''_\theta$$

$$\frac{1}{\tau(\kappa_d + \varepsilon')}\left(N'_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}N'_w - \frac{\kappa}{RT_*}N'_\phi\right)_w \equiv N''_\phi$$

- third combine $N''_C, N''_\theta, N''_\phi$ into N_P

$$N''_C - \left(\hat{\delta}_\zeta N''_\theta + \overline{\breve{N}''_\theta}^{\zeta} + \overrightarrow{\varepsilon' N''_\phi}^{\zeta}\right) \equiv N_P$$

- fourth make following extra combinations at the last level

$$\frac{1}{\varpi_*^+}\left[(N''_\theta)_{N+\frac{1}{4}} - \varpi_*^-(N''_\theta)_{N-\frac{1}{2}}\right] = (N''_\theta)_{N+\frac{1}{2}}$$

$$(N''_\theta)_{N+\frac{1}{2}} = (N'''_\theta)_{N+\frac{1}{2}}$$

$$(N_P)_N + C''_S(N'''_\theta)_{N+\frac{1}{2}} = (N'_P)_N$$

- and obtain final Right-Hand Side of the Elliptic Problem $R'_P - N'_P$

- **sol:** solve the Elliptic Problem (**section 16 & appendix 4**)

$$L'_P = R'_P - N'_P = \nabla^2_\zeta P + \left[\mathbf{D\Gamma D} + \mathbf{M\Gamma D} - \mathbf{D}(\Gamma\varepsilon') - (1-\kappa_d)\mathbf{M}(\Gamma\varepsilon')\mathbf{M}\right]P$$

- **bac:** back substitution: compute variables for next iteration/time step (**section 18**)

$$w: \qquad w = \frac{\tau\, RT_*}{g}[R''_\phi - N''_\phi + \Gamma(\delta_\zeta P + \kappa_d \breve{\overline{P}}^\zeta)]$$

$$q: \; \delta_\zeta q + \breve{\overline{q}}^\zeta = -\frac{1}{g}\left[R_w - N'_w - \frac{w}{\tau_{nh}}\right]; \qquad q_1 = 0$$

...

$$u: \qquad u = \tau\left[R_u - N_u - \delta_X P\right]$$

$$v: \qquad v = \tau\left[R_v - N_v - \delta_Y P\right]$$

$$s: \qquad s = \frac{P_S - \phi_S}{RT_{*S}} - \boldsymbol{r}' q_S$$

$$\dot{\zeta}: \qquad \dot{\zeta} = -\tau[R''_\theta - N''_\theta + \Gamma(\delta_\zeta P - \varepsilon' \overline{P}^\zeta)] - \frac{\overline{B}^\zeta s + \boldsymbol{r}'\overline{q}^\zeta}{\tau}; \; \dot{\zeta}_S = 0$$

$$\phi': \qquad \phi' = P - R\overline{T_*}^\zeta (Bs + \boldsymbol{r}' q)$$

end inner loop

end outer loop

end timestep loop

Appendix 7. The hydrostatic option

We start with the final form of the equations given in **section 5**:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\mu)\nabla_\zeta\phi' = 0$$

$$\frac{d}{dt}\left[Bs + \ln(1+\partial_\zeta Bs)\right] + \nabla_\zeta\cdot\mathbf{V}_h + (\partial_\zeta+1)\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(Bs+q)\right] - \kappa\dot{\zeta} + \dot{\zeta}\,\partial_\zeta \ln T_* = 0$$

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$\frac{T}{T_*} + e^q\,\frac{\partial_\zeta\phi / RT_*}{1+\partial_\zeta Bs} = 0$$

…

$$\frac{dw}{dt} - g\mu = 0$$

$$1 + \mu - e^q\left(1 + \frac{\partial_\zeta q}{1+\partial_\zeta Bs}\right) = 0$$

The hydrostatic approximation may be considered to consist in neglecting non-hydrostatic pressure effects, therefore assuming q=μ=0. Then the vertical acceleration *dw/dt* is neglected. No equations after the **…** are required in the solution system. Therefore, we only need to solve (using the switch *Schm_hydro_L=.true.*):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta\phi' = 0$$

$$\frac{d}{dt}\left[Bs + \ln(1+\partial_\zeta Bs)\right] + \nabla_\zeta\cdot\mathbf{V}_h + (\partial_\zeta+1)\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa Bs\right] - \kappa\dot{\zeta} + \dot{\zeta}\,\partial_\zeta \ln T_* = 0$$

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$\frac{T}{T_*} + e^q\,\frac{\partial_\zeta\phi / RT_*}{1+\partial_\zeta Bs} = 0$$

Appendix 8. The autobarotropic model

We build an autobarotropic model (Dutton, *The Ceaseless Wind*, pp 186-7) from the three-dimensional code of GEM in order to simulate a barotropic model. We do that in

i) eliminating the physical effects,
ii) making the hydrostatic hypothesis,
iii) introducing a key $\delta_{autobarot}$=0 to eliminate the pressure tendency $d(Bs)/dt$ in both the thermodynamic and continuity equations,
iv) initializing with barotropic conditions :
$\mathbf{V}_h \neq \mathbf{V}_h(\zeta);\ T = T_* = const;\ \dot{\zeta} = \dot{\xi} = 0;\ \phi' + RT_* Bs = \phi'_T = \phi_S + RT_* s$,
conditions which will be maintained afterwards, hence the name autobarotropic model.

From the hydrostatic equations (**Appendix 7**):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta \phi' = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \partial_\zeta Bs\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \left(\partial_\zeta + 1\right)\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa Bs\right] - \kappa\dot{\zeta} + \dot{\zeta}\,\partial_\zeta \ln T_* = 0$$

$$\frac{T}{T_*} + \frac{\partial_\zeta \phi / RT_*}{1 + \partial_\zeta Bs} = 0$$

with B defined simply as $B = \dfrac{\zeta - \zeta_T}{\zeta_S - \zeta_T}$ and considering barotropic initial conditions $\left[\mathbf{V}_h \neq \mathbf{V}_h(\zeta);\ T = T_* = const;\ \dot{\zeta} = 0\right]$, we derive from the hydrostatic equation that P is uniform in the vertical:

$$P = \phi' + RT_* Bs = \phi'_T = \phi_S + RT_* s \neq P(\zeta)$$

and we may define $s = \dfrac{\phi'_T - \phi_S}{RT_*}$. Indeed, $\dfrac{\partial\left(\phi'/RT_* + Bs\right)}{\partial(\zeta + Bs)} = \dfrac{\partial P}{\partial \zeta}\dfrac{\partial \zeta}{\partial(\zeta + Bs)} = 0$,hence $\dfrac{\partial P}{\partial \zeta} = 0$.

We therefore have in the momentum equation:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + \nabla_\zeta \phi_T = 0$$

and since $P = \phi_T \neq P(\zeta)$, then $\mathbf{V}_h$ stays $\mathbf{V}_h \neq \mathbf{V}_h(\zeta)$.

Now, even though $\dot{\zeta} = 0$ and $T = T_* = const$ initially, temperature will change unless the tendency *d(Bs)/dt* is set to vanish. Multiplying *d(Bs)/dt* by $\delta_{autobarot} = 0$ whenever it occurs does the job though. If $\dot{\zeta} = T' = 0$ is maintained true after a time step, the equations

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta \phi' = 0$$

$$\delta_{autobarot}\frac{d}{dt}Bs + \frac{d}{dt}\ln\left(1 + \partial_\zeta Bs\right) + \nabla_\zeta \cdot \mathbf{V}_h + \left(\partial_\zeta + 1\right)\dot{\zeta} = 0$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\delta_{autobarot}\frac{d}{dt}Bs - \kappa\dot{\zeta} = 0$$

$$\frac{T}{T_*} + \frac{\partial_\zeta \phi / RT_*}{1 + \partial_\zeta Bs} = 0$$

are then effectively equivalent to

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h \nabla_\zeta\left(\phi' + RT_* Bs\right) = 0$$

$$\frac{d}{dt}\ln\left(1 + \partial_\zeta Bs\right) + \nabla_\zeta \cdot \mathbf{V}_h = 0$$

$$\partial_\zeta\left(\phi' + RT_* Bs\right) = 0$$

and since $B = \dfrac{\zeta - \zeta_T}{\zeta_S - \zeta_T}$ and with $\phi_{*T} = RT_*\left(\zeta_S - \zeta_T\right)$, then

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + \nabla_\zeta \phi'_T = 0$$

$$\frac{d}{dt}\ln\left(1 + \frac{\phi'_T - \phi_S}{\phi_{*T}}\right) + \nabla_\zeta \cdot \mathbf{V}_h = 0$$

and finally with $\phi_T = \phi_{*T} + \phi'_T$,

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + \nabla_\zeta \phi_T = 0$$

$$\frac{d}{dt}\ln\left(\phi_T - \phi_S\right) + \nabla_\zeta \cdot \mathbf{V}_h = 0$$

And these relations are invariant in the vertical. Hence, the model equations with a vertical structure (a few levels, at least 3: e.g. hyb = 0.5, 0.7, 0.9, to satisfy the operations), but starting with barotropic conditions, simulates the barotropic equations.

The model is autobarotropic.

Appendix 9. *Open top* boundary conditions

The goal is to develop an open boundary condition at the top, i.e. a condition with $X_{OpenT} \neq 0$. First, let us deal with the linear system (**Appendix 4**) at the stage where we have obtained L_P at all levels starting from k_0>1:

$$\begin{aligned}(L_P)_{k_0} &= \delta_{XX} P_{k_0} + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\,\delta_Y P_{k_0}\right)\\ &+\left(\frac{1}{\Delta\zeta_{k_0}}+\varpi^+_{k_0}\right)\Gamma_{k_0+\frac12}\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac12}}-\left(\frac{1}{\Delta\zeta_{k_0}}-\varpi^-_{k_0}\right)\Gamma_{k_0-\frac12}\frac{P_{k_0}-P_{k_0-1}}{\Delta\zeta_{k_0-\frac12}}-\frac{P_{k_0}}{\Delta\zeta_{k_0}}\left[(\Gamma\varepsilon')_{k_0+\frac12}-(\Gamma\varepsilon')_{k_0-\frac12}\right]\\ &-(1-\kappa_d)\left[\varpi^+_{k_0}(\Gamma\varepsilon')_{k_0+\frac12}\frac{P_{k_0+1}+P_{k_0}}{2}+\varpi^-_{k_0}(\Gamma\varepsilon')_{k_0-\frac12}\frac{P_{k_0}+P_{k_0-1}}{2}\right]\end{aligned}$$

and suppose we wanted to impose a closed upper boundary condition at the level $k_0 - \frac{1}{2}$. We might then have used, since (the thermodynamic level being) available, the relation:

$$(L''_\theta)_{k_0-\frac12}+\frac{1}{\tau}X_{k_0-\frac12}=-\Gamma_{k_0-\frac12}\left(\frac{P_{k_0}-P_{k_0-1}}{\Delta\zeta_{k_0-\frac12}}-\varepsilon'_{k_0-\frac12}\frac{P_{k_0}+P_{k_0-1}}{2}\right)$$

assuming $X_{k_0-\frac12}$=0 to obtain a relation for P_{k_0-1}:

$$P_{k_0-1}=\alpha_T P_{k_0}+C_T(L''_\theta)_{k_0-\frac12}$$

where

$$C_T=\frac{1}{\dfrac{\Gamma_{k_0-\frac12}}{\Delta\zeta_{k_0-\frac12}}+\dfrac{(\Gamma\varepsilon')_{k_0-\frac12}}{2}};\qquad \alpha_T=\left(\frac{\Gamma_{k_0-\frac12}}{\Delta\zeta_{k_0-\frac12}}-\frac{(\Gamma\varepsilon')_{k_0-\frac12}}{2}\right)C_T$$

Hence we might write

$$\begin{aligned}(L'_P)_{k_0} &= \delta_{XX} P_{k_0} + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\,\delta_Y P_{k_0}\right)\\ &+\left(\frac{1}{\Delta\zeta_{k_0}}+\varpi^+_{k_0}\right)\Gamma_{k_0+\frac12}\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac12}}-\left(\frac{1}{\Delta\zeta_{k_0}}-\varpi^-_{k_0}\right)\Gamma_{k_0-\frac12}\frac{(1-\alpha_T)P_{k_0}}{\Delta\zeta_{k_0-\frac12}}-\frac{P_{k_0}}{\Delta\zeta_{k_0}}\left[(\Gamma\varepsilon')_{k_0+\frac12}-(\Gamma\varepsilon')_{k_0-\frac12}\right]\\ &-(1-\kappa_d)\left[\varpi^+_{k_0}(\Gamma\varepsilon')_{k_0+\frac12}\frac{P_{k_0+1}+P_{k_0}}{2}+\varpi^-_{k_0}(\Gamma\varepsilon')_{k_0-\frac12}\frac{(1+\alpha_T)P_{k_0}}{2}\right]\end{aligned}$$

where

$$(L'_P)_{k_0}=(L_P)_{k_0}-C''_T(L''_\theta)_{k_0-\frac12}$$

with

$$C_T'' = \left[\left(\frac{1}{\Delta\zeta_{k_0}} - \varpi_{k_0}^-\right)\frac{\Gamma_{k_0-\frac{1}{2}}}{\Delta\zeta_{k_0-\frac{1}{2}}} - (1-\kappa_d)\frac{\varpi_{k_0}^-}{2}(\Gamma\varepsilon')_{k_0-\frac{1}{2}}\right]C_T$$

and thus be able to solve the elliptic problem. This in fact exactly corresponds to the original so-called numerically un-truncated boundary condition, whereby a thermodynamic equation existed at level $\frac{1}{2}$, viz. above the first momentum level. We could therefore still impose the original top boundary condition but only starting at index $\frac{3}{2}$ (momentum levels starting at index 2).

In order to impose an open top boundary condition, we may proceed in a similar manner. We must however find a different closure assumption, calculating, instead of imposing, $X_{k_0-\frac{1}{2}}$. This may be provided by the relation,

$$(L_\theta)_{k_0-\frac{1}{2}} = \frac{T'_{k_0-\frac{1}{2}}}{\tau T_{*k_0-\frac{1}{2}}} - \kappa_d X_{k_0-\frac{1}{2}},$$

suggesting simply imposing temperature at the top. In effect, the following combination

$$(L_\theta''')_{k_0-\frac{1}{2}} = (L_\theta'')_{k_0-\frac{1}{2}} - \frac{(L_\theta)_{k_0-\frac{1}{2}}}{\kappa_d\tau} + \frac{T'_{openT}}{\kappa_d\tau^2 T_{*k_0-\frac{1}{2}}} = -\Gamma_{k_0-\frac{1}{2}}\left(\frac{P_{k_0} - P_{k_0-1}}{\Delta\zeta_{k_0-\frac{1}{2}}} - \varepsilon'_{k_0-\frac{1}{2}}\frac{P_{k_0} + P_{k_0-1}}{2}\right)$$

provides a new relation by which P_{k_0-1} may be obtained:

$$P_{k_0-1} = \alpha_T P_{k_0} + C_T (L_\theta''')_{k_0-\frac{1}{2}}$$

.

$(L_\theta''')_{k_0-\frac{1}{2}} \equiv L_B = R_B - N_B$ simply replacing $(L_\theta'')_{k_0-\frac{1}{2}}$ in the relation. All of this is trivial then, except for the calculation of the right-hand sides corresponding to L_B, i.e. R_B and N_B:

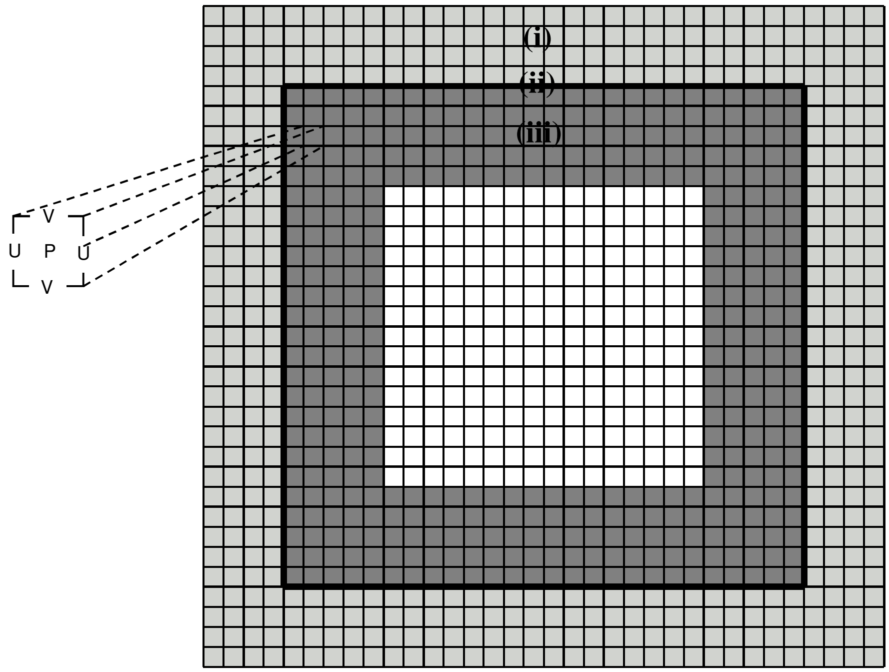
$$R_B = (R''_\theta)_{k_0-\frac{1}{2}} - \frac{1}{\kappa_d\tau}\left[(R'_\theta)_{k_0-\frac{1}{2}} - \frac{T'_{openT}}{\tau T_{*k_0-\frac{1}{2}}}\right]$$

$$N_B = (N''_\theta)_{k_0-\frac{1}{2}} - \frac{1}{\kappa_d\tau}\frac{(N_\theta)_{k_0-\frac{1}{2}}}{T_{*k_0-\frac{1}{2}}}$$

In the non-hydrostatic case, another condition is needed, namely q_{openT}.

Appendix 10. Lateral boundary conditions

A limited area (LAM) version of GEM exists. It requires lateral boundary conditions. These are provided by three sets of grid point values:



(i) The *first* set is external to the LAM domain and allows the semi-Lagrangian scheme to function as if no boundary existed, i.e. a sufficient number of points exist outside of the domain so that the upwind values of all relevant fields can be obtained by interpolation provided a predetermined Courant number is not exceeded. The relevant fields are the R_i's, the Right-Hand Sides terms calculated from the previous timestep history carrying model variables. If the values provided to the LAM come from a global host-model identical to the LAM in all respects (space and time resolutions, physical parameterizations, etc) then the host-model results for the R_i's are reproduced.

(ii) The *second* set is the boundary set proper: it comprises exclusively the *wind component normal to the boundary and at the boundary* itself. These grid point values serve to close the elliptic problem in the horizontal. In effect, the so-called elliptic equation will contain in particular (see **section 17**) the following terms:

$$\left(\delta_X L_u\right)_{i_0 jk} + \left[\tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta L_v\right)\right]_{i_0 jk} - \frac{\left(L_C\right)_{i_0 jk}}{\tau} + \ldots \equiv \left(L_P\right)_{i_0 jk} = \left(\delta_{XX} P\right)_{i_0 jk} + \left[\tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\delta_Y P\right)\right]_{i_0 jk} + \ldots$$

To the left, the L's must be known quantities. To the right, there is only the unknown P. Here we consider, as an example, the grid points with the label i_0. This is the X-direction and we assume that i_0 is the first internal model cell on its left-hand side. Developing the relevant terms, we obtain

$$\frac{(L_u)_{i_0+\frac{1}{2}jk} - (L_u)_{i_0-\frac{1}{2}jk}}{\Delta X_j} + \ldots \equiv (L_P)_{i_0 jk} = \frac{(\delta_X P)_{i_0+\frac{1}{2}jk} - (\delta_X P)_{i_0-\frac{1}{2}jk}}{\Delta X_j} + \ldots$$

But note, the equation

$$(L_u)_{i_0-\frac{1}{2}jk} = \frac{u_{i_0-\frac{1}{2}jk}}{\tau} + (\delta_X P)_{i_0-\frac{1}{2}jk}$$

which should have served to eliminate $u_{i_0-\frac{1}{2}jk}$ from the continuity equation does not exists. $(L_u)_{i_0-\frac{1}{2}jk}$ is in fact an unknown quantity. Let us then restore $u_{i_0-\frac{1}{2}jk}$ in the previous equation:

$$\frac{(L_u)_{i_0+\frac{1}{2}jk} - u_{i_0-\frac{1}{2}jk}/\tau}{\Delta X_j} + \ldots \equiv (L_P)_{i_0 jk} = \frac{(\delta_X P)_{i_0+\frac{1}{2}jk} - 0}{\Delta X_j} + \ldots$$

Thus the elliptic problem may be solved if we provide the normal wind component on the boundary, $u_{i_0-\frac{1}{2}jk}$, as a boundary condition. The elliptic problem may appear as if we had set $(\delta_X P)_{i_0-\frac{1}{2}jk} = 0$ as a boundary condition on *P* to the left of the system. In fact we have replaced an unknown quantity, $(L_u)_{i_0-\frac{1}{2}jk}$, by a known one, $u_{i_0-\frac{1}{2}jk}/\tau$, the true boundary condition, to the right of the system. The same procedure is applied to the normal wind components on all the boundaries of the LAM. Again, if the normal wind components provided to the LAM come from an identical global host-model, then the host-model results are reproduced. Since the solution of the elliptic problem corresponds to a future timestep, the set of boundary winds must come from the timestep following that from which came the external set.

(iii) Finally, a *third* set of grid point values are internal to the LAM domain. They allow for a gradual relaxation of LAM-fields to the HOST-fields as we approach the boundary. All history carrying variables are relaxed this way. Of course, if the host-model is identical (the *acid test*), this third step of the procedure is redundant.

In GEM presently, physical parameterization is added (split mode) after the dynamics, i.e. after the relaxation step just mentioned. Thus for the LAM to reproduce the host-model results, the *future* values provided in steps (ii) and (iii) must come from the host-model after the dynamics *prior* physical parameterization while the *past* values provided in step (i) must come from the host-model *after* physical parameterization.

N.B. As soon as horizontal winds are modified by space and time interpolation, i.e. when not performing the acid test, the vertical motion field $\dot{\zeta}$ should be diagnosed (see **Appendix 17**)

Appendix 11. Time varying topography

The initial conditions as well as the lateral conditions (see **Appendix 10**) of a LAM are frequently provided by a host-model or by an analysis made on the host-model grid with much coarser horizontal resolution, typically at least a factor of three coarser. And the information usually comes in terrain-following vertical coordinates. Then the bottom surfaces, the topography, of the host and LAM may differ considerably. Straightforward interpolation-extrapolation often results in poorly balanced fields: a point fairly high in the host may have relatively strong winds which may find themselves near the surface in the LAM; vice versa a surface point with light winds in the host may find itself fairly high in the LAM. For the first two sets of lateral conditions, i.e. outside and on the boundary of the LAM domain, the host topography may be kept, but for the third set, the relaxation zone, the problem cannot be avoided. One may only attenuate the problem by relaxing the topography in essentially the same way that the other model fields are relaxed and then interpolating-extrapolating the variables. As for the initial imbalances, it has been found desirable to initialize the LAM with the coarser host topography, gradually modifying it to reach the finer LAM topography after a suitable interval of integration time: the LAM then having a so-called time-varying topography field. Artificial though it may be for the atmosphere, this is a perfectly acceptable mathematical procedure and, provided the induced vertical motions remain small, the meteorological consequences *may* remain acceptable (a 10 cm/s topography velocity is able to lift the terrain by more than 1 km in 3 hours).

Examining the equations, we find that a local tendency of geopotential is provided and calculated implicitly by the equation:

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

A surface level is present in the vertical discretization ($\dot{\boldsymbol{\zeta}}_S = 0$):

$$\frac{d\phi_S}{dt} - gw_S = 0$$

After time discretization, we have:

$$\frac{(\phi_S)^A}{\tau} - g(w_S)^A = \frac{(\phi_S)^D}{\tau} + g\beta(w_S)^D$$

$(\phi_S)^A$ is the surface geopotential at the arrival point, i.e. at the grid point at the future time. It is an external parameter which may be externally specified. In hydrostatic-pressure coordinate, the **time varying topography option** is just and *only just* that: modifying ϕ_S at the appropriate place in the model code.

Appendix 12. Trajectory calculations.

The essence of semi-Lagrangian advection resides in trajectory calculations. These serve to estimate the upstream position of fields to be advected. Once these positions are found, the upstream values of the fields are obtained by (cubic) interpolation. The three-dimensional equation to be solved is:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

as follows:

$$\Delta\mathbf{r} = \mathbf{v}\Delta t + \frac{d\mathbf{v}}{dt}\frac{\Delta t^2}{2} + \ldots$$

Three methods may be used in GEM: the **mid-point rule**, the **trapezoidal rule** and **SETTLS** scheme. The mid-point rule (a time mean followed by a space interpolation) can be described as follows:

$$\Delta\mathbf{r}^i = \frac{\mathbf{v}\left(t, \mathbf{r} - \Delta\mathbf{r}^{i-1}/2\right) + \mathbf{v}\left(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{i-1}/2\right)}{2}\Delta t = \mathbf{v}_M \Delta t$$

where i is for iterations being made due to the non-linear nature of the process. We write the trapezoidal rule (a space interpolation followed by a space-time mean) with the off-centering possibility as follows:

$$\Delta\mathbf{r}^i = \left[b^A\mathbf{v}(t, \mathbf{r}) + \left(1 - b^A\right)\mathbf{v}\left(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{i-1}\right)\right]\Delta t = \left[b^A\mathbf{v}^A + \left(1 - b^A\right)\mathbf{v}^D\right]\Delta t$$

Finally, SETTLS scheme may be written

$$\Delta\mathbf{r}^i = \mathbf{v}\left(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{i-1}\right)\Delta t + \frac{\mathbf{v}(t - \Delta t, \mathbf{r}) - \mathbf{v}\left(t - 2\Delta t, \mathbf{r} - \Delta\mathbf{r}^{i-1}\right)}{\Delta t}\frac{\Delta t^2}{2}$$

$$\Delta\mathbf{r}^i = \frac{\mathbf{v}^A + \mathbf{v}^D}{2}\Delta t; \quad \mathbf{v}^A = \mathbf{v}(t - \Delta t, \mathbf{r}); \quad \mathbf{v}^D = 2\mathbf{v}\left(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{i-1}\right) - \mathbf{v}\left(t - 2\Delta t, \mathbf{r} - \Delta\mathbf{r}^{i-1}\right)$$

The first two methods assume that winds at time t have been estimated and so require an iterative outer (so-called Crank-Nicholson) step. SETTLS scheme, using winds at t-Δt and t-$2\Delta t$, do not. The scheme performs satisfactorily in barotropic mode. Recently, after the elimination of logarithmic tendencies and with the introduction of a vertically variable T_*, stable integrations in full NWP mode were achieved. The scheme is more efficient. So far though, the meteorological performance remains inferior.

Traditionally, GEM was using the mid-point rule and linear interpolation in trajectory calculations. However, noise in Schär's mountain case could only be eliminated if using the trapezoidal rule combined with cubic interpolation: a case of consistency between semi-Lagrangian trajectory calculations and following semi-Lagrangian advection calculations. Moreover, improved forecasts, improved trajectories, were found to result when the trapezoidal rule combined with cubic interpolation were substituted to the mid-point rule with linear interpolation and so now this is the favored method.

Initially, the trapezoidal rule was implemented, like the mid-point rule and SETTLS scheme, without off-centering. This is though a necessary feature of GEM for the advective part of semi-Lagrangian calculations when running in NWP mode. Off-centering is neither required nor used in Schär's mountain case as the expected smoothing is unwanted. However, if off-centering is activated in the advection calculations but not in the trajectory calculations, as traditionally done in GEM, a noisy mountain wave solution reappears due to this renewed inconsistency. Applying off-centering consistently, viz. applying off-centering simultaneously in trajectory as well as advection calculations is therefore required for consistency: hence its introduction in the trapezoidal method. So far, off-centering is a necessary evil in this model and better applying it consistently …

Appendix 13. The option of vertically varying T_*

Is there an interest in using a vertically varying reference thermal profile, T_*, close to the actual atmospheric profile in GEM? According to Bénard (MWR 2004, pp. 1319-1324), there might be some advantage in doing so, in spite of the fact that traditional schemes have been shown to be less robust if using such variable T_* than when using a (warm) fixed value. The reduced stability when using a variable T_* is apparently not due to the magnitude of the difference between T and T_*, T', but rather to the reduction of static stability in the linear thermodynamic equation,

$$\frac{d}{dt}\left(\frac{T'}{T_*}\right)-\kappa\left(\frac{d}{dt}(Bs+q)+\dot{\zeta}\right)+\dot{\zeta}\,\partial_\zeta \ln T_* = 0,$$

when $\partial_\zeta \ln T_* > 0$. Bénard's proposition is to exclude $\dot{\zeta}\,\partial_\zeta \ln T_*$ from the linear terms when having a variable T_*, thus maintaining and perhaps in certain cases improving robustness of the scheme while keeping the advantage of reduced non-linear terms involving temperature since T' is reduced.

Very little changes are required in GEM to implement this option, *besides having variable* T_* *and variable parameters involving* T_*. Basically, only the thermodynamic equation is modified (**section 7**). Nevertheless, the fact that T_* varies has an important impact on the code up to the elliptic problem which now reads (see **Appendix 4**):

$$L'_P = \nabla_\zeta^2 P + \left[\mathbf{D}\Gamma\mathbf{D} + \mathbf{M}\Gamma\mathbf{D} - \mathbf{D}(\Gamma\varepsilon') - (1-\kappa_d)\mathbf{M}(\Gamma\varepsilon')\mathbf{M}\right]P$$

where **D** and **M** symbolically represent Difference and Mean operators instead of the simpler

$$L'_P = \nabla_\zeta^2 P + \Gamma\left[\mathbf{DD} + \mathbf{MD} - (1-\kappa_d)\varepsilon'\mathbf{MM}\right]P$$

when Γ and ε' are constant.

Appendix 14. The modified epsilon, $\varepsilon' = \boldsymbol{r}\varepsilon$, option

Bénard (MWR 2004 pp. 1319-1324) found increased stability of the non-hydrostatic equations when the structure equation,

$$\left(N_*^2 + \frac{1}{\tau^2}\right)\nabla_\zeta^2 P + \frac{\delta_{\zeta\zeta}P + \overline{\delta_\zeta P}^\zeta}{\tau^2 H_*^2} - \frac{1}{\tau^4 c_*^2}\overline{P}^{\zeta\zeta} = 0$$

is modified as follows

$$\left(\frac{N_*^2}{\boldsymbol{r}} + \frac{1}{\tau^2}\right)\nabla_\zeta^2 P + \frac{\delta_{\zeta\zeta}P + \overline{\delta_\zeta P}^\zeta}{\boldsymbol{r}\tau^2 H_*^2} - \frac{1}{\tau^4 c_*^2}\overline{P}^{\zeta\zeta} = 0; \quad r \le 1$$

Substituting the values of the constants (here we assume that T_*=const.),

$$N_*^2 => \frac{g^2}{c_p T_*}; \;\; c_*^2 => \frac{RT_*}{1-\kappa}; \;\; H_* => \frac{RT_*}{g}; \;\; \frac{RT_*}{g^2\tau^2} => \varepsilon$$

We get successively

$$\left(\frac{g^2}{\boldsymbol{r}c_p T_*} + \frac{1}{\tau^2}\right)\nabla_\zeta^2 P + \frac{\delta_{\zeta\zeta}P + \overline{\delta_\zeta P}^\zeta}{\boldsymbol{r}\tau^2 [RT_*]^2 / g^2} - \frac{1-\kappa}{\tau^4 RT_*}\overline{P}^{\zeta\zeta} = 0$$

$$(\kappa + \boldsymbol{r}\varepsilon)\nabla_\zeta^2 P + \frac{\delta_{\zeta\zeta}P + \overline{\delta_\zeta P}^\zeta - \boldsymbol{r}\varepsilon(1-\kappa)\overline{P}^{\zeta\zeta}}{\tau^2 RT_*} = 0$$

$$\nabla_\zeta^2 P + \frac{1}{(\kappa + \varepsilon')\tau^2 RT_*}\left[\delta_{\zeta\zeta}P + \overline{\delta_\zeta P}^\zeta - \varepsilon'(1-\kappa)\overline{P}^{\zeta\zeta}\right] = 0$$

recovering our own structure equation (when T_*=const.) with

$$\varepsilon' = \boldsymbol{r}\varepsilon \, .$$

The argument is that T_* is reduced to rT_* in certain terms of the equations and this has a stabilizing effect with respect to sound waves.

There is in fact a well known way to modify ε, ε', which is the off-centering. In effect, as noted, $\varepsilon = RT_* / g^2\tau^2$, and $\tau = \Delta t b^A$ but this particular τ can be shown to come from the non-hydrostatic side of the system, which could be dealt with a different, larger, off-centering, $\tau_{nh} = \Delta t b_{nh}^A$. Hence, we will consider a different ε, $\varepsilon = RT_* / g^2\tau_{nh}^2$, in addition to a modify ε, $\varepsilon' = \boldsymbol{r}\varepsilon$.

Going into the equations further backward (**section 17**), we find:

$$\delta_X L_u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta L_V\right) - \frac{1}{\tau}\left(L'_C - \frac{\boldsymbol{r}}{g\tau_{nh}}\overrightarrow{L'_w}^{\zeta}\right) \equiv L''_C = \delta_{XX}P + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta\delta_Y P\right) - \frac{1}{\tau}\left(\delta_\zeta X + \overline{X}^{\zeta}\right) + \frac{g\varepsilon'}{\tau R T_*}\vec{w}^{\zeta}$$

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(L'_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}L'_w + \frac{\boldsymbol{r}}{g^2\tau_{nh}^2}L'_\phi\right) \equiv \breve{L}''_\theta = -\Gamma\left(\delta_\zeta P - \varepsilon'\breve{\overline{P}}^{\zeta}\right) - \frac{\breve{X}}{\tau}$$

...

$$\frac{1}{\tau(\kappa_d+\varepsilon')}\left(L'_\theta + \frac{\boldsymbol{r}}{g\tau_{nh}}L'_w - \frac{\kappa_d}{RT_*}L'_\phi\right) \equiv L''_\phi = -\Gamma\left(\delta_\zeta P + \kappa_d\breve{\overline{P}}^{\zeta}\right) + \frac{gw}{\tau R T_*}$$

all the non-hydrostatic equations having to be multiplied by $\boldsymbol{r}$. Finally, going into the initial linear system (**section14**), we must have

$$L_u = \frac{u}{\tau} + \delta_X[\phi' + RT_*(Bs + \boldsymbol{r}'q)]$$

$$L_V = \frac{V}{\tau} + \delta_Y[\phi' + RT_*(Bs + \boldsymbol{r}'q)]$$

$$L_C = \frac{\overline{B}^{\zeta\zeta}s + \delta_\zeta\overline{B}^{\zeta}s}{\tau} + \delta_X u + \tfrac{1}{\cos\theta}\delta_Y\left(\cos\theta V\right) + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}$$

$$L_\theta = \frac{T'}{\tau T_*} - \kappa_d\left(\breve{\dot{\zeta}} + \frac{\breve{\overline{B}}^{\zeta}s + \boldsymbol{r}'\breve{\overline{q}}^{\zeta}}{\tau}\right)$$

$$L_\phi = \frac{\breve{\overline{\phi'}}^{\zeta}}{\tau} - RT_*\breve{\dot{\zeta}} - gw$$

$$L_H = \frac{T'}{T_*} + \frac{\delta_\zeta[\phi' + RT_*(Bs + \boldsymbol{r}'q)]}{RT_*} - \boldsymbol{r}'\left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right)$$

...

$$L_w = \frac{w}{\tau_{nh}} - g\mu$$

$$L_\mu = \mu - \left(\delta_\zeta q + \breve{\overline{q}}^{\zeta}\right)$$

In other words, whenever q modifies the hydrostatic subset of equations, it will have to be multiplied by $\boldsymbol{r}'$ and, of course, τ_{nh} will serve in L_w. This is not quite un-expected.

Appendix 15. Dynamics-Physics Interface

a) *Vertical coordinate transformation*

GEM dynamics is defined in log-hydrostatic pressure coordinate, ζ:

$$\ln\pi = \zeta + Bs$$

where π is hydrostatic pressure, $s = \ln(\pi_S / p_{ref})$, π_S being hydrostatic surface pressure and p_{ref} a reference pressure value. B is a parameter varying between one at the model surface and zero at the model top.

What is the coordinate used by GEM physics? It is not ζ. While it remains of the hydrostatic-pressure type, it is a traditional sigma (σ) coordinate:

$$\sigma = \pi / \pi_S$$

It means then that a coordinate transformation is involved in GEM interface between Dynamics and Physics calculations. We note that the transformation is not independent of π_S. It is therefore not independent of the time variation of π_S. When we enter the Physics calculation two values of π_S are available [π_S^-, beginning of time step, before Dynamics step, or π_S^*, after Dynamics step]. Which one should we use in the coordinate transformation $\sigma(\zeta,\pi_S)$?

Since Physics calculations are done in σ-coordinate, of course we want σ to remain constant during the Physics calculations. The quantity σ, a dependent variable for the Dynamics, becomes an independent variable for the Physics and π_S^* is the value to use in the transformation of coordinates. It leads to values of σ which we will not further change during the Physics step. Many physical processes, thermodynamic processes etc, the processes in which mass is conserved, may be considered to occur at constant pressure in fact. Even the processes which affect the mass of the atmosphere, like the in/out-fluxes of water, which then further affect the surface pressure, do not affect the values of σ.

b) *Water vapor and precipitation fluxes: source of mass*

In the atmosphere, there are no sources/sinks of mass. Therefore the equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0$$

is strictly valid. There are though, in GEM, fluxes of water (condensates as well as vapor) through the earth's surface and these have so far not been accounted for in the total mass budget. Adding water in the atmosphere while keeping the mass constant amounts to changing dry air into water. One way to account for the addition of mass is to add a "*source*" of surface pressure,

π_S. In effect, π_S / g corresponds to the mass of the atmosphere over a unit surface; $s = \ln(\pi_S / p_{ref})$ is a model variable and π given by

$$\frac{\partial \pi}{\partial z} = -g\rho$$

is linked to the model coordinate through $\ln \pi = \zeta + Bs$. Integrating

$$g \int_{z_S}^{z_T} \rho dz = \int_{\pi_T}^{\pi_S} d\pi = \pi_S - \pi_T$$

Considering mass changes, $\delta\rho$, they must give rise to a surface pressure change,

$$g \int_{z_S}^{z_T} \delta\rho dz = \Delta\pi_S$$

mass changes specifically due to water changes, $\delta\rho_w$

$$\delta\rho = \delta\rho_w = \delta(\rho q_w) = \rho \frac{\delta q_w}{1 - q_w}; \qquad q_w = q_v + \sum q_i$$

Hence, the variation of total mass of the atmosphere in GEM may be calculated as follows:

$$\boxed{\Delta\pi_S = g \int_{z_S}^{z_T} \frac{\delta q_w}{1 - q_w} \rho dz = \int_{\pi_T}^{\pi_S} \frac{\delta q_w}{1 - q_w} d\pi}$$

Note that q_w includes cloud water and precipitations. Note that $\Delta\pi_S$ is a 2 dimensional variable. Discretely, with indices *A* and *B* standing respectively for after and before a change, it is exact to calculate $\delta\rho$ as follows:

$$\delta\rho = \frac{\rho_B \delta q_w}{1 - q_{wA}}$$

In effect,

$$\begin{aligned}\delta\rho = \delta\rho_w &= \delta(\rho q_w) = \rho_A q_{wA} - \rho_B q_{wB} \\ &= \rho_B (q_{wA} - q_{wB}) + (\rho_A - \rho_B) q_{wA} \\ &= \rho_B \delta q_w + \delta\rho \, q_{wA}\end{aligned}$$

Here then is the way to take care of the real sources of mass due to fluxes of the water substance through the earth's surface.

Appendix 16. Diagnostic calculation of vertical motion at initial time

There are two vertical motion fields required at initial time. The first, $\dot{\zeta}$, is truly diagnostic fields. The second, *w*, is a diagnostic field only when the hydrostatic approximation is made; in the non-hydrostatic case, *w* could become an analyzed field.

a) *Diagnostic calculation of* $\dot{\zeta}$

From the continuity equation (**section 4**):

$$\frac{d}{dt}\ln\left(\pi\frac{\partial\ln\pi}{\partial\zeta}\right)+\nabla_\zeta\cdot\mathbf{V}_h+\frac{\partial\dot{\zeta}}{\partial\zeta}=0$$

transformed as follows

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial\pi}{\partial t}+\dot{\zeta}\frac{\partial\pi}{\partial\zeta}\right)+\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)=0$$

and integrated

$$\frac{\partial\pi}{\partial t}+\dot{\zeta}\frac{\partial\pi}{\partial\zeta}+\int_{\zeta_T}^{\zeta}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta=0 \qquad (1)$$

$$\frac{\partial\pi_S}{\partial t}+\int_{\zeta_T}^{\zeta_S}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta=0$$

we derive an explicit relation for $\dot{\zeta}$ **:**

$$\dot{\zeta}\frac{\partial\ln\pi}{\partial\zeta}=\frac{B}{\pi_S}\int_{\zeta_T}^{\zeta_S}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta-\frac{1}{\pi}\int_{\zeta_T}^{\zeta}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta$$

In effect, $\ln\pi=\zeta+Bs$, hence

$$\frac{\partial\ln\pi}{\partial t}=B\frac{\partial s}{\partial t}=B\frac{\partial\ln\pi_S}{\partial t}$$

and

$$\frac{\partial\ln\pi}{\partial\zeta}=1+\frac{\partial B}{\partial\zeta}s$$

In discrete form we have

$$\dot{\zeta}_{k-\frac{1}{2}}\left(\frac{\partial\ln\pi}{\partial\zeta}\right)_{k-\frac{1}{2}}=\frac{B_{k-\frac{1}{2}}\mathcal{J}_{N+\frac{1}{2}}}{\pi_S}-\frac{\mathcal{J}_{k-\frac{1}{2}}}{\pi_{k-\frac{1}{2}}};\qquad \mathcal{J}_{k-\frac{1}{2}}=\sum_{l=1}^{k-1}\left[\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)\right]_l\Delta\zeta_l$$

b) *Di*

agnostic calculation of w

We use the approximation:

$$w \approx -\frac{\dot{\pi}}{g\rho}$$

The approximation seems acceptable in general but note: at the model top $\dot{\pi} = 0$ by construction while $w \neq 0$; similarly at the bottom, when the terrain is flat, $w = 0$ while $\dot{\pi} \neq 0$ in general. We obtain an explicit relation for $\dot{\pi}$ again from the integrated continuity equation as follows:

$$\dot{\pi} = \frac{\partial \pi}{\partial t} + \mathbf{V}_h \cdot \nabla_\zeta \pi + \dot{\zeta}\frac{\partial \pi}{\partial \zeta}$$

$$\dot{\pi} = \mathbf{V}_h \cdot \nabla_\zeta \pi - \int_{\zeta_T}^{\zeta} \nabla_\zeta \cdot \left(\frac{\partial \pi}{\partial \zeta}\mathbf{V}_h\right) d\zeta$$

$$\dot{\pi} = \pi B \mathbf{V}_h \cdot \nabla_\zeta s - \int_{\zeta_T}^{\zeta} \nabla_\zeta \cdot \left(\frac{\partial \pi}{\partial \zeta}\mathbf{V}_h\right) d\zeta$$

and it is convenient to replace the advection term by the difference of two divergences:

$$\dot{\pi} = \pi B\left[\nabla_\zeta \cdot s\mathbf{V}_h - s\nabla_\zeta \cdot \mathbf{V}_h\right] - \int_{\zeta_T}^{\zeta} \nabla_\zeta \cdot \left(\frac{\partial \pi}{\partial \zeta}\mathbf{V}_h\right) d\zeta$$

In discrete form, we have:

$$\dot{\pi}_{ijk-\frac{1}{2}} = \pi_{ijk-\frac{1}{2}} B_{k-\frac{1}{2}} \left[\nabla_\zeta \cdot s\overline{\mathbf{V}_h}^{\zeta} - s\nabla_\zeta \cdot \overline{\mathbf{V}_h}^{\zeta}\right]_{ijk-\frac{1}{2}} - \mathcal{J}_{ijk-\frac{1}{2}} \qquad \begin{aligned}(\mathbf{V}_h)_0 &= (\mathbf{V}_h)_1 \\ (\mathbf{V}_h)_{N+1} &= (\mathbf{V}_h)_N\end{aligned}$$

$$\text{N.B.} \quad \dot{\pi} = -g\rho w + \underbrace{\left(\frac{\partial \pi}{\partial t} + \mathbf{V}_h \cdot \nabla \pi\right)}_{\text{(neglected tern)}} + \rho\underbrace{\left(\frac{\partial \phi}{\partial t} + \mathbf{V}_h \cdot \nabla \phi\right)}_{\text{(neglected tern)}}$$

Table 1. The equations of GEM in 4 transformations

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$
$$\frac{dT}{dt} - \frac{RT}{c_p}\frac{d\ln p}{dt} = \frac{Q+f}{c_p}$$
$$\frac{d\ln\rho}{dt} + \nabla\cdot\mathbf{V} = 0$$
$$\rho = \frac{p}{RT}$$

Vertical coordinate transformation: z to ζ (unspecified)

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$
$$\frac{\partial}{\partial z} \equiv \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

(1) →

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p - \nabla_\zeta z \frac{\partial\zeta}{\partial z}\frac{\partial \ln p}{\partial\zeta}\right) = \mathbf{F}_h$$
$$\frac{dw}{dt} + RT\frac{\partial\zeta}{\partial z}\frac{\partial\ln p}{\partial\zeta} + g = F_w$$
$$\frac{dT}{dt} - \frac{RT}{c_p}\frac{d\ln p}{dt} = \frac{Q+f}{c_p}$$
$$\frac{d}{dt}\ln\left(\rho\frac{\partial z}{\partial\zeta}\right) + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot\zeta}{\partial\zeta} = 0$$
$$\frac{dz}{dt} - w = 0$$
$$\rho = \frac{p}{RT}$$
$$z = z(\mathbf{r}_h, \zeta, t)$$

Vertical coordinate transformation: z to ζ (specified)

$$RT = -\frac{p}{\pi}\frac{\partial\phi}{\partial\ln\pi}$$
$$\phi = gz$$
$$\mu = \frac{p}{\pi}\frac{\partial\ln p}{\partial\ln\pi} - 1$$

(2) →

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\mu)\nabla_\zeta\phi = \mathbf{F}_h$$
$$\frac{d}{dt}\ln\left(\pi\frac{\partial\ln\pi}{\partial\zeta}\right) + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot\zeta}{\partial\zeta} = 0$$
$$\frac{d\ln T}{dt} - \frac{R}{c_p}\frac{d\ln p}{dt} = \frac{Q+f}{c_p}$$
$$\frac{d\phi}{dt} - gw = 0$$
$$RT + \frac{p}{\pi}\frac{\partial\phi}{\partial\ln\pi} = 0$$
$$\ln\pi = \ln\pi(\mathbf{r}_h, \zeta, t)$$
$$\dots$$
$$\frac{dw}{dt} - g\mu = F_w$$
$$1 + \mu - \frac{p}{\pi}\frac{\partial\ln p}{\partial\ln\pi} = 0$$

Going to model thermodynamic variables $\phi', q, s, \dot{\zeta}$

$$\phi' = \phi - \phi_*$$
$$\ln p = \ln \pi + q$$
$$\ln \pi = \zeta + Bs$$

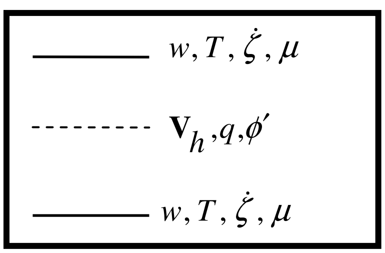
③ →

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\times\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\mu)\nabla_\zeta\phi' = \mathbf{F}_h$$
$$\frac{d}{dt}\left[Bs + \ln(1+\partial_\zeta Bs)\right] + \nabla_\zeta\cdot\mathbf{V}_h + (\partial_\zeta + 1)\dot{\zeta} = 0$$
$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(Bs+q)\right] - \kappa\dot{\zeta} + \dot{\zeta}\partial_\zeta \ln T_* = \frac{Q+f}{c_pT}$$
$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$
$$\frac{T}{T_*} - e^q\left(\frac{1-\partial_\zeta\phi'/RT_*}{1+\partial_\zeta Bs}\right) = 0$$
$$\cdots$$
$$\frac{dw}{dt} - g\mu = F_w$$
$$1+\mu - e^q\left(1+\frac{\partial_\zeta q}{1+\partial_\zeta Bs}\right) = 0$$

Discretizing in the vertical

$$\overline{(\ )}^{\zeta};\ (\breve{\ })$$
$$\delta_\zeta(\ )$$

④ →



$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{h\zeta}\nabla_\zeta(Bs+q) + \left(1+\overline{\mu}^{h\zeta}\right)\nabla_\zeta\phi' = \mathbf{F}_H$$
$$\frac{d}{dt}\left[Bs + \ln(1+\delta_\zeta\overline{B}^{\zeta}s)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta} = 0$$
$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(\breve{\overline{B}}^{\zeta}s + \breve{\overline{q}})\right] - \kappa\breve{\dot{\zeta}} + \breve{\dot{\zeta}}\frac{\delta_\zeta\overline{T_*}^{\zeta}}{T_*} = \frac{Q+f}{c_pT}$$
$$\frac{d\breve{\overline{\phi'}}^{\zeta}}{dt} - RT_*\breve{\dot{\zeta}} - gw = 0$$
$$\frac{T}{T_*} - e^{\breve{\overline{q}}^{\zeta}}\,\frac{1-\delta_\zeta\phi'/RT_*}{1+\delta_\zeta Bs} = 0$$
$$\cdots$$
$$\frac{dw}{dt} - g\mu = F_w$$
$$1+\mu - e^{\breve{\overline{q}}^{\zeta}}\left(1+\frac{\delta_\zeta q}{1+\delta_\zeta Bs}\right) = 0$$

Table 2. The Equations of GEM *vertically discretized* on **Charney-Phillips grid**

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{h\zeta}\nabla_\zeta(Bs+q) + (1+\overline{\mu}^{h\zeta})\nabla_\zeta\phi' = \mathbf{F}_H \qquad \text{------}$$

$$\frac{d}{dt}\left[Bs+\ln\left(1+\delta_\zeta\overline{B}^\zeta s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^\zeta = 0 \qquad \text{------}$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(\breve{\overline{B}}^\zeta s + \breve{q})\right] - \kappa\breve{\dot{\zeta}} + \breve{\dot{\zeta}}\frac{\delta_\zeta\overline{T_*}^\zeta}{T_*} = \frac{Q+f}{c_pT} \qquad \text{———}$$

$$\frac{d\breve{\overline{\phi'}}^\zeta}{dt} - RT_*\breve{\dot{\zeta}} - gw = 0 \qquad \text{———}$$

$$\frac{T}{T_*} - e^{\breve{\overline{q}}^\zeta}\frac{1-\delta_\zeta\phi'/RT_*}{1+\delta_\zeta Bs} = 0 \qquad \text{———}$$

...

$$\frac{dw}{dt} - g\mu = F_w \qquad \text{———}$$

$$1+\mu - e^{\breve{\overline{q}}^\zeta}\left(1+\frac{\delta_\zeta q}{1+\delta_\zeta Bs}\right) = 0 \qquad \text{———}$$

——— $w, T, \dot{\zeta}, \mu$

------ $\mathbf{V}_h, \phi', q$

——— $w, T, \dot{\zeta}, \mu$

$\mathbf{V}_h$: *horizontal wind*; f : *Coriolis parameter*

w : *vertical velicity*; $\kappa = R/c_p$ is variable

T : *temperature*; $T' = T\text{-}T_*$; $T_*(\zeta)$

ϕ : *geopotential*; $\phi' = \phi - \phi_*$; $\delta_\zeta\phi_* = -RT_*$

$q = \ln(p/\pi)$: *non-hydrostatic log-pressure deviation*

p : *pressure*; π : *hydrostatic pressure*; $\partial\phi/\partial\pi = -RT/p$

$\mu = \partial p/\partial\pi - 1$: *ratio of vertical acceleration to gravitational acceleration*

$s = \ln(\pi_S/p_{ref})$: *log-surface-pressure*; B : *metric parameter*

$\dot{\zeta} = d\zeta/dt$; ζ : *model vertical coordinate*

$\overline{(\ )}^\zeta, (\breve{\ })$: *averaging operators*; δ_ζ : *differencing operator*

$p_T/p_{ref} = \eta_T < \eta < 1$: *specified* π*-like model* levels; $p_{ref} = 10^5\,Pa$

$$\zeta = \zeta_S + \ln(\eta)$$

$$\ln p_T = \zeta_T \le \zeta \le \zeta_S = \ln p_{ref} : \textit{calculated } \ln\pi\text{-like } \textit{model} \text{ levels}$$

$$\ln\pi = A + Bs$$

$$A = \zeta; \quad B = \lambda^r; \quad 0 < r = r_{\max} - (r_{\max} - r_{\min})\lambda < 50; \quad \lambda = \max\left[\frac{\zeta-\zeta_U}{\zeta_S-\zeta_U}, 0\right]; \quad \zeta_U \ge \zeta_T$$

Boundary Conditions: $\dot{\zeta}_S = \dot{\zeta}_T = 0 \quad [q_T = \ln(p_T/\pi_T) = 0; \phi_S = gz_{topo}; p_T = const]$