GEM4.2: A non-hydrostatic atmospheric model (Euler equations)

in terrain-following vertical coordinate ($\zeta = \zeta_S + \ln\eta$; $\zeta_S = \ln p_{ref}$)
of the **log-hydrostatic-pressure** type ($\ln\pi = \zeta + Bs$; $s = \ln(\pi_S / p_{ref})$)

vertically discretized on a Charney-Phillips grid
with simple differences and means

Claude Girard and André Plante

other significant contributions:

Sylvie Gravel: Semi-Lagrangian Scheme
Abdessamad Qaddouri: Non-symmetric Elliptic Solver
Stéphane Chamberland: Physics Interface
Lubos Spacek: Physics
Vivian Lee: Input/Output, Cascade, Acid test
Michel Desgagné: Coordination

Revision 2

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N.B. In this document, only ***vertical*** *discretization is discussed in dept; semi-Lagrangian* ***time*** *discretization is sketched in section 8; aspects of* ***horizontal*** *discretization are presented in* **Appendix 10**.

PREFACE

GEM4.2: A work nearly and satisfactorily completed …

Reduction of noise in GEM was the main motivation for the present project consisting in *the introduction of vertical staggering* (Charney-Phillips grid). It was deemed the first and primary ingredient to achieve this goal. In effect, there are numerical modes which were theoretically diagnosed on the previous un-staggered grid which are absent from the new one. As a first step therefore in this project, only the grid was changed. Everything else, the equations, the independent as well as the dependent variables, were kept unchanged. Very *positive results* were obtained *with respect to noise*. But there remain problems, in particular an accuracy problem in the hydrostatic relation at upper levels when the true resolution (in terms of height) is insufficient.

Improving the accuracy of the hydrostatic relation using *logarithmic differencing* wherever appropriate was therefore the goal of a second step. The *results* from this modification of the code were very satisfying with *improved scores in the stratosphere*.

With this incentive, it was tempting to try and implement *a full log-hydrostatic-pressure coordinate*, ζ. A theoretical advantage of ζ is its linear relationship with ln*p*, [$\ln p = \ln(p/\pi) + \ln(\pi/\pi_*) + \ln\pi_* = q + Bs + \zeta$]. Along with the fact that $q = \ln(p/\pi)$ and $s = \ln(\pi_S / p_{ref})$ are already model variables, this greatly simplifies the linearization of model equations. Again the accuracy of the hydrostatic equation is improved since the finite differences not only are calculated logarithmically but also become defined at logarithmic mid-points. This third step though has *little impact* on model performance.

An important development: it was discovered that the initial staggered version of the semi-Lagrangian scheme, linear vertical interpolation of the departure positions for variables arriving on thermodynamic levels, resulted in significant loss of kinetic energy. *Cubic interpolation* is rather the thing to do.

A secondary motivation for the project was the resolution of accuracy and noise problems encountered in the simulation of non-hydrostatic mountain waves, specifically what we call Schär's case. Well, a completely satisfactory solution has been achieved, not via staggering though but again through modifications of the semi-Lagrangian scheme: tri-dimensional cubic interpolation of the departure positions replacing linear ones combined with trapezoidal means of the velocities instead of the mid-point rule.

Many new appendices appear in GEM4.2, notably Appendices 13 and 14. Appendix 13 constitutes a major development affecting the code substantially since the top thermodynamic level, level 3/4, is being eliminated. Theoretical analysis as well as experimental testing has shown that this level was essentially dynamically disconnected from the rest of the model. Improved simulations resulted from the change. Appendix 14 describes trapezoidal means.

Older versions of this document, GEM4.0 and GEM4.1, remain available.

1) The meteorological equations

- 4 independent variables: t, $\mathbf{r}=(\mathbf{r}_h, z)$
- 6 dependent variables: $\mathbf{V}=(\mathbf{V}_h, w)$, T, ρ, p
- 6 scalar equations: $$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$ $$\frac{d\ln T}{dt} - \frac{R}{c_p}\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$ $$\frac{d\ln\rho}{dt} + \nabla\cdot\mathbf{V} = 0$$ $$\rho = \frac{p}{RT}$$

- There are: 5 prognostic equations (momentum + energy + mass conservation),
  1 diagnostic equation (perfect gas law).

N.B. The Coriolis force is approximated (traditional meteorological approximations apply).

N.B. Many more approximations are implied if we consider that the atmospheric substance contains, in addition to dry air, not only a variable quantity of water vapor but also condensed water and precipitations. The above equations are valid under the assumptions of *dynamic* (precipitations falling at terminal velocity) and *thermodynamic* (neglecting temperature differences between air and hydrometeors) *equilibrium* and neglecting precipitation fluxes. Equations for the displacement and evolution of the hydrometeors are required to complete the system.

N.B. In the above equations the coefficients R and c_p are variable. The introduction of *virtual temperature* (replacing RT by R_dT_v where R_d is now constant) and approximating the ratio $\kappa=R/c_p$ by the constant ratio $\kappa_d=R_d/c_{pd}$ lead to further simplifications (see **Appendix 1**).

2) The equations transformed to generalized η-coordinate

- Note the necessary decomposition of vector equations into their horizontal/vertical components due to the different horizontal/vertical transformation rules.

2 transformation rules: $\nabla_z \equiv \nabla_\eta - \nabla_\eta z \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial \eta}; \quad \dfrac{\partial}{\partial z} \equiv \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial \eta}; \quad \dfrac{\partial}{\partial t_z} \equiv \dfrac{\partial}{\partial t_\eta} - \dfrac{\partial z}{\partial t_\eta}\dfrac{\partial}{\partial \eta}$

- 4 independent variables : t, $\mathbf{r}_h$, η

- 8 dependent variables: $\mathbf{V}_h$, w, T, ρ, p, $\dot{\eta}$, z

- 8 equations (6 prognostic and 2 diagnostic):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\eta \ln p - \nabla_\eta z \frac{\partial \eta}{\partial z}\frac{\partial \ln p}{\partial \eta}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial \eta}{\partial z}\frac{\partial \ln p}{\partial \eta} + g = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\ln\left(\rho\frac{\partial z}{\partial \eta}\right) + \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\eta}}{\partial \eta} = 0$$

$$\frac{dz}{dt} - w = 0$$

$$\rho = \frac{p}{RT}$$

$$z \equiv z(\eta, \mathbf{r}_h, t)$$

- Were added then: 1 prognostic equation (***dz/dt=w***) for varying height in space and time,
1 diagnostic equation (yet to be specified) defining the coordinate η.

- the continuity equation is the only one requiring more than simple manipulation:

$$w = \frac{dz}{dt} = \frac{\partial z}{\partial t} + \mathbf{V}_h \cdot \nabla_\eta z + \dot{\eta}\frac{\partial z}{\partial \eta}$$

$$\frac{\partial \eta}{\partial z}\frac{\partial w}{\partial \eta} = \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial t} + \mathbf{V}_h \cdot \nabla_\eta z + \dot{\eta}\frac{\partial z}{\partial \eta}\right) = \frac{\partial \eta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \eta} \cdot \nabla_\eta z + \frac{\partial \dot{\eta}}{\partial \eta} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \eta}\right)$$

hence

$$\nabla_z \cdot \mathbf{V}_h + \frac{\partial w}{\partial z} = \nabla_\eta \cdot \mathbf{V}_h - \frac{\partial \eta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \eta} \cdot \nabla_\eta z + \frac{\partial \eta}{\partial z}\frac{\partial w}{\partial \eta} = \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\eta}}{\partial \eta} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \eta}\right)$$

- See **Appendix 2** for some details on transformation rules.

3) Eliminating ρ introducing $\ln\pi$, log-hydrostatic pressure, eliminating z defining the geopotential ϕ and adding μ (ratio of vertical acceleration to gravitational acceleration)

$$\frac{\partial \pi}{\partial z} = -g\rho\,; \quad RT = -\frac{p}{\pi}\frac{\partial \phi}{\partial \ln \pi}; \quad \phi = gz\,; \quad \mu = \frac{p}{\pi}\frac{\partial \ln p}{\partial \ln \pi} - 1$$

- 9 dependent variables: $\mathbf{V}_h, w, T, p, \dot{\eta}, \phi, \mu, \pi$

- 9 equations (added diagnostic equation for μ):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\eta \ln p + (1+\mu)\nabla_\eta \phi = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\mu = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\ln\left(\pi\frac{\partial \ln \pi}{\partial \eta}\right) + \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\eta}}{\partial \eta} = 0$$

$$\frac{d\phi}{dt} - gw = 0$$

$$1 + \mu - \frac{p}{\pi}\frac{\partial \ln p}{\partial \ln \pi} = 0$$

$$RT + \frac{p}{\pi}\frac{\partial \phi}{\partial \ln \pi} = 0$$

$$\ln\pi \equiv \ln\pi(\eta, \mathbf{r}_h, t)$$

N.B. At this point η is still a general coordinate of the hydrostatic-pressure type: in the next section we will specify both ζ and η.

N.B. For the rest of the presentation, the ***physical forcings*** $\mathbf{F}_h$, F_w, Q will be excluded and the ***parameters*** R and κ will be treated as constants.

4) The new ζ-coordinate for GEM4 is $\ln\pi$-like

$$\zeta = \zeta_S + \ln\eta; \qquad \zeta_S = \ln p_{ref}; \quad p_{ref} = 10^5$$

$$\ln\pi = A(\zeta) + B(\zeta)s; \qquad s = \ln\pi_S - \zeta_S = \ln(\pi_S / p_{ref})$$

$$A = \zeta; \quad B = \lambda^r; \quad \lambda = \max\left[\frac{\zeta - \zeta_U}{\zeta_S - \zeta_U}, 0\right] \quad \begin{cases} \zeta_U \geq \zeta_T; \quad \zeta_T = \ln p_T \\ 0 \leq r = r_{\max} - (r_{\max} - r_{\min})\lambda \leq 200 \end{cases}$$

$$\ln\pi = \zeta + B(\zeta)s$$

transformation rules: $\nabla_\eta = \nabla_\zeta; \quad \frac{\partial}{\partial\eta} = \frac{1}{\eta}\frac{\partial}{\partial\zeta}; \quad \frac{\partial}{\partial t_\eta} = \frac{\partial}{\partial t_\zeta}$

- 9 dependent variables: $\mathbf{V}_h$, w, T, p, $\dot{\zeta}$, ϕ, μ, π

- 9 equations:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\mu)\nabla_\zeta\phi = 0$$

$$\frac{dw}{dt} - g\mu = 0$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = 0$$

$$\frac{d}{dt}\ln\left(\pi\frac{\partial\ln\pi}{\partial\zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} = 0$$

$$\frac{d\phi}{dt} - gw = 0$$

$$1 + \mu - \frac{p}{\pi}\frac{\partial\ln p}{\partial\ln\pi} = 0$$

$$RT + \frac{p}{\pi}\frac{\partial\phi}{\partial\ln\pi} = 0$$

$$\ln\pi \equiv \zeta + Bs$$

- **obviously,** at this point, the form of the equations in ζ and η coordinates is identical

N.B. $p_{top} / p_{ref} < \eta < 1$ is now but a label characterizing model ζ-levels. Another way to characterize the levels would be to use H, a number having the units of height and corresponding approximately to *model level height* (above sea level):

$$\zeta = \zeta_S - H / H_{ref}; \qquad H_{ref} = \left(\frac{RT}{g}\right)_{ref} = \frac{16000}{\ln 10} \approx 6950m$$

See **Appendix 3** for more information on the metric parameter B.

5) Perturbation thermodynamic variables, T', ϕ', q, and simplifications

Introducing the logarithm of the non-hydrostatic pressure perturbation $q=\ln(p/\pi)$ and perturbation variables T' and ϕ'. Eliminating p, ϕ and π. We keep T for convenience.

$$
\begin{aligned}
&T' = T - T_*; && T_* = const \\
&\phi' = \phi - \phi_*; && \phi_*(\zeta) = -RT_*\left(\zeta - \zeta_S\right) \\
&\ln p = \ln\pi + q = \zeta + Bs + q &&
\end{aligned}
$$

- **8 variables**: $\mathbf{V}_h, w, T$ or $T', q, \left(\dot{\zeta}, s\right), \phi', \mu$, **final number**

- **8 equations** [6 prognostic & 2 diagnostic], **final form** ready for linearization:

$$
\begin{aligned}
&\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\mu)\nabla_\zeta\phi' = 0 \\
&\frac{dw}{dt} - g\mu = 0 \\
&\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(Bs+q)\right] - \kappa\dot{\zeta} = 0 \\
&\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \left(\frac{\partial}{\partial\zeta}+1\right)\dot{\zeta} = 0 \\
&\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0 \\
&1 + \mu - e^q\left(1 + \frac{\partial q}{\partial(\zeta+Bs)}\right) = 0 \\
&\frac{T}{T_*} - e^q\left(1 - \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta+Bs)}\right) = 0
\end{aligned}
$$

N. B. The variable S is 2-D only and $\dot{\zeta}$ vanishes at the surface. The combination ($\dot{\zeta}$, S) may therefore be considered to constitute a single 3-D variable.

N.B. $P = \phi' + RT_*(Bs+q) = \phi' + RT_*(\ln p - \zeta) = \phi + RT_*\ln(p/p_{ref})$, which may be called *generalized pressure*, is a variable which will be convenient to invoke later on.

6) Boundary Conditions

The model top (subscript $_T$) and bottom (subscript $_S$ for earth's surface when talking of the bottom of the atmosphere), are defined to be material surfaces. Therefore we have the following **top and bottom boundary conditions**:

$$\dot{\zeta}_T = \dot{\zeta}(\zeta_T) = 0; \qquad \dot{\zeta}_S = \dot{\zeta}(\zeta_S) = 0$$

In addition, the behavior of theses surfaces must be specified and this will lead to **an additional** condition in the non-hydrostatic case. The bottom surface is assumed to be terrain-following and not moving. In effect, the bottom geopotential ϕ_S is specified, varying with position but *usually* fixed in time: $\partial\phi_S / \partial t = 0$. This though does not imply a vertical velocity that necessarily vanishes at the surface. In effect, $gw_S = [d\phi / dt]_S \neq 0$ generally. At the top, we consider a *flexible surface* whereby the top pressure:

$$p_T = \pi_T$$

is assumed to remain constant. This is automatic in the hydrostatic case since the top surface pressure cannot be anything other than a material hydrostatic pressure surface. In the non-hydrostatic case, to maintain a constant top pressure equal to the constant top hydrostatic pressure surface provides a **top boundary specification** for pressure. In terms of the non-hydrostatic pressure variable *q*, this becomes:

$$q_T = \ln(p_T / \pi_T) = 0$$

The top surface is then assumed free to move, constrained only by this artificially imposed pressure p_T (the atmosphere above exerting its weight only).

N.B. *Open top boundary conditions* are of course a possibility: see **Appendix 9**.

N.B. For the Limited Area version of the Model (LAM), there are **lateral boundary conditions**. See **Appendix 11**.

N.B. *Time varying topography*, $\partial\phi_S / \partial t \neq 0$, is also an option: see **Appendix 12**. In effect, when adapting a given atmospheric state to a higher resolution topography inter(extra)polation is required. Artificially varying ϕ_S in time for a short period is an attractive alternative.

N.B. Initial conditions are time boundary conditions. At initial time, $\mathbf{V}_h, T$ and s are analyzed fields; $\dot{\zeta}$, ϕ' and *w* (in the hydrostatic case) are diagnosed: see **Appendix 15** for the calculation of $\dot{\zeta}$ and the estimation of *w*. In the non-hydrostatic case, *w* and *q* could be analyzed but usually *w* is estimated and *q* set to vanish; μ is diagnosed.

7) Vertical discretization with staggering

For vertical discretization, the following choice is made:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\mu}^{\zeta}\right)\nabla_{\zeta}\phi' = 0$$

$$\frac{dw}{dt} - g\mu = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa\overline{(Bs+q)}^{\zeta}\right] - \kappa\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1+\delta_{\zeta}\overline{B}^{\zeta}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \delta_{\zeta}\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta} = 0$$

$$\frac{d\overline{\phi'}^{\zeta}}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$1 + \mu - e^{\overline{q}^{\zeta}}\left[1 + \frac{\delta_{\zeta}q}{\delta_{\zeta}(\zeta+Bs)}\right] = 0$$

$$\frac{T}{T_*} - e^{\overline{q}^{\zeta}}\left[1 - \frac{\delta_{\zeta}(\phi'/RT_* + Bs)}{\delta_{\zeta}(\zeta+Bs)}\right] = 0$$

In other words, the derivatives are replaced by simple finite differences represented by the operator δ_{ζ} and averaging operators represented by over bars are introduced where required. From the notation, it may be gathered that $\mathbf{V}_h, q, \phi'$ are defined on the same levels to be called *full* or **momentum** levels. They are staggered with respect to $w, T, \mu, \dot{\zeta}$ placed on *half* or **thermodynamic** levels. With this staggering, double operations on dependent variables are severely reduced. No difference is calculated over more than two levels. The number of averaging operators is minimized. In the horizontal momentum equations, they occur on non-linear terms only; in the hydrostatic case (with q=μ=0 and w dropping out of the system), only one averaging operator remains on linear terms, namely on $\dot{\zeta}$ in the continuity equation. *Details of the discretization* are given in **Appendices 4a, 4b, 5** and **6** (but first read the rest of the main document). Taking into account the boundary conditions, it is natural to have half levels rather than full levels coincide with the top and bottom. This essentially, though not fully, completes the description of the *vertical grid.* See **Figure 1**, next page.

N. B. The metric parameter B is exactly calculated on full levels only. It is averaged for the half levels.

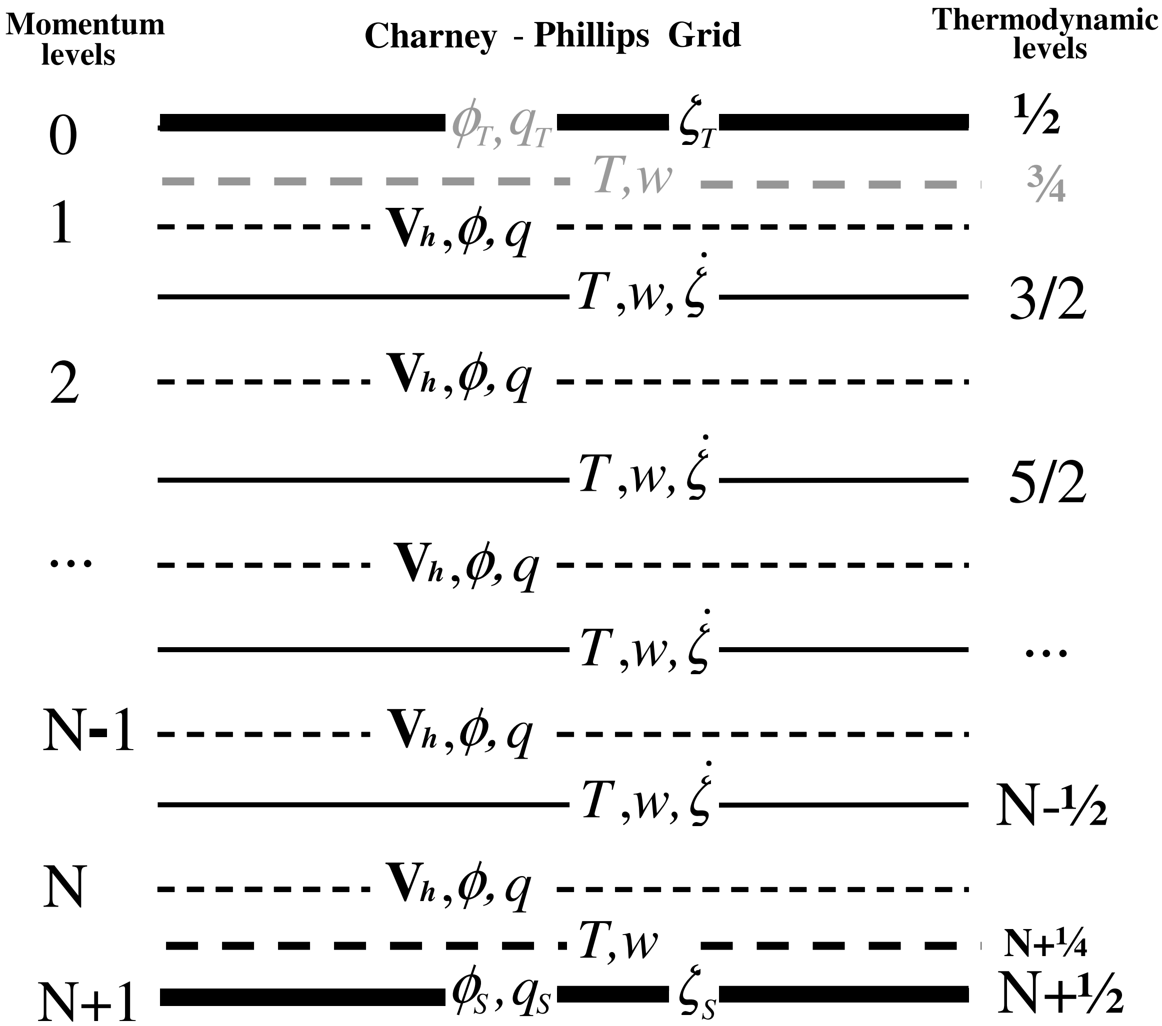


Figure 1. The Charney-Phillips grid, giving the position occupied by each variable in the vertical domain. The model is composed of N layers, inside of which (in the middle of which only if the layers are equal) are the momentum levels [1,2,…,N] where the wind components *U* and *V*, the geopotential ϕ and *q* are positioned. ϕ and *q* are also defined on the boundaries (top level 0 and surface level N+1). These N layers are delimited by N-1 interfaces corresponding to N-1 so-called thermodynamic levels [3/2,…,N-1/2] where are positioned the temperature *T* and the two vertical motion fields *w* and $\dot{\zeta}$, exactly in the middle of the momentum levels. $\dot{\zeta}$ also has 2 additional levels [½ and N+½] corresponding to the top and bottom surfaces. *T* and *w* also have 2 additional levels [¾ and N+¼] positioned exactly in between, respectively, the top surface and first momentum level and the last momentum level and bottom surface. [**N.B.** The ¾ thermodynamic level (variables in gray) is absent from the ***numerically truncated top***: see **Appendix 13**]

8) Semi-Lagrangian **Implicit** time discretization (n.b. not Semi-Implicit)

- Approximating the ***substantial derivatives*** and averaging the ***dynamical forcings***, each of the equations (index *i*) may be formally written as follows:

$$\boxed{\frac{dF_i}{dt} + G_i = 0}$$

$$\frac{dF_i}{dt} \approx \frac{F_i^A - F_i^D}{\Delta t}; \quad G_i \approx b^A G_i^A + \left(1 - b^A\right) G_i^D; \qquad 0.5 \le b^A \le 0.6 \ \ \text{(off-centering)}$$

$$\frac{F_i^A - F_i^D}{\Delta t} + b^A G_i^A + \left(1 - b^A\right) G_i^D = 0$$

$A: \ (\mathbf{r}, t)$ Arrival

$D: \ (\mathbf{r} - \Delta\mathbf{r}, t - \Delta t)$ Departure

- Separating the time levels ($\tau = \Delta t b^A; \beta = \left(1 - b^A\right) / b^A$)

$$\frac{F_i^A}{\tau} + G_i^A = \frac{F_i^D}{\tau} - \beta G_i^D \equiv R_i$$

- Decomposing the left-hand side into linear and residual non-linear parts

$$\frac{F_i^A}{\tau} + G_i^A = L_i + N_i = R_i$$

$$L_i \equiv \left(\frac{F_i^A}{\tau} + G_i^A\right)_{lin}; \qquad N_i \equiv \frac{F_i^A}{\tau} + G_i^A - \left(\frac{F_i^A}{\tau} + G_i^A\right)_{lin}$$

- Defining the solution method (a Crank-Nicholson scheme)

$$L_i = R_i - N_i$$

Iterating (*jter*: *departure loop, iter*: *non-linear loop*) :

Do jter=1,2
Do iter=1,2

$$\left(L_i\right)^{iter,jter} = \left(R_i\right)^{jter} - \left(N_i\right)^{iter-1,jter}; \qquad \left(N_i\right)^{0,1} = N_i\left(\mathbf{r}, t - \Delta t\right)$$

end do
end do

$$\left(R_i\right)^{jter} = R_i\left(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{jter}\right); \quad \Delta\mathbf{r}^{jter} = \tfrac{\Delta t}{2}\left(\mathbf{v}(t - \Delta t) + \mathbf{v}(t)^{jter}\right)\left(\mathbf{r} - \tfrac{\Delta\mathbf{r}^{jter-1}}{2}\right)$$

$$\mathbf{v}(t)^1 = \mathbf{v}(t - \Delta t); \ \Delta\mathbf{r}^0 \ \text{from previous timestep}$$

N.B. The displacement $\Delta\mathbf{r}^{jter}$ is here calculated by the *mid-point* rule. A different option consists in using the *trapezoidal* rule: see **Appendix 14**.

9) The *F's* and the *G's*

$$\mathbf{F}_h \equiv \mathbf{V}_h \qquad \mathbf{G}_h \equiv f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\mu}^{\zeta}\right)\nabla_{\zeta}\phi'$$

$$F_w \equiv w \qquad G_w \equiv -g\mu$$

$$F_\theta \equiv \ln\left(\frac{T}{T_*}\right) - \kappa\overline{(Bs+q)}^{\zeta} \qquad G_\theta \equiv -\kappa\dot{\zeta}$$

$$F_C \equiv Bs + \ln\left(1+\delta_\zeta \overline{B}^{\zeta} s\right) \qquad G_C \equiv \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}$$

$$F_\phi \equiv \overline{\phi'}^{\zeta} \qquad G_\phi \equiv -RT_*\dot{\zeta} - gw$$

$$F_\mu \equiv 0 \qquad G_\mu \equiv 1+\mu - e^{\overline{q}^{\zeta}}\left[1+\frac{\delta_\zeta q}{\delta_\zeta(\zeta+Bs)}\right] = 0$$

$$F_H \equiv 0 \qquad G_H \equiv \frac{T}{T_*} - e^{\overline{q}^{\zeta}}\left[1-\frac{\delta_\zeta(\phi'/RT_* + Bs)}{\delta_\zeta(\zeta+Bs)}\right] = 0$$

N.B. Since $F_\mu = F_H = 0$ and $G_\mu = G_H = 0$, then of course $R_\mu = R_H = 0$.

The role of diagnostic equations is to abbreviate other equations. If, in the 6 prognostic equations, we replace the symbols μ and T by their definitions, the diagnostic equations as well as the associated variables vanish.

10) The Left-Hand Side terms: $L_i + N_i$

$$\boxed{L_i + N_i \equiv \frac{F_i^A}{\tau} + G_i^A}$$

Prognostic (dropping the superscript *A*):

$$\mathbf{L}_h + \mathbf{N}_h = \frac{\mathbf{V}_h}{\tau} + f\mathbf{k}\times\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\mu}^{\zeta}\right)\nabla_{\zeta}\phi'$$

$$L_w + N_w = \frac{w}{\tau} - g\mu$$

$$L_\theta + N_\theta = \frac{1}{\tau}\ln\left(\frac{T}{T_*}\right) - \kappa\left(\dot{\zeta} + \frac{\overline{(Bs+q)}^{\zeta}}{\tau}\right)$$

$$L_C + N_C = \frac{1}{\tau}\left[Bs + \ln\left(1+\delta_{\zeta}\overline{B}^{\zeta}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \delta_{\zeta}\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}$$

$$L_\phi + N_\phi = \frac{\overline{\phi'}^{\zeta}}{\tau} - RT_*\dot{\zeta} - gw$$

Diagnostic:

$$L_\mu + N_\mu = 1 + \mu - e^{\overline{q}^{\zeta}}\left[1 + \frac{\delta_{\zeta}q}{\delta_{\zeta}(\zeta+Bs)}\right] = 0$$

$$L_H + N_H = \frac{T}{T_*} - e^{\overline{q}^{\zeta}}\left[1 - \frac{\delta_{\zeta}(\phi'/RT_* + Bs)}{\delta_{\zeta}(\zeta+Bs)}\right] = 0$$

11) The linear Left-Hand Side terms: L_i

$$L_i \equiv \left[\frac{F_i^A}{\tau} + G_i^A \right]_{linear}$$

Linearizing (approximating the logarithms [$\ln(1+\alpha) \approx \alpha$], the exponentials [$e^{\alpha} \approx 1+\alpha$] and the products [$(1+\alpha)(1+\beta)^{\pm 1} \approx 1+\alpha \pm \beta$]; note the Coriolis term $f\mathbf{k} \times \mathbf{V}_h$ is treated as if it was a non-linear term) yields:

$$\mathbf{L}_h = \frac{\mathbf{V}_h}{\tau} + \nabla_{\zeta}\left[\phi' + RT_*(Bs+q)\right]$$

$$L_w = \frac{w}{\tau} - g\mu$$

$$L_\theta = \frac{T'}{\tau T_*} - \kappa\left(\dot{\zeta} + \frac{\overline{(Bs+q)}^{\zeta}}{\tau} \right)$$

$$L_C = \frac{1}{\tau}\left[\overline{B}^{\zeta\zeta} s + \delta_{\zeta}\overline{B}^{\zeta} s\right] + \nabla_{\zeta} \cdot \mathbf{V}_h + \delta_{\zeta}\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}$$

$$L_\phi = \frac{\overline{\phi'}^{\zeta}}{\tau} - RT_*\dot{\zeta} - gw$$

$$L_\mu = \mu - \left(\delta_{\zeta} q + \overline{q}^{\zeta}\right) \neq 0$$

$$L_H = \frac{T'}{T_*} - \overline{q}^{\zeta} + \frac{\delta_{\zeta}(\phi' + RT_* Bs)}{RT_*} \neq 0$$

12) The non-linear Left-Hand side terms, N_i, are the left-over differences

$$\boxed{N_i = \left[\frac{F_i^A}{\tau} + G_i^A\right] - \left[\frac{F_i^A}{\tau} + G_i^A\right]_{linear} = \left[\frac{F_i^A}{\tau} + G_i^A\right] - L_i}$$

and therefore:

$$
\begin{aligned}
\mathbf{N}_h &= f\mathbf{k} \times \mathbf{V}_h + R\overline{T'}^{\zeta}\nabla_{\zeta}\left(Bs + q\right) + \overline{\mu}^{\zeta}\nabla_{\zeta}\phi' \\
N_w &= 0 \\
N_{\theta} &= \frac{1}{\tau}\left[\ln\left(\frac{T}{T_*}\right) - \frac{T'}{T_*}\right] \\
N_C &= \frac{1}{\tau}\left[Bs + \ln\left(1 + \delta_{\zeta}\overline{B}^{\zeta}s\right) - \overline{B}^{\zeta\zeta}s - \delta_{\zeta}\overline{B}^{\zeta}s\right] \\
N_{\phi} &= 0 \\
N_{\mu} &= -\left(\mu - \delta_{\zeta}q - \overline{q}^{\zeta}\right) = -L_{\mu} \\
N_H &= -\left(\frac{T'}{T_*} - \overline{q}^{\zeta} + \frac{\delta_{\zeta}\left(\phi' + RT_*Bs\right)}{RT_*}\right) = -L_H
\end{aligned}
$$

13) Elimination of the diagnostic equations from the solution system

As noted above, R_μ=R_H=0. It is then convenient to immediately eliminate the two diagnostic equations, involving the diagnostic variables μ and T', from the Left-Hand side terms, i.e. to eliminate L_μ, L_H and N_μ, N_H. We are left with 6 *basic* equations for the linear system:

$$\mathbf{L}_h = \frac{\mathbf{V}_h}{\tau} + \nabla_\zeta\left[\phi' + RT_*(Bs+q)\right]$$

$$L_w + gL_\mu \equiv L'_w = \frac{w}{\tau} - g\left(\delta_\zeta q + \overline{q}^\zeta\right)$$

$$L_\theta - \frac{L_H}{\tau} \equiv L'_\theta = \frac{\overline{q}^\zeta}{\tau} - \frac{\delta_\zeta(\phi' + RT_* Bs)}{\tau RT_*} - \kappa\left(\dot{\zeta} + \frac{\overline{(Bs+q)}^\zeta}{\tau}\right)$$

$$L_C = \frac{1}{\tau}\left[\overline{B}^{\zeta\zeta} s + \delta_\zeta \overline{B}^\zeta s\right] + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^\zeta$$

$$L_\phi = \frac{\overline{\phi'}^\zeta}{\tau} - RT_*\dot{\zeta} - gw$$

Similarly for the non-linear system we have

$$\mathbf{N}_h = f\mathbf{k}\times\mathbf{V}_h + R\overline{T'}^\zeta \nabla_\zeta(Bs+q) + \overline{\mu}^\zeta \nabla_\zeta \phi'$$

$$N_w + gN_\mu \equiv N'_w = -g\left(\mu - \delta_\zeta q - \overline{q}^\zeta\right)$$

$$N_\theta - \frac{N_H}{\tau} \equiv N'_\theta = \frac{1}{\tau}\left[\ln\left(\frac{T}{T_*}\right) - \overline{q}^\zeta + \frac{\delta_\zeta(\phi' + RT_* Bs)}{RT_*}\right]$$

$$N_C = \frac{1}{\tau}\left[\ln\left(1 + \delta_\zeta \overline{B}^\zeta s\right) - \delta_\zeta \overline{B}^\zeta s\right]$$

$$N_\phi = 0$$

14) The Previous time step on the Right-Hand Sides: R_i

$$R_i \equiv \frac{F_i^D}{\tau} - \beta G_i^D$$

(dropping the superscript *D*)

$$\mathbf{R}_h = \frac{\mathbf{V}_h}{\tau} \qquad - \beta\left[f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\mu}^{\zeta}\right)\nabla_{\zeta}\phi'\right]$$

$$R_w = \frac{w}{\tau} \qquad - \beta(-g\mu)$$

$$R_{\theta} = \frac{1}{\tau}\left[\ln\left(\frac{T}{T_*}\right) - \kappa\overline{(Bs+q)}^{\zeta}\right] \qquad - \beta\left(-\kappa\dot{\zeta}\right)$$

$$R_C = \frac{1}{\tau}\left[Bs + \ln\left(1+\delta_{\zeta}\overline{B}^{\zeta}s\right)\right] \qquad - \beta\left(\nabla_{\zeta}\cdot\mathbf{V}_h + \delta_{\zeta}\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta}\right)$$

$$R_{\phi} = \frac{\overline{\phi'}^{\zeta}}{\tau} \qquad - \beta\left(-RT_*\dot{\zeta} - gw\right)$$

15) The elliptic problem

Introducing $P \equiv \phi' + RT_*(Bs+q)$ and $X = \dot{\zeta} + \frac{\overline{(Bs+q)}^{\zeta}}{\tau}$, the linear system takes the form:

$$
\begin{aligned}
\mathbf{L}_h &= \frac{\mathbf{V}_h}{\tau} + \nabla_\zeta P \\
L'_w &= \frac{w}{\tau} - g\left(\delta_\zeta q + \overline{q}^{\zeta}\right) \\
L'_\theta &= \frac{1}{\tau}\left(\delta_\zeta q + \overline{q}^{\zeta}\right) - \frac{\delta_\zeta P}{\tau RT_*} - \kappa X \\
L_C &= -\frac{1}{\tau}\overline{\left(\delta_\zeta q + \overline{q}^{\zeta}\right)}^{\zeta} + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta X + \overline{X}^{\zeta} \\
L_\phi &= \frac{\overline{P}^{\zeta}}{\tau} - RT_* X - gw
\end{aligned}
$$

The number of equations and dependent variables, $\mathbf{V}_h$, w, P, q, X, is easily reduced to 3 thus (variables left: P, w, X):

$$
\begin{aligned}
\nabla_\zeta \cdot \mathbf{L}_h - \frac{1}{\tau}\left(L_C - \frac{\varepsilon\tau}{H_*}\overline{L'_w}^{\zeta}\right) &\equiv L''_C = \nabla_\zeta^2 P - \frac{1}{\tau}\left(\delta_\zeta X + \overline{X}^{\zeta}\right) + \varepsilon\frac{\overline{w}^{\zeta}}{\tau w_*} \\
\frac{\gamma}{\kappa\tau}\left(L'_\theta + \frac{\varepsilon\tau}{H_*}L'_w + \frac{\varepsilon}{RT_*}L_\phi\right) &\equiv L''_\theta = -\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon\overline{P}^{\zeta}\right) - \frac{X}{\tau} \\
\frac{\gamma}{\kappa\tau}\left(L'_\theta + \frac{\varepsilon\tau}{H_*}L'_w - \frac{\kappa}{RT_*}L_\phi\right) &\equiv L''_\phi = -\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa\overline{P}^{\zeta}\right) + \frac{w}{\tau H_*}
\end{aligned}
$$

with $\varepsilon = \frac{H_*}{g\tau^2} = \frac{RT_*}{g^2\tau^2}$ and $\gamma = \frac{\kappa}{\kappa + \varepsilon}$.

Here note: we have assumed $\delta_\zeta \overline{q}^{\zeta} \equiv \overline{\delta_\zeta q}^{\zeta}$, i.e. we have assumed commutation of the mean and difference operators. See **Appendix 5** for details on averaging operators and commutation.

Finally, these three equations are combined to give the *structure* equation:

$$
L''_C - \left(\delta_\zeta L''_\theta + \overline{L''_\theta}^{\zeta}\right) - \varepsilon\overline{L''_\phi}^{\zeta} \equiv L_P = \nabla_\zeta^2 P + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta^2 P + \overline{\delta_\zeta P}^{\zeta} - \varepsilon(1-\kappa)\overline{P}^{\zeta\zeta}\right)
$$

again provided commutation holds.

This is the elliptic problem to be solved with boundary conditions (on P) given by

$$L''_\theta = -\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon \overline{P}^\zeta\right) - \frac{1}{\tau}\left(\dot{\zeta} + \frac{\overline{(Bs+q)}^\zeta}{\tau}\right)$$

applied at both top and bottom as follows:

$$\left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon \overline{P}^\zeta\right)\right]_T = -\left(L''_\theta\right)_T$$

$$\left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa \overline{P}^\zeta\right)\right]_S = -\left(L''_\theta\right)_S + \frac{\phi_S}{\tau^2 RT_*} = -\left(L'''_\theta\right)_S$$

In effect

$$\left(\dot{\zeta} + \frac{\overline{(Bs+q)}^\zeta}{\tau}\right)_T = 0$$

since $\dot{\zeta}_T = 0$, $B_T = 0$ and $q_T = 0$ at the top and (noting that $X \equiv \dot{\zeta} + \frac{\overline{P-\phi'}^\zeta}{\tau RT_*}$)

$$\left(\dot{\zeta} + \frac{\overline{(Bs+q)}^\zeta}{\tau}\right)_S = \frac{1}{\tau}\left(q_S + s\right) = \frac{P_S - \phi_S}{\tau RT_*}$$

since $\dot{\zeta}_S = 0$ and $B_S = 1$ at the bottom. ϕ_S is a known quantity.

N.B. These are closed boundary conditions. Open top boundary conditions are considered in **Appendix 9**.

N.B. The top boundary condition is X_T=0. It is therefore *not necessary* to eliminate $X_{\frac{1}{2}} = X_T$ from the top continuity equation, $\left(L''_C\right)_1$, using a top thermodynamic equation, $\left(L''_\theta\right)_{\frac{3}{4}}$, i.e. it is not necessary to have a top thermodynamic level ¾ . See **Appendix 13** for such a *Numerically truncated top boundary condition*.

N.B. Using the scale height, $H_* = RT_*/g$, the square of the Brünt-Väisälä frequency, $N_*^2 = g^2/c_pT_*$, and the square of the speed of sound, $c_*^2 = \left(c_p/c_v\right)RT_*$, the *structure* equation takes a more familiar form:

$$\left(N_*^2 + \frac{1}{\tau^2}\right)L_P = \left(N_*^2 + \frac{1}{\tau^2}\right)\nabla_\zeta^2 P + \frac{1}{\tau^2}\left(\frac{\delta_\zeta^2 P + \overline{\delta_\zeta P}^\zeta}{H_*^2} - \frac{1}{c_*^2\tau^2}\overline{P}^{\zeta\zeta}\right)$$

16) The non-linear problem

To find the solution to the non-linear problem we need to perform the following operations iteratively

$$
\begin{aligned}
(\mathbf{L'}_h)^{1+iter,jter} &= (\mathbf{R}_h)^{jter} - (\mathbf{N'}_h)^{iter,jter} \\
(L'_w)^{1+iter,jter} &= (R_w)^{jter} - (N'_w)^{iter,jter} \\
(L'_\theta)^{1+iter,jter} &= (R_\theta)^{jter} - (N'_\theta)^{iter,jter} \\
(L_\phi)^{1+iter,jter} &= (R_\phi)^{jter} - (N_\phi)^{iter,jter}
\end{aligned}
$$

In order to obtain $R_P, R''_\theta, R''_\phi$ and $N_P, N''_\theta, N''_\phi$, we transform the *R*'s and *N*'s, like was done for the *L*'s to obtain $L_P, L''_\theta, L''_\phi$, i.e. we compute:

$$
\begin{aligned}
&\nabla_\zeta \cdot \mathbf{R}_h - \frac{1}{\tau}\left(R_C - \frac{\varepsilon\tau}{H_*}\overline{R_w}^\zeta \right) \equiv R''_C
&&\nabla_\zeta \cdot \mathbf{N}_h - \frac{1}{\tau}\left(N_C - \frac{\varepsilon\tau}{H_*}\overline{N'_w}^\zeta \right) \equiv N''_C \\
&\frac{\gamma}{\kappa\tau}\left(R_\theta + \frac{\varepsilon\tau}{H_*}R_w + \frac{\varepsilon}{RT_*}R_\phi \right) \equiv R''_\theta
&&\frac{\gamma}{\kappa\tau}\left(N'_\theta + \frac{\varepsilon\tau}{H_*}N'_w + \frac{\varepsilon}{RT_*}N_\phi \right) \equiv N''_\theta \\
&\frac{\gamma}{\kappa\tau}\left(R_\theta + \frac{\varepsilon\tau}{H_*}R_w - \frac{\kappa}{RT_*}R_\phi \right) \equiv R''_\phi
&&\frac{\gamma}{\kappa\tau}\left(N'_\theta + \frac{\varepsilon\tau}{H_*}N'_w - \frac{\kappa}{RT_*}N_\phi \right) \equiv N''_\phi \\
&R''_C - \left(\delta_\zeta R''_\theta + \overline{R''_\theta}^\zeta \right) - \varepsilon \overline{R''_\phi}^\zeta \equiv R_P
&&N''_C - \left(\delta_\zeta N''_\theta + \overline{N''_\theta}^\zeta \right) - \varepsilon \overline{N''_\phi}^\zeta \equiv N_P
\end{aligned}
$$

Note that we have R_w, R_θ on the left and N'_w, N'_θ on the right and remember that $N_\phi = 0$.

17) Back substitution

The following equations give in a straight forward manner the 6 prognostic variables $\mathbf{V}_h, w, q,\ \left(s, \dot{\zeta}\right)$ and ϕ':

$$\mathbf{V}_h: \qquad \frac{\mathbf{V}_h}{\tau} = \left[\mathbf{R}_h - \mathbf{N}_h - \nabla_\zeta P\right]$$

$$w: \qquad \frac{w}{\tau H_*} = \left[R''_\phi - N''_\phi + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa\overline{P}^\zeta\right)\right]$$

$$q: \quad \delta_\zeta q + \overline{q}^\zeta = -\frac{\varepsilon\tau^2}{H_*}\left[R_w - N'_w - \frac{w}{\tau}\right]; \qquad q_T = 0$$

$$s: \qquad s = \frac{P_S - \phi_S}{RT_*} - q_S$$

$$\dot{\zeta}: \qquad \frac{\dot{\zeta}}{\tau} = -\left[R''_\theta - N''_\theta + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon\overline{P}^\zeta\right)\right] - \frac{\overline{(Bs+q)}^\zeta}{\tau^2}; \quad \dot{\zeta}_T = \dot{\zeta}_S = 0$$

$$\phi': \qquad \phi' = P - RT_*(q + Bs)$$

Finally we may compute μ and T diagnostically:

$$1 + \mu = e^{\overline{q}^\zeta}\left[1 + \frac{\delta_\zeta q}{\delta_\zeta(\zeta + Bs)}\right]$$

$$\frac{T}{T_*} = e^{\overline{q}^\zeta}\left[1 - \frac{\delta_\zeta(\phi'/RT_* + Bs)}{\delta_\zeta(\zeta + Bs)}\right]$$

For a brief description of *The Dynamic Core Code*, see **Appendix 6.**

There is THE HYDROSTATIC OPTION. For a description, see **Appendix 7.**

There is THE AUTOBAROTROPIC OPTION. For a description, see **Appendix 8**.

Aspects of HORIZONTAL DISCRETIZATION are given in **Appendix 13**.

See **Table 1**, page 72, for a summary of the model equations and transformations.

See **Table 2**, page 74, for a summary of the equations, variables, etc.

THE END

Appendix 1. Virtual temperature

In presence of water vapor q_v and various types of hydrometeors q_i, the density of atmospheric substance is given by

$$\rho = \rho\left(q_d + q_v + \sum q_i\right)$$

where q_d is the dry air specific mass. The equation of state is given by

$$\begin{aligned} p &= \rho\left(R_d q_d + R_v q_v\right)T \\ &= \rho R_d\left(1 + \delta q_v - \sum q_i\right)T \end{aligned}$$

where $\delta = R_v / R_d - 1 \approx 0.6$ and we rewrite the equation of state as follows:

$$p = \rho R_d T_v$$

defining virtual temperature thus

$$T_v = T\left(1 + \delta q_v - \sum q_i\right)$$

Rewriting the equations to appear in terms of virtual temperature and approximating the ratio $\kappa = R/c_p$ by $\kappa_d = R_d/c_{pd}$, the equations of **section 1** may then be replaced by the following:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + R_d T_v \nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d \ln T_v}{dt} - \kappa_d \frac{d \ln p}{dt} = \frac{Q_v}{c_{pd}T_v} = \frac{Q}{c_{pd}T_v} + \frac{T}{T_v}\left(\delta \frac{dq_v}{dt} - \sum \frac{dq_i}{dt}\right)$$

$$\frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{V} = 0$$

$$\rho = \frac{p}{R_d T_v}$$

From the point of view of the pure dynamics, these equations are formally identical to those in **section 1** in which R and c_p would take the dry air constant values, temperature be replaced by virtual temperature and appropriate source terms be added in the thermodynamic equation. The advantage of this formulation is of course the fact that the parameters R and c_p no longer varies while all of the virtual effects, including *water vapor buoyancy* and *condensed water loading* effects, are implicitly taken into account. The only approximation made here, the replacement of κ by κ_d in the thermodynamic equation, is facultative.

Appendix 2. Coordinate transformation rules

Appendix 2a. Invariance of the total derivative

By the *chain rule* we first verify the invariance of the total derivative *df/dt* under a general coordinate transformation. In effect, if we consider $f(x,y,z,t)$, then:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{x,y,z} + \left(\frac{\partial f}{\partial x}\right)_{y,z,t} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{x,z,t} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{x,y,t} \frac{dz}{dt}$$

while for $f(x,y,\zeta,t)$, we naturally have:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{x,y,\zeta} + \left(\frac{\partial f}{\partial x}\right)_{y,\zeta,t} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{x,\zeta,t} \frac{dy}{dt} + \left(\frac{\partial f}{\partial \zeta}\right)_{x,y,t} \frac{d\zeta}{dt}$$

Here we only have changed the vertical coordinate from z to ζ with the result that the horizontal components of the velocity $(dx/dt, dy/dt) = (U,V) = \mathbf{V}_h$ remain unchanged. The vertical motion though has transformed from $dz/dt = w$ into $d\zeta / dt = \dot{\zeta}$. Shortening the notation, we also write the above relations respectively as follows:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_z + U\left(\frac{\partial f}{\partial x}\right)_z + V\left(\frac{\partial f}{\partial y}\right)_z + w\frac{\partial f}{\partial z} = \frac{\partial f}{\partial t} + \mathbf{V}_h \cdot \nabla_z f + w\frac{\partial f}{\partial z}$$

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_\zeta + U\left(\frac{\partial f}{\partial x}\right)_\zeta + V\left(\frac{\partial f}{\partial y}\right)_\zeta + \dot{\zeta}\frac{\partial f}{\partial \zeta} = \frac{\partial f}{\partial t} + \mathbf{V}_h \cdot \nabla_\zeta f + \dot{\zeta}\frac{\partial f}{\partial \zeta}$$

Thus we minimized the indices. We also introduced the vector notation for the 'horizontal' part of the advection operator. Note though that the new coordinate ζ is generally curvilinear and non-orthogonal and the scalar product must be interpreted with care (see appendix 2c)

Appendix 2b. Transformation rules for derivatives.

It is remarkable that not only can all these rules *be recovered* from the invariance of the total derivative but also that these derivative transformation rules suffice to transform the Euler equations. In effect, the three velocity components may be treated as three independent scalars ('pseudo-scalars'), the velocity vector not being transformed. We are left though with a 'hybrid' system since maintaining two vertical velocities w and $\dot{\eta}$ or $\dot{\zeta}$ and therefore needing an additional [prognostic when $(\partial z / \partial t)_\zeta \neq 0$), diagnostic otherwise] equation. A complete transformation to a time-varying non-orthogonal curvilinear coordinate, a complete elimination of w, is of course possible but then the notions of four-dimensional tensor calculus is very useful (see appendix 2d).

The transformation rules may be obtained by equating the above two relations. In effect, we must have

$$0=\left(\frac{\partial f}{\partial t}\right)_z-\left(\frac{\partial f}{\partial t}\right)_\zeta+U\left[\left(\frac{\partial f}{\partial x}\right)_z-\left(\frac{\partial f}{\partial x}\right)_\zeta\right]+V\left[\left(\frac{\partial f}{\partial y}\right)_z-\left(\frac{\partial f}{\partial y}\right)_\zeta\right]+w\frac{\partial f}{\partial z}-\dot{\zeta}\frac{\partial f}{\partial \zeta}$$

and since

$$w=\frac{dz}{dt}=\left(\frac{\partial z}{\partial t}\right)_\zeta+U\left(\frac{\partial z}{\partial x}\right)_\zeta+V\left(\frac{\partial z}{\partial y}\right)_\zeta+\dot{\zeta}\frac{\partial z}{\partial \zeta}$$

then

$$\begin{aligned}0=&\left[\left(\frac{\partial f}{\partial t}\right)_z-\left(\frac{\partial f}{\partial t}\right)_\zeta+\left(\frac{\partial z}{\partial t}\right)_\zeta\frac{\partial f}{\partial z}\right]+U\left[\left(\frac{\partial f}{\partial x}\right)_z-\left(\frac{\partial f}{\partial x}\right)_\zeta+\left(\frac{\partial z}{\partial x}\right)_\zeta\frac{\partial f}{\partial z}\right]\\&+V\left[\left(\frac{\partial f}{\partial y}\right)_z-\left(\frac{\partial f}{\partial y}\right)_\zeta+\left(\frac{\partial z}{\partial y}\right)_\zeta\frac{\partial f}{\partial z}\right]+\dot{\zeta}\left[\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \zeta}\right]\end{aligned}$$

Each bracket must vanish independently. Therefore the rules are:

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_z&=\left(\frac{\partial f}{\partial t}\right)_\zeta-\left(\frac{\partial z}{\partial t}\right)_\zeta\frac{\partial f}{\partial z}\\\left(\frac{\partial f}{\partial x}\right)_z&=\left(\frac{\partial f}{\partial x}\right)_\zeta-\left(\frac{\partial z}{\partial x}\right)_\zeta\frac{\partial f}{\partial z}\\\left(\frac{\partial f}{\partial y}\right)_z&=\left(\frac{\partial f}{\partial y}\right)_\zeta-\left(\frac{\partial z}{\partial y}\right)_\zeta\frac{\partial f}{\partial z}\\\frac{\partial f}{\partial z}&=\frac{\partial \zeta}{\partial z}\frac{\partial f}{\partial \zeta}\end{aligned}$$

Appendix 2c. Vectors in non-orthogonal curvilinear coordinates

In non-orthogonal curvilinear coordinates $\hat{\mathbf{x}}=\left(\hat{x}^1,\hat{x}^2,\hat{x}^3\right)$ (see Dutton, John A, *The Ceaseless Wind*, chapters 5 and 7), there appear two sets of basis vectors (usually not even of unit length) and two sets of vector components. Applying the chain rule, we obtain the following two expansions (summation convention):

$$\begin{aligned}d\mathbf{x}&=\frac{\partial \mathbf{x}}{\partial \hat{x}^j}d\hat{x}^j & \mathrm{d}\hat{x}^i&=\frac{\partial \hat{x}^i}{\partial x^j}dx^j=\left(\nabla\hat{x}^i\right)\cdot d\mathbf{x}\\&=\boldsymbol{\tau}_j d\hat{x}^j & &=\boldsymbol{\eta}^{\mathrm{i}}\cdot d\mathbf{x}\end{aligned}$$

where $\boldsymbol{\tau}_j$ is covariant: *tangent to the curve along which only* $\hat{x}^j$ *varies* and $\boldsymbol{\eta}^{\mathrm{i}}$ is contra-variant: *normal to the surface* $\hat{x}^i$ = const. and we have the orthogonality relation

$$\boldsymbol{\tau}_j \boldsymbol{\eta}^{\mathrm{i}} = \delta_j^i$$

Representing a vector **A** as

$$\mathbf{A} = A^k \boldsymbol{\tau}_k = A_k \boldsymbol{\eta}^{\mathrm{k}}$$

we may recover the components [A_k (A^k): covariant (contravariant) components] using the above orthogonality relation:

$$A^i = \mathbf{A} \cdot \boldsymbol{\eta}^{\mathrm{i}} = A^j \boldsymbol{\tau}_j \cdot \boldsymbol{\eta}^{\mathrm{i}}$$
$$A_j = \mathbf{A} \cdot \boldsymbol{\tau}_j = A_i \boldsymbol{\eta}^{\mathrm{i}} \cdot \boldsymbol{\tau}_j$$

The scalar product is

$$\mathbf{A} \cdot \mathbf{B} = A^k B_k = A_k B^{\mathrm{k}}$$

Therefore in generalized vertical coordinate $\hat{\mathbf{x}} = (x, y, \zeta)$ the basis vectors become [the original orthogonal Cartesian coordinate being $\mathbf{x} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$]

$$\boldsymbol{\eta}^1 = \nabla x = \mathbf{i} \qquad \boldsymbol{\tau}_1 = \frac{\partial \mathbf{x}}{\partial x} = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}$$
$$\boldsymbol{\eta}^2 = \nabla y = \mathbf{j} \qquad \boldsymbol{\tau}_2 = \frac{\partial \mathbf{x}}{\partial y} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}$$
$$\boldsymbol{\eta}^3 = \nabla \zeta \qquad \boldsymbol{\tau}_3 = \frac{\partial \mathbf{x}}{\partial \zeta} = \frac{\partial z}{\partial \zeta}\mathbf{k}$$

The contravariant components of the velocity vector $V^i = \mathbf{V} \cdot \boldsymbol{\eta}^{\mathrm{i}}$ are found to be

$$U,\ V,\ \mathbf{V} \cdot \nabla \boldsymbol{\zeta} = \dot{\boldsymbol{\zeta}}$$

While the covariant components of the gradient $\partial f / \partial \hat{x}^j = \nabla f \cdot \boldsymbol{\tau}_i$ are found to be

$$\left(\frac{\partial f}{\partial x}\right)_\zeta,\ \left(\frac{\partial f}{\partial y}\right)_\zeta,\ \frac{\partial f}{\partial \zeta}$$

And the vector product $\mathbf{V} \cdot \nabla f$ may be computed as follows:

$$\mathbf{V} \cdot \nabla f = V^i \boldsymbol{\tau}_i \cdot \boldsymbol{\eta}^j \frac{\partial f}{\partial \hat{x}^j}$$
$$= \left[U\left(\mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}\right) + V\left(\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}\right) + \dot{\zeta}\frac{\partial z}{\partial \zeta}\mathbf{k} \right] \cdot \left[\left(\frac{\partial f}{\partial x}\right)_\zeta \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_\zeta \mathbf{j} + \frac{\partial f}{\partial \zeta}\nabla \zeta \right]$$

$$= U\left[\left(\frac{\partial f}{\partial x}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\mathbf{i}\cdot\nabla\zeta + \frac{\partial z}{\partial x}\mathbf{k}\cdot\nabla\zeta\right)\right] + V\left[\left(\frac{\partial f}{\partial y}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\mathbf{j}\cdot\nabla\zeta + \frac{\partial z}{\partial y}\mathbf{k}\cdot\nabla\zeta\right)\right] + \dot{\zeta}\,\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial \zeta}\mathbf{k}\cdot\nabla\zeta$$

$$= U\left[\left(\frac{\partial f}{\partial x}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial \zeta}{\partial z}\right)\right] + V\left[\left(\frac{\partial f}{\partial y}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial \zeta}{\partial z}\right)\right] + \dot{\zeta}\,\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}$$

$$= U\left(\frac{\partial f}{\partial x}\right)_\zeta + V\left(\frac{\partial f}{\partial y}\right)_\zeta + \dot{\zeta}\,\frac{\partial f}{\partial \zeta}$$

since

$$\left(\frac{\partial \zeta}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_z \frac{\partial \zeta}{\partial z} = \left(\frac{\partial \zeta}{\partial x}\right)_\zeta = 0$$

$$\left(\frac{\partial \zeta}{\partial y}\right)_z + \left(\frac{\partial z}{\partial y}\right)_z \frac{\partial \zeta}{\partial z} = \left(\frac{\partial \zeta}{\partial y}\right)_\zeta = 0$$

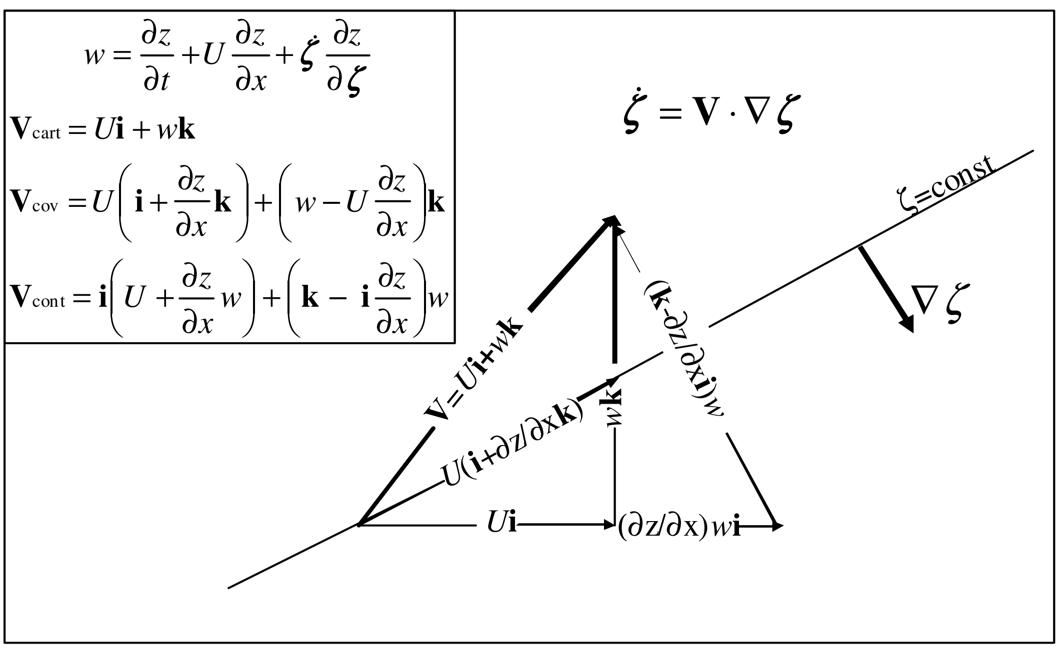


Figure 2. Representation of the wind vector in both orthogonal *z*-coordinate and oblique *ζ*-coordinate

Appendix 2d. Complete elimination of *w*.

Neglecting the Coriolis force and physical forcings, the four equations of motion in η-coordinate (*see page 5*) may be written:

$$\frac{d\mathbf{V}_h}{dt}+\frac{1}{\rho}\left(\nabla_\eta p-\nabla_\eta z\left(\frac{\partial z}{\partial\eta}\right)^{-1}\frac{\partial p}{\partial\eta}\right)=0 \qquad \text{(A2.1)}$$

$$\frac{dw}{dt}+\frac{1}{\rho}\left(\frac{\partial z}{\partial\eta}\right)^{-1}\frac{\partial p}{\partial\eta}+g=0 \qquad \text{(A2.2)}$$

$$\frac{dz}{dt}=w \qquad \text{(A2.3)}$$

with

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}_h\cdot\nabla_\eta+\dot{\eta}\frac{\partial}{\partial\eta} \qquad \text{(A2.4)}$$

Inserting (A2.3) in (A2.2) using (A2.1) and (A2.4), we obtain (Einstein summation convention):

$$\frac{d\dot{\eta}}{dt}+\Gamma^3_{\alpha\beta}u^\alpha u^\beta+\frac{1}{\rho}h^{3\alpha}\frac{\partial p}{\partial x^\alpha}+\left(\frac{\partial z}{\partial\eta}\right)^{-1}g=0$$

with $x^\alpha=(t,x,y,\eta)$ and $u^\alpha=(1,u,v,\dot{\eta})$, and where

$$\Gamma^3_{\alpha\beta}=\left(\frac{\partial z}{\partial\eta}\right)^{-1}\frac{\partial^2 z}{\partial x^\alpha\partial x^\beta}$$

is a Christoffel symbol and where

$$h^{30}=0;\; h^{31}=-\left(\frac{\partial z}{\partial\eta}\right)^{-1}\frac{\partial z}{\partial x};\; h^{32}=-\left(\frac{\partial z}{\partial\eta}\right)^{-1}\frac{\partial z}{\partial y};\; h^{33}=\left(\frac{\partial z}{\partial\eta}\right)^{-2}\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$

is a special tensor related to the metric tensor as follows:

$$h^{\mu\nu}=g^{\mu\nu}-g^{\mu 0}g^{0\nu}$$

(see Charron et al. 2013 in QJRMS for all the beautiful details).

Appendix 3. The metric parameter *B*.

Before investigating the relation defining the hydrostatic pressure π in ζ-coordinate (GEM4), let us review the behavior of the similar relation in η-*coordinate* (GEM3) which is given by:

$$\pi = A(\eta) + B(\eta)\pi_S$$

$$A = (\eta - B)p_{ref} \qquad B = \left(\frac{\eta - \eta_T}{1-\eta_T}\right)^r; \qquad r \ge 1,\ const$$

$$\pi = \eta p_{ref} - B(p_{ref} - \pi_S)$$

B is the relevant parameter although p_{ref} also plays a role. We have $0 \le B \le 1$; we calculate its derivative

$$\frac{\partial B}{\partial \eta} = \frac{rB}{\eta - \eta_T} \ge 0$$

Monotonicity is essential, requiring that

$$\frac{\partial \pi}{\partial \eta} = p_{ref} - \frac{\partial B}{\partial \eta}(p_{ref} - \pi_S) > 0$$

Smallest values occur at the surface where $\partial B / \partial \eta$ is maximum and over high ground where π_S is minimum. Hence

$$\frac{\partial \pi}{\partial \eta} = p_{ref} - \frac{r}{1-\eta_T}(p_{ref} - \pi_S) > 0$$

i.e.

$$\frac{\pi_S}{p_{ref}} > 1 - \left(\frac{\partial B}{\partial \eta}\right)^{-1} = \frac{r - 1 + \eta_T}{r} \approx \frac{r-1}{r}$$

The concern here is model layer thicknesses, $\Delta z = -(RT/g)\Delta \ln \pi$, and unfortunately these are also smallest near the surface. Furthermore, the temperature is lower over high ground. Let us therefore compute the **surface thinning factor**, $thfs = (\Delta \ln \pi)_{top} / (\Delta \ln \pi)_{msl}$, the minimum ratio of model layer thicknesses over high ground (say $\pi_{S\,top} \approx 500\,\text{hPa}$) to those at sea level (say $\pi_{S\,msl} \approx 1000\,\text{hPa}$). With the top pressure $p_T = 10\,\text{Pa}$, $\eta_T = .0001$ can be neglected and

$$\pi \approx (\eta - \eta^r)p_{ref} + \eta^r \pi_S$$

Since η is close to 1, we write $\eta = 1 - \Delta\eta$. We may expand, with the result

$$\pi \approx (r-1)\Delta\eta p_{ref} + (1 - r\Delta\eta)\pi_S$$

$$\Delta \ln \pi \approx \frac{\pi_S - \pi}{\pi_S} \approx \Delta\eta \left[r - (r-1)\frac{p_{ref}}{\pi_S} \right]$$

$$thfs = \frac{r - (r-1)p_{ref} / \pi_{S\,top}}{r - (r-1)p_{ref} / \pi_{S\,msl}}$$

With p_{ref} = 1000 hPa, $thfs = 2 - r$; thus for *r*=1.6, *thfs*=0.4. With p_{ref} = 800 hPa, $thfs = [r - (r-1)8/5]/[r - (r-1)8/10]$ and for *r*=1.6, *thfs*=0.57. In both cases though *thfs*=1 for *r*=1. Note that for *r*=1.6, $\pi_{\min}(0.2) \approx 162$ hPa when p_{ref} = 1000 hPa while $\pi_{\min}(0.2) \approx 137$ hPa when p_{ref} = 800 hPa. Here it is important to note that, in addition to allow for an increase in *thfs*, a lower p_{ref} forces *a decrease of pressure for* (*a lifting of*) *all levels except the surface*. For *r*=1.6, there results a 12% increase of the thickness of the bottom level.

In ζ-*coordinate*, the hydrostatic pressure will be given by

$$\ln \pi = A(\zeta) + B(\zeta)s; \qquad s = \ln \pi_S - \zeta_S = \ln(\pi_S / p_{ref}); \quad p_{ref} = 1000\,\text{hPa}$$

$$A = \zeta \qquad B = \lambda^r$$

$$\lambda = \frac{\zeta - \zeta_U}{\zeta_S - \zeta_U} \geq 0; \quad \zeta_U \geq \zeta_T; \quad r = r_{\max} - (r_{\max} - r_{\min})\lambda$$

$$\ln \pi = \zeta + B[\ln \pi_S - \zeta_S]$$

Here *B* is the unique relevant parameter as p_{ref} is not allowed to change. We note two remaining differences from the similar relation in η-coordinate: the logarithmic character of the relation and the introduction of a variable exponent *r*. We again have $0 \leq B \leq 1$ and a positive derivative:

$$\frac{\partial \ln B}{\partial \lambda} = \frac{\partial r \ln \lambda}{\partial \lambda} = \frac{1}{\lambda}[r - \Delta r \lambda \ln \lambda] \geq 0; \qquad \Delta r = r_{\max} - r_{\min}$$

Monotonicity requires that

$$\frac{\partial \ln \pi}{\partial \zeta} = 1 + \frac{\partial B}{\partial \lambda}[\ln \pi_S - \zeta_S]\frac{\partial \lambda}{\partial \zeta} > 0$$

$$\frac{\ln \pi_S}{\zeta_S} > K; \qquad K = 1 - \left(\frac{\partial B}{\partial \lambda}\right)_{\max}^{-1}\left(1 - \frac{\zeta_T}{\zeta_S}\right)$$

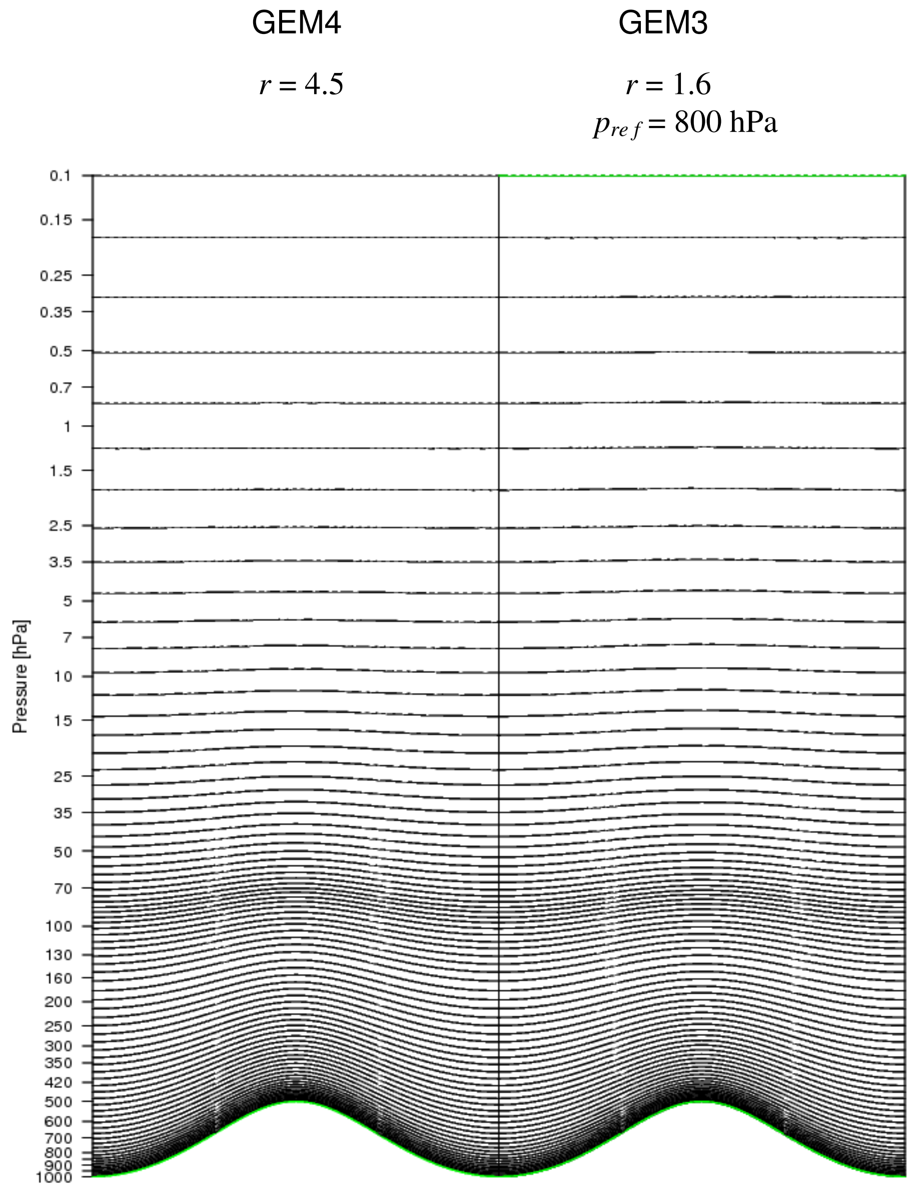


Figure 3. The structure of the 79 levels of GEM4 (r = 4.5) compared to the 80 levels of GEM3 (r = 1.6; p_{ref} = 800 hPa).

When r is constant (Δr=0), $(\partial B/\partial\lambda)_{\max} = r$ at the model surface λ=1. K=1-1/$r\left(1-\zeta_T/\zeta_S\right)$ and the monotonicity requirement is $r < \ln\left(p_{ref}/p_T\right)/\ln\left(p_{ref}/\pi_S\right)$. For $\pi_{S\,high} \approx p_{ref}/2$ and p_{top}=10 Pa, this implies $r < 4\ln 10/\ln 2 \approx 13.2$ and for p_{top}=10 hPa, $r < 2\ln 10/\ln 2 \approx 6.6$.

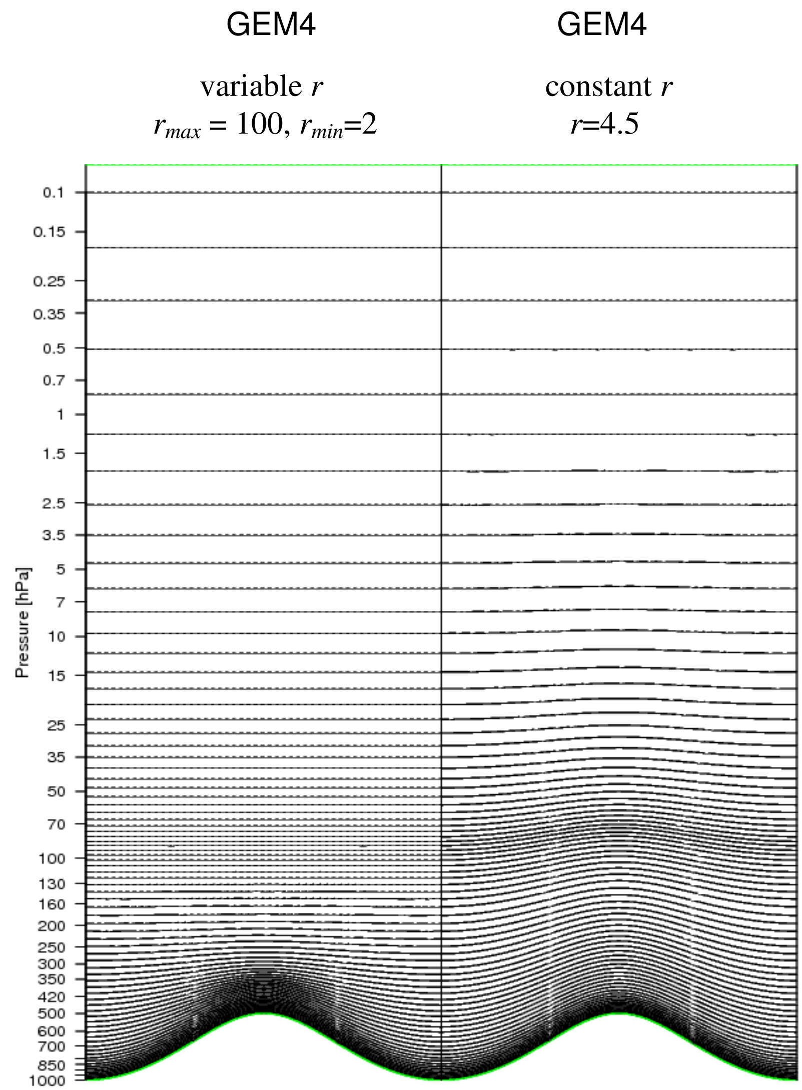


Figure 4. GEM4 with variable r (r_{max}=100, r_{min}=2) compared to GEM4 with constant r=4.5.

Larger admitted exponents do not necessarily mean better coordinate straightening though and we must keep worrying about the ratio of model layer thicknesses. Considering

$$\left(\frac{\partial \ln \pi}{\partial \zeta}\right)_{\min} = 1 - \left(\frac{\partial B}{\partial \lambda}\right)_{\max} \frac{\ln\left(p_{ref} / \pi_S\right)}{\ln\left(p_{ref} / p_{top}\right)}$$

we get, for constant r with p_{top} =10 Pa,

$$thfs = 1 - r\frac{\ln 2}{\ln\left(p_{ref} / p_{top}\right)} \approx \left(1 - .075r\right)$$

Hence, for r=1, $thfs \approx .925$ already. This ratio is 8% smaller than in η-coordinate (15% smaller with p_{top}=10 hPa). The value *thfs*=0.4 is reached for r=8 and *thfs*=0.57 is reached for r= 5.7 meanwhile $\pi(0.2) \approx 172$ hPa with r=8 and $\boldsymbol{\pi}(0.2) \approx 159$ hPa with r=5.7, slightly better but no doubt insufficient rectification. Hence the need to keep r close to 0 near the surface while faster coordinate rectification requires increasingly larger values of r aloft and this is what we may attempt to achieve with the present formulation.

Three figures are shown above and below. In the first, **Figure 3**, we compare the 79 momentum levels of GEM4 (r=4.5) to the 80 levels of GEM3 (r=1.6) used operationally in its global configuration (year 2011). Basically, as can be seen, GEM4 levels have been adjusted such that the pressures above π_S=1000 hPa correspond one by one to GEM3 levels. In **Figures 4** and **5**, we compare GEM4 with variable r (r_{max}=100, r_{min}=2) to GEM4 with constant r=4.5, clearly showing the rectifying possibilities inherent in variable r. The basic idea here is to essentially eliminate topography induced coordinate variation above 200 hPa.

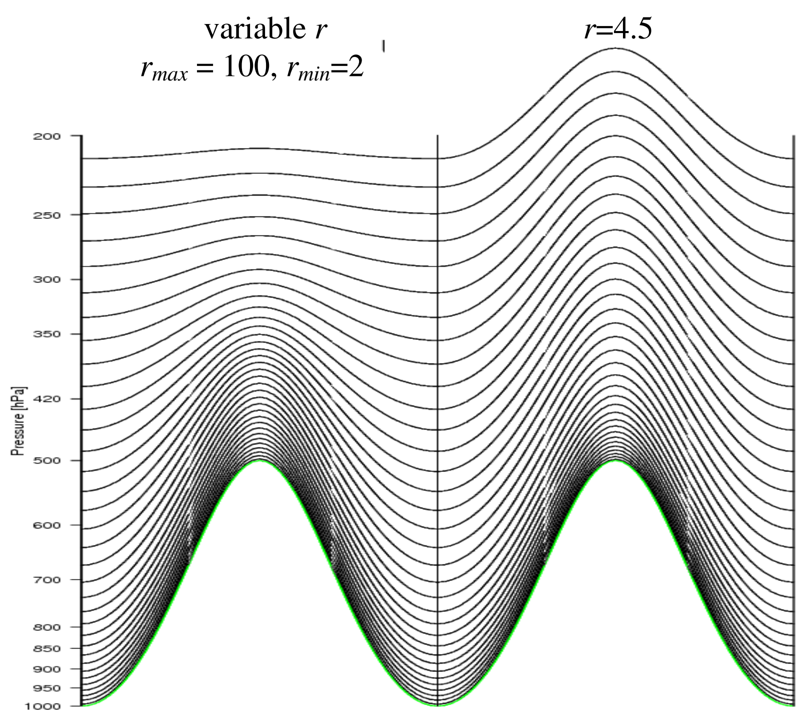


Figure 5. GEM4 with variable r (r_{max}=100, r_{min}=2) compared to GEM4 with constant r=4.5 **below 200 hPa**.

Appendix 4a. Detailed spatial discretization with staggering: *the linear terms*

In **section 7**, we described the vertical discretization succinctly. In **section 15**, we examined the Elliptic Problem. We showed that all variables could be readily eliminated in favor of P. We now go back and examine the discrete linear system leading to the elliptic problem in full details. As mentioned earlier, the finite differences replacing the derivatives are made as simple as possible, i.e.

$$\left(\delta_\zeta F\right)_{k+\frac{1}{2}} = \frac{F_{k+1} - F_k}{\Delta\zeta_{k+\frac{1}{2}}} \quad (k=0,N) \quad ; \quad \left(\delta_\zeta G\right)_k = \frac{G_{k+\frac{1}{2}} - G_{k-\frac{1}{2}}}{\Delta\zeta_k} \quad (k=1,N)$$

$$\Delta\zeta_{k+\frac{1}{2}} = \zeta_{k+1} - \zeta_k \qquad \Delta\zeta_k = \zeta_{k+\frac{1}{2}} - \zeta_{k-\frac{1}{2}}$$

with the top $\zeta_0 = \zeta_{\frac{1}{2}} = \zeta_T$, the surface $\zeta_{N+1} = \zeta_{N+\frac{1}{2}} = \zeta_S$ and the momentum levels ζ_k $\left(1 \le k \le N\right)$ specified while the intermediate thermodynamic levels $\zeta_{k+\frac{1}{2}} = \dfrac{\zeta_{k+1} + \zeta_k}{2}$ $\left(1 \le k \le N-1\right)$ are calculated. The use of averaging is minimized. Starting with $\left(L_C\right)_k$, $\left(\mathbf{V}_h\right)_k$ and $X_{k+\frac{1}{2}}$ are chosen and therefore $\left(\mathbf{L}_h\right)_k$ and P_k. The hydrostatic case (q being absent) suggests $\left(L'_\theta\right)_{k+\frac{1}{2}}$, therefore $\left(L_H\right)_{k+\frac{1}{2}}$ and $T'_{k+\frac{1}{2}}$ (**section 11**). In $\left(L'_\theta\right)_{k+\frac{1}{2}}$ and $\left(L_H\right)_{k+\frac{1}{2}}$, we now introduce q_k. This in turn suggests $\left(L'_w\right)_{k+\frac{1}{2}}$, $w_{k+\frac{1}{2}}$ and finally $\left(L_\phi\right)_{k+\frac{1}{2}}$. Hence

$$\left(\mathbf{L}_h\right)_k = \left(\frac{\mathbf{V}_h}{\tau} + \nabla_\zeta P\right)_k \qquad \left(k=1,N\right)$$

$$\left(L'_w\right)_{k+\frac{1}{2}} = \left(\frac{w}{\tau} - g\left(\delta_\zeta q + \overline{q}^\zeta\right)\right)_{k+\frac{1}{2}} \qquad \left(k=0,N\right)$$

$$\left(L'_\theta\right)_{k+\frac{1}{2}} = \left(\frac{\delta_\zeta q + \overline{q}^\zeta}{\tau} - \frac{\delta_\zeta P}{\tau R T_*} - \kappa X\right)_{k+\frac{1}{2}} \qquad \left(k=0,N\right)$$

$$\left(L_C\right)_k = \left(-\frac{\overline{\delta_\zeta q + \overline{q}^\zeta}^\zeta}{\tau} + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta X + \overline{X}^\zeta\right)_k \qquad \left(k=1,N\right)$$

$$\left(L_\phi\right)_{k+\frac{1}{2}} = \left(\frac{\overline{P}^\zeta}{\tau} - R T_* X - g w\right)_{k+\frac{1}{2}} \qquad \left(k=0,N\right)$$

In the vertical, this leads to Charney-Phillips grid (**Figure 1**, page 11). For the two vertical means, we formally write

$$\left(\overline{F}^{\zeta}\right)_{k+\frac{1}{2}} = \varpi^{+}_{k+\frac{1}{2}} F_{k+1} + \varpi^{-}_{k+\frac{1}{2}} F_{k} \qquad \text{and} \qquad \left(\overline{G}^{\zeta}\right)_{k} = \varpi^{+}_{k} G_{k+\frac{1}{2}} + \varpi^{-}_{k} G_{k-\frac{1}{2}}.$$

$$\varpi^{-}_{k+\frac{1}{2}} = 1 - \varpi^{+}_{k+\frac{1}{2}} \qquad\qquad \varpi^{-}_{k} = 1 - \varpi^{+}_{k}$$

The first one, averaging variables from momentum or full levels toward thermodynamic or half-levels, follows the rule of calculation for the half-levels, i.e.

$$\varpi^{+}_{\frac{1}{2}} = 0, \quad \varpi^{+}_{k+\frac{1}{2}} = \frac{1}{2}\ \left(1 \le k \le N-1\right), \quad \varpi^{+}_{N+\frac{1}{2}} = 1$$

This ensures optimal (second-order) accuracy for the hydrostatic equation L_H in particular. For the second one, averaging variables from thermodynamic levels toward momentum levels, three choices were considered: linear interpolation, simple average, average commuting with difference. Due to lack of sensibility, the last was adopted because it simplifies the code:

$$\varpi^{+}_{k} = \frac{\Delta\zeta_{k+\frac{1}{2}}}{2\Delta\zeta_{k}} = \frac{\zeta_{k+1} - \zeta_{k}}{\zeta_{k+1} - \zeta_{k-1}}\ \left(1 \le k \le N-1\right)$$

$$\varpi^{+}_{N} = \frac{\Delta\zeta_{N+\frac{1}{2}}}{\Delta\zeta_{N}} = \frac{\zeta_{N+1} - \zeta_{N}}{\zeta_{N+\frac{1}{2}} - \zeta_{N-\frac{1}{2}}}$$

More explicitly,

$$\left(\mathbf{L}_{h}\right)_{k} = \frac{\mathbf{V}_{h\,k}}{\tau} + \nabla_{\zeta} P_{k} \qquad (k = 1, N)$$

$$\left(L'_{w}\right)_{k+\frac{1}{2}} = \frac{w_{k+\frac{1}{2}}}{\tau} - g\left(\frac{q_{k+1} - q_{k}}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi^{+}_{k+\frac{1}{2}} q_{k+1} + \varpi^{-}_{k+\frac{1}{2}} q_{k}\right) \qquad (k = 0N)$$

$$\left(L'_{\theta}\right)_{k+\frac{1}{2}} = \frac{1}{\tau}\left(\frac{q_{k+1} - q_{k}}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi^{+}_{k+\frac{1}{2}} q_{k+1} + \varpi^{-}_{k+\frac{1}{2}} q_{k}\right) - \frac{1}{\tau R T_{*}} \frac{P_{k+1} - P_{k}}{\Delta\zeta_{k+\frac{1}{2}}} - \kappa X_{k+\frac{1}{2}} \qquad (k = 0, N)$$

$$\left(L_{C}\right)_{k} = -\frac{1}{\tau}\left[\varpi^{+}_{k}\left(\frac{q_{k+1} - q_{k}}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi^{+}_{k+\frac{1}{2}} q_{k+1} + \varpi^{-}_{k+\frac{1}{2}} q_{k}\right) + \varpi^{-}_{k}\left(\frac{q_{k} - q_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}} + \varpi^{+}_{k-\frac{1}{2}} q_{k} + \varpi^{-}_{k-\frac{1}{2}} q_{k-1}\right)\right]$$

$$+ \nabla_{\zeta} \cdot \mathbf{V}_{h\,k} + \frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\zeta_{k}} + \varpi^{+}_{k} X_{k+\frac{1}{2}} + \varpi^{-}_{k} X_{k-\frac{1}{2}} \qquad (k = 1, N)$$

$$\left(L_{\phi}\right)_{k+\frac{1}{2}} = \frac{1}{\tau}\left(\varpi^{+}_{k+\frac{1}{2}} P_{k+1} + \varpi^{-}_{k+\frac{1}{2}} P_{k}\right) - R T_{*} X_{k+\frac{1}{2}} - g w_{k+\frac{1}{2}} \qquad (k = 0, N)$$

We have in the vertical direction $3N$ [$\mathbf{L}_h, L_C$] + $3(N+1)$ [L'_w, L'_T, L_ϕ], i.e. $6N+3$ equations and $2N$ $[U,V]_{1,N}$ + $2(N+1)$ $[w,X,]_{1/2,N+1/2}$ + $2(N+2)$ $[P,q]_{0,N+1}$, i.e. $6N+6$ variables. As expected (section 7), we will need 3 boundary conditions in the vertical to close the problem. Now, 3 variables ($\mathbf{V}_h$, q) can easily be eliminated by combining the equations as follows:

$$\nabla_\zeta \cdot (\mathbf{L}_h)_k - \frac{1}{\tau}\left[(L_C)_k - \frac{\varepsilon\tau}{H_*}\left(\varpi_k^+ (L'_w)_{k+\frac{1}{2}} + \varpi_k^- (L'_w)_{k-\frac{1}{2}}\right)\right] \equiv (L''_C)_k$$

$$\frac{\gamma}{\kappa\tau}\left[(L'_T)_{k+\frac{1}{2}} + \frac{\varepsilon\tau}{H_*}(L'_w)_{k+\frac{1}{2}} + \frac{\varepsilon}{RT_*}(L_\phi)_{k+\frac{1}{2}}\right] \equiv (L''_\theta)_{k+\frac{1}{2}}$$

$$\frac{\gamma}{\kappa\tau}\left[(L'_T)_{k+\frac{1}{2}} + \frac{\varepsilon\tau}{H_*}(L'_w)_{k+\frac{1}{2}} - \frac{\kappa}{RT_*}(L_\phi)_{k+\frac{1}{2}}\right] \equiv (L''_\phi)_{k+\frac{1}{2}}$$

to give

$$(L''_C)_k = \nabla_\zeta^2 P_k - \frac{1}{\tau}\left[\frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\zeta_k} + \left(\varpi_k^+ X_{k+\frac{1}{2}} + \varpi_k^- X_{k-\frac{1}{2}}\right)\right] + \frac{\varepsilon}{\tau H_*}\left(\varpi_k^+ w_{k+\frac{1}{2}} + \varpi_k^- w_{k-\frac{1}{2}}\right)$$

$$(L''_\theta)_{k+\frac{1}{2}} = -\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \varepsilon\left(\varpi_{k+\frac{1}{2}}^+ P_{k+1} + \varpi_{k+\frac{1}{2}}^- P_k\right)\right] - \frac{X_{k+\frac{1}{2}}}{\tau}$$

$$(L''_\phi)_{k+\frac{1}{2}} = -\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} + \kappa\left(\varpi_{k+\frac{1}{2}}^+ P_{k+1} + \varpi_{k+\frac{1}{2}}^- P_k\right)\right] + \frac{w_{k+\frac{1}{2}}}{\tau H_*}$$

By further forming

$$\begin{aligned}
-\frac{(L''_\theta)_{k+\frac{1}{2}} - (L''_\theta)_{k-\frac{1}{2}}}{\Delta\zeta_k} - \varpi_k^+ (L''_\theta)_{k+\frac{1}{2}} - \varpi_k^- (L''_\theta)_{k-\frac{1}{2}} = \\
\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{1}{\Delta\zeta_k}\left(\frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right) + \varpi_k^+ \frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi_k^- \frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right] \\
- \frac{\gamma\varepsilon}{\kappa\tau^2 RT_*}\frac{1}{\Delta\zeta_k}\left[\left(\varpi_{k+\frac{1}{2}}^+ P_{k+1} + \varpi_{k+\frac{1}{2}}^- P_k\right) - \left(\varpi_{k-\frac{1}{2}}^+ P_k + \varpi_{k-\frac{1}{2}}^- P_{k-1}\right)\right] \\
- \frac{\gamma\varepsilon}{\kappa\tau^2 RT_*}\left[\varpi_k^+\left(\varpi_{k+\frac{1}{2}}^+ P_{k+1} + \varpi_{k+\frac{1}{2}}^- P_k\right) + \varpi_k^-\left(\varpi_{k-\frac{1}{2}}^+ P_k + \varpi_{k-\frac{1}{2}}^- P_{k-1}\right)\right] \\
+ \frac{1}{\tau}\left(\frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\zeta_k} + \varpi_k^+ X_{k+\frac{1}{2}} + \varpi_k^- X_{k-\frac{1}{2}}\right)
\end{aligned}$$

and

$$-\varepsilon\left(\varpi_k^+\left(L''_\phi\right)_{k+\frac{1}{2}}+\varpi_k^-\left(L''_\phi\right)_{k-\frac{1}{2}}\right)=\frac{\gamma\varepsilon}{\kappa\tau^2RT_*}\left[\varpi_k^+\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}}+\varpi_k^-\frac{P_k-P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right]$$
$$+\frac{\gamma\varepsilon\kappa}{\kappa\tau^2RT_*}\left[\varpi_k^+\left(\varpi_{k+\frac{1}{2}}^+P_{k+1}+\varpi_{k+\frac{1}{2}}^-P_k\right)+\varpi_k^-\left(\varpi_{k-\frac{1}{2}}^+P_k+\varpi_{k-\frac{1}{2}}^-P_{k-1}\right)\right]$$
$$-\frac{g\varepsilon}{\tau RT_*}\left(\varpi_k^+w_{k+\frac{1}{2}}+\varpi_k^-w_{k-\frac{1}{2}}\right)$$

and finally

$$\left(L''_C\right)_k-\frac{\left(L''_\theta\right)_{k+\frac{1}{2}}-\left(L''_\theta\right)_{k-\frac{1}{2}}}{\Delta\zeta_k}-\varpi_k^+\left[\left(L''_\theta\right)_{k+\frac{1}{2}}+\varepsilon\left(L''_\phi\right)_{k+\frac{1}{2}}\right]-\varpi_k^-\left[\left(L''_\theta\right)_{k-\frac{1}{2}}+\varepsilon\left(L''_\phi\right)_{k-\frac{1}{2}}\right]=\left(L_P\right)_k$$

we succeed in eliminating X and w. In effect, we have

$$\left(L_P\right)_k=\nabla_\zeta^2P_k+\frac{\gamma}{\kappa\tau^2RT_*}\left[\frac{1}{\Delta\zeta_k}\left(\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}}-\frac{P_k-P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right)+\varpi_k^+\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}}+\varpi_k^-\frac{P_k-P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right]$$
$$+\frac{\gamma\varepsilon}{\kappa\tau^2RT_*}\left[\varpi_k^+\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}}+\varpi_k^-\frac{P_k-P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}-\frac{\varpi_{k+\frac{1}{2}}^+P_{k+1}+\varpi_{k+\frac{1}{2}}^-P_k-\varpi_{k-\frac{1}{2}}^+P_k-\varpi_{k-\frac{1}{2}}^-P_{k-1}}{\Delta\zeta_k}\right]$$
$$-\frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2RT_*}\left[\varpi_k^+\left(\varpi_{k+\frac{1}{2}}^+P_{k+1}+\varpi_{k+\frac{1}{2}}^-P_k\right)+\varpi_k^-\left(\varpi_{k-\frac{1}{2}}^+P_k+\varpi_{k-\frac{1}{2}}^-P_{k-1}\right)\right]$$

The second bracket corresponds to the difference $\overline{\delta_\zeta P}^\zeta-\delta_\zeta\overline{P}^\zeta$ which vanishes by construction (commuting average, see **Appendix 5**). Therefore the final result is

$$\left(L_P\right)_k=\nabla_\zeta^2P_k+\frac{\gamma}{\kappa\tau^2RT_*}\left[\frac{1}{\Delta\zeta_k}\left(\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}}-\frac{P_k-P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right)+\varpi_k^+\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}}+\varpi_k^-\frac{P_k-P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right]$$
$$-\frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2RT_*}\left[\varpi_k^+\left(\varpi_{k+\frac{1}{2}}^+P_{k+1}+\varpi_{k+\frac{1}{2}}^-P_k\right)+\varpi_k^-\left(\varpi_{k-\frac{1}{2}}^+P_k+\varpi_{k-\frac{1}{2}}^-P_{k-1}\right)\right]$$

N equations and N+2 unknowns.

Appendix 4b. Detailed spatial discretization: *matrices of the elliptic problem*

The matrix of the elliptic problem is composed of the previous equations (k=1,N):

$$\Delta\zeta_k\left(L_P\right)_k = \Delta\zeta_k\nabla_\zeta^2 P_k + \frac{\gamma}{\kappa\tau^2 RT_*}\left[\left(\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right) + \Delta\zeta_k\left(\boldsymbol{\varpi}_k^+\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}} + \boldsymbol{\varpi}_k^-\frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right)\right]$$
$$-\frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2 RT_*}\Delta\zeta_k\left[\boldsymbol{\varpi}_k^+\left(\boldsymbol{\varpi}_{k+\frac{1}{2}}^+ P_{k+1} + \boldsymbol{\varpi}_{k+\frac{1}{2}}^- P_k\right) + \boldsymbol{\varpi}_k^-\left(\boldsymbol{\varpi}_{k-\frac{1}{2}}^+ P_k + \boldsymbol{\varpi}_{k-\frac{1}{2}}^- P_{k-1}\right)\right]$$

(note that it has been multiplied through by $\Delta\zeta_k$) plus the boundary equations:

$$\left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon\overline{P}^\zeta\right)\right]_{k_0-\frac{1}{2}} = \frac{\gamma}{\kappa\tau^2 RT_*}\left(\frac{P_{k_0}-P_{k_0-1}}{\Delta\zeta_{k_0-\frac{1}{2}}} - \varepsilon\left(\boldsymbol{\varpi}_{k_0-\frac{1}{2}}^+ P_{k_0} + \boldsymbol{\varpi}_{k_0-\frac{1}{2}}^- P_{k_0-1}\right)\right) = -\left(L''_\theta\right)_{k_0-\frac{1}{2}} = -L_B$$

$$\left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa\overline{P}^\zeta\right)\right]_{N+\frac{1}{2}} = \frac{\gamma}{\kappa\tau^2 RT_*}\left(\frac{P_{N+1}-P_N}{\Delta\zeta_{N+\frac{1}{2}}} + \kappa\left(\boldsymbol{\varpi}_{N+\frac{1}{2}}^+ P_{N+1} + \boldsymbol{\varpi}_{N+\frac{1}{2}}^- P_N\right)\right) = -\left(L'''_\theta\right)_{N+\frac{1}{2}}$$

which are used to reduce the number of unknowns from N+2 to N. [**Note:** *here and below we have replace* 1 ***by*** $\boldsymbol{k_0}$ *and introduce* $\boldsymbol{L_B}$ *for reasons that will become clear in* **Appendix 9**]. In effect, we find:

$$P_{k_0-1} = \alpha_T P_{k_0} + C_T L_B$$
$$P_{N+1} = \alpha_S P_N - C_S\left(L'''_\theta\right)_{N+\frac{1}{2}}$$

with

$$\alpha_T = \frac{1/\Delta\zeta_{k_0-\frac{1}{2}} - \varepsilon\boldsymbol{\varpi}_{k_0-\frac{1}{2}}^+}{1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon\boldsymbol{\varpi}_{k_0-\frac{1}{2}}^-}; \qquad C_T = \frac{\kappa\tau^2 RT_*}{\gamma}\frac{1}{1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon\boldsymbol{\varpi}_{k_0-\frac{1}{2}}^-}$$

$$\alpha_S = \frac{1/\Delta\zeta_{N+\frac{1}{2}} - \kappa\boldsymbol{\varpi}_{N+\frac{1}{2}}^-}{1/\Delta\zeta_{N+\frac{1}{2}} + \kappa\boldsymbol{\varpi}_{N+\frac{1}{2}}^+}; \quad C_S = \frac{\kappa\tau^2 RT_*}{\gamma}\frac{1}{1/\Delta\zeta_{N+\frac{1}{2}} + \kappa\boldsymbol{\varpi}_{N+\frac{1}{2}}^+}$$

Therefore we may rewrite the equations for $\left(L_P\right)_{k_0}$ and $\left(L_P\right)_N$ as follows

$$\Delta\zeta_{k_0}\left(L'_P\right)_{k_0} = \Delta\zeta_{k_0}\left(L_P\right)_{k_0} - C''_T\, L_B$$
$$\Delta\zeta_N\left(L'_P\right)_N = \Delta\zeta_N\left(L_P\right)_N + C''_S\left(L'''_\theta\right)_{N+\frac{1}{2}}$$

to get respectively

$$\Delta\zeta_{k_0}\left(L'_P\right)_{k_0} = \Delta\zeta_{k_0}\left(\nabla_\zeta^2 P\right)_{k_0}$$
$$+\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}} - \frac{(1-\alpha_T)P_{k_0}}{\Delta\zeta_{k_0-\frac{1}{2}}} + \frac{\Delta\zeta_{k_0}\varpi^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}\left(P_{k_0+1}-P_{k_0}\right) + \frac{\Delta\zeta_{k_0}\varpi^-_{k_0}}{\Delta\zeta_{k_0-\frac{1}{2}}}(1-\alpha_T)P_{k_0}\right]$$
$$-\frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2 RT_*}\left\{\Delta\zeta_{k_0}\varpi^+_{k_0}\varpi^+_{k_0+\frac{1}{2}}P_{k_0+1} + \Delta\zeta_{k_0}\left[\varpi^+_{k_0}\varpi^-_{k_0+\frac{1}{2}} + \varpi^-_{k_0}\left(\varpi^+_{k_0-\frac{1}{2}} + \varpi^-_{k_0-\frac{1}{2}}\alpha_T\right)\right]P_{k_0}\right\}$$

and

$$\Delta\zeta_N\left(L'_P\right)_N = \Delta\zeta_N\left(\nabla_\zeta^2 P\right)_N + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\frac{(\alpha_S-1)P_N}{\Delta\zeta_{N+\frac{1}{2}}} - \frac{P_N-P_{N-1}}{\Delta\zeta_{N-\frac{1}{2}}} + \frac{\Delta\zeta_N\varpi^+_N}{\Delta\zeta_{N+\frac{1}{2}}}(\alpha_S-1)P_N + \frac{\Delta\zeta_N\varpi^-_N}{\Delta\zeta_{N-\frac{1}{2}}}\left(P_N-P_{N-1}\right)\right)$$
$$-\frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2 RT_*}\left\{\Delta\zeta_N\left[\varpi^+_N\left(\varpi^+_{N+\frac{1}{2}}\alpha_S + \varpi^-_{N+\frac{1}{2}}\right) + \varpi^-_N\varpi^+_{N-\frac{1}{2}}\right]P_N + \Delta\zeta_N\varpi^-_N\varpi^-_{N-\frac{1}{2}}P_{N-1}\right\}$$

with

$$C''_T = \frac{1/\Delta\zeta_{k_0-\frac{1}{2}} - \Delta\zeta_{k_0}\varpi^-_{k_0}\left(1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon(1-\kappa)\varpi^-_{k_0-\frac{1}{2}}\right)}{1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon\varpi^-_{k_0-\frac{1}{2}}}$$

$$C''_S = \frac{1/\Delta\zeta_{N+\frac{1}{2}} + \Delta\zeta_N\varpi^+_N\left(1/\Delta\zeta_{N+\frac{1}{2}} - \varepsilon(1-\kappa)\varpi^+_{N+\frac{1}{2}}\right)}{1/\Delta\zeta_{N+\frac{1}{2}} + \kappa\varpi^+_{N+\frac{1}{2}}}$$

The vertical matrix problem may be decomposed into a combination of a diagonal $\mathbf{P}$ and a set of tri-diagonal matrices, $\mathbf{P}_{\delta\delta}, \mathbf{P}_{\delta\mu} = \mathbf{P}_{\mu\delta}, \mathbf{P}_{\mu\mu}$, representing respectively a double difference, a mean followed by a difference *or* a difference followed by a mean and a double mean as follows:

$$\mathbf{P}\left(L'_P\right) = \mathbf{P}\nabla_\zeta^2 P + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\mathbf{P}_{\delta\delta} + \mathbf{P}_{\delta\mu} - \varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)P$$

After solving the elliptic problem and therefore knowing P_{k_0} to P_N, we calculate P_{k_0-1} and P_{N+1} using the relations:

$$P_{k_0-1} = \alpha_T P_{k_0} + C_T L_B$$
$$P_{N+1} = \alpha_S P_N - C_S\left(L'''_\theta\right)_{N+\frac{1}{2}}$$

The tri-diagonal matrix elements are as follows:

$$\mathbf{P} = \begin{vmatrix} \Delta\zeta_{k_0} & 0 & 0 \\ 0 & \Delta\zeta_k & 0 \\ 0 & 0 & \Delta\zeta_N \end{vmatrix}$$

$$\mathbf{P}_{\delta\delta} = \begin{vmatrix} -\left(\frac{1-\alpha_T}{\Delta\zeta_{k_0-\frac{1}{2}}} + \frac{1}{\Delta\zeta_{k_0+\frac{1}{2}}}\right) & \frac{1}{\Delta\zeta_{k_0+\frac{1}{2}}} & 0 \\ \frac{1}{\Delta\zeta_{k-\frac{1}{2}}} & -\left(\frac{1}{\Delta\zeta_{k-\frac{1}{2}}} + \frac{1}{\Delta\zeta_{k+\frac{1}{2}}}\right) & \frac{1}{\Delta\zeta_{k+\frac{1}{2}}} \\ 0 & \frac{1}{\Delta\zeta_{N-\frac{1}{2}}} & -\left(\frac{1}{\Delta\zeta_{N-\frac{1}{2}}} + \frac{1-\alpha_S}{\Delta\zeta_{N+\frac{1}{2}}}\right) \end{vmatrix}$$

$$\mathbf{P}_{\delta\mu} = \begin{vmatrix} \left(\frac{(1-\alpha_T)\boldsymbol{\varpi}^-_{k_0}}{\Delta\zeta_{k_0-\frac{1}{2}}} - \frac{\boldsymbol{\varpi}^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}\right)\Delta\zeta_{k_0} - \text{"}\boldsymbol{\varepsilon}\,\textbf{NOTOP}\text{"} & \frac{\boldsymbol{\varpi}^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}\Delta\zeta_{k_0} & 0 \\ -\frac{\boldsymbol{\varpi}^-_k}{\Delta\zeta_{k-\frac{1}{2}}}\Delta\zeta_k & \left(\frac{\boldsymbol{\varpi}^-_k}{\Delta\zeta_{k-\frac{1}{2}}} - \frac{\boldsymbol{\varpi}^+_k}{\Delta\zeta_{k+\frac{1}{2}}}\right)\Delta\zeta_k & \frac{\boldsymbol{\varpi}^+_k}{\Delta\zeta_{k+\frac{1}{2}}}\Delta\zeta_k \\ 0 & -\frac{\boldsymbol{\varpi}^-_N}{\Delta\zeta_{N-\frac{1}{2}}}\Delta\zeta_N & \left(\frac{\boldsymbol{\varpi}^-_N}{\Delta\zeta_{N-\frac{1}{2}}} - \frac{(1-\alpha_S)\boldsymbol{\varpi}^+_N}{\Delta\zeta_{N+\frac{1}{2}}}\right)\Delta\zeta_N \end{vmatrix}$$

$$\mathbf{P}_{\mu\mu} = \begin{vmatrix} \left[\boldsymbol{\varpi}^-_{k_0}\left(\boldsymbol{\alpha}_T\boldsymbol{\varpi}^-_{k_0-\frac{1}{2}} + \boldsymbol{\varpi}^+_{k_0-\frac{1}{2}}\right) + \boldsymbol{\varpi}^+_{k_0}\boldsymbol{\varpi}^-_{k_0+\frac{1}{2}}\right]\Delta\zeta_{k_0} & \boldsymbol{\varpi}^+_{k_0}\boldsymbol{\varpi}^+_{k_0+\frac{1}{2}}\Delta\zeta_{k_0} & 0 \\ \boldsymbol{\varpi}^-_k\boldsymbol{\varpi}^-_{k-\frac{1}{2}}\Delta\zeta_k & \left(\boldsymbol{\varpi}^-_k\boldsymbol{\varpi}^+_{k-\frac{1}{2}} + \boldsymbol{\varpi}^+_k\boldsymbol{\varpi}^-_{k+\frac{1}{2}}\right)\Delta\zeta_k & \boldsymbol{\varpi}^+_k\boldsymbol{\varpi}^+_{k+\frac{1}{2}}\Delta\zeta_k \\ 0 & \boldsymbol{\varpi}^-_N\boldsymbol{\varpi}^-_{N-\frac{1}{2}}\Delta\zeta_N & \left[\boldsymbol{\varpi}^-_N\boldsymbol{\varpi}^+_{N-\frac{1}{2}} + \boldsymbol{\varpi}^+_N\left(\boldsymbol{\varpi}^-_{N+\frac{1}{2}} + \boldsymbol{\varpi}^+_{N+\frac{1}{2}}\boldsymbol{\alpha}_S\right)\right]\Delta\zeta_N \end{vmatrix}$$

A term, $-\text{"}\boldsymbol{\varepsilon}\,\textbf{NOTOP}\text{"}$, has been added arbitrarily in the matrix $\mathbf{P}_{\delta\mu}$. For an explanation read **Appendix 13**. Here though this term vanishes.

Appendix 5. How were chosen the averaging operators and note about commutation

Let us consider two variables, *G* and *H*, defined on separate staggered grids as follows:

$$G_{k+\frac{1}{2}} = G(\zeta_{k+\frac{1}{2}}) \quad ; \quad H_k = H(\zeta_k)$$

indicating that *G* is defined on half-levels while *H* is defined on full ones. Only the independent variable ζ could and was defined on both types of levels and thus take the two types of indices. The metric parameter could also sometimes be defined on both types of level, hence two different symbols (*B* on full and B on half levels). To obtain the variables *G* and *H* on their alternative grids, averaging operators α and *a* such that:

$$(\alpha G)_k = \alpha_k G_{k+\frac{1}{2}} + (1-\alpha_k) G_{k-\frac{1}{2}} \quad ; \quad (aH)_{k+\frac{1}{2}} = a_{k+\frac{1}{2}} H_{k+1} + \left(1 - a_{k+\frac{1}{2}}\right) H_k$$

are introduced. In the following discussion, difference operators will be needed and we define them:

$$(\delta G)_k = \frac{G_{k+\frac{1}{2}} - G_{k-\frac{1}{2}}}{\zeta_{k+\frac{1}{2}} - \zeta_{k-\frac{1}{2}}} = \frac{G_{k+\frac{1}{2}} - G_{k-\frac{1}{2}}}{\Delta\zeta_k} \quad ; \quad (\delta H)_{k+\frac{1}{2}} = \frac{H_{k+1} - H_k}{\zeta_{k+1} - \zeta_k} = \frac{H_{k+1} - H_k}{\Delta\zeta_{k+\frac{1}{2}}}$$

Now, let us consider the discretized elliptic equation derived in **section 15** and which we write formally as follows:

$$\nabla_\zeta^2 P_k + \left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta^2 + \alpha\delta + \varepsilon(\alpha\delta - \delta a) - \varepsilon(1-\kappa)\alpha a\right)P\right]_k = (L_P)_k$$

There is a term, $\varepsilon(\alpha\delta - \delta a)$, which was assumed to vanish, which has no analytic equivalent but which vanishes only if the mean and difference operators commute. Let us impose this condition and examine the consequences. We get

$$(\alpha\delta P)_k = \alpha_k (\delta P)_{k+\frac{1}{2}} + (1-\alpha_k)(\delta P)_{k-\frac{1}{2}} = \alpha_k \frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} + (1-\alpha_k)\frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}$$

$$(\delta a P)_k = \frac{(aP)_{k+\frac{1}{2}} - (aP)_{k-\frac{1}{2}}}{\Delta\zeta_k} = \frac{a_{k+\frac{1}{2}} P_{k+1} + \left(1 - a_{k+\frac{1}{2}}\right) P_k - a_{k-\frac{1}{2}} P_k - \left(1 - a_{k-\frac{1}{2}}\right) P_{k-1}}{\Delta\zeta_k}$$

Implying that

$$\frac{\Delta\zeta_k}{\Delta\zeta_{k+\frac{1}{2}}} = \frac{a_{k+\frac{1}{2}}}{\alpha_k}; \qquad \frac{\Delta\zeta_k}{\Delta\zeta_{k-\frac{1}{2}}} = \frac{1 - a_{k-\frac{1}{2}}}{1-\alpha_k}$$

and either

$$\text{(a)} \qquad \frac{\Delta\zeta_{k+1}}{\Delta\zeta_k} = \frac{\alpha_k}{1-\alpha_{k+1}} \frac{1-a_{k+\frac{1}{2}}}{a_{k+\frac{1}{2}}} \quad \text{or} \quad \text{(b)} \; \frac{\Delta\zeta_{k+\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}}} = \frac{\alpha_k}{1-\alpha_k} \frac{1-a_{k-\frac{1}{2}}}{a_{k+\frac{1}{2}}}$$

If the relation between the half and full levels is given, for example if, as we have chosen:

$$\zeta_{k+\frac{1}{2}} = \frac{\zeta_{k+1} + \zeta_k}{2}$$

then we most likely want

$$a_{k+\frac{1}{2}} = \frac{1}{2}$$

From (b) we get

$$\alpha_k = \frac{\Delta\zeta_{k+\frac{1}{2}}}{\Delta\zeta_{k-\frac{1}{2}} + \Delta\zeta_{k+\frac{1}{2}}} = \frac{\Delta\zeta_{k+\frac{1}{2}}}{2\Delta\zeta_k}$$

and thus

$$(\alpha G)_k = \frac{\Delta\zeta_{k+\frac{1}{2}} G_{k+\frac{1}{2}} + \Delta\zeta_{k-\frac{1}{2}} G_{k-\frac{1}{2}}}{2\Delta\zeta_k} \quad ; \quad (aH)_{k+\frac{1}{2}} = \frac{H_{k+1} + H_k}{2}$$

Instead of choosing $a_{k+\frac{1}{2}}$ off-hand as we have done, we might have imposed another condition such as the symmetry of matrix M formed by the product of the matrix obtained from the double averaging operator αa and the diagonal matrix with elements $\Delta\zeta_k$, i.e. if we had imposed that the tri-diagonal matrix M whose elements are

$$\begin{aligned}(\Delta\zeta\alpha a P)_k &= \Delta\zeta_k \alpha_k \left(a_{k+\frac{1}{2}} P_{k+1} + \left(1 - a_{k+\frac{1}{2}}\right) P_k \right) + (1-\alpha_k)\Delta\zeta_k \left(a_{k-\frac{1}{2}} P_k + \left(1 - a_{k-\frac{1}{2}}\right) P_{k-1} \right) \\ &= M_{k+1,k} P_{k+1} + M_{k,k} P_k + M_{k-1,k} P_{k-1}\end{aligned}$$

be symmetric, i.e. setting $M_{k+1,k} = M_{k,k+1}$, i.e. $\Delta\zeta_k \alpha_k a_{k+\frac{1}{2}} = (1-\alpha_{k+1})\Delta\zeta_{k+1}\left(1 - a_{k+\frac{1}{2}}\right)$, i.e.

$$\text{(c)} \qquad \frac{\Delta\zeta_{k+1}}{\Delta\zeta_k} = \frac{\alpha_k}{1-\alpha_{k+1}} \frac{a_{k+\frac{1}{2}}}{1-a_{k+\frac{1}{2}}}$$

Then, combining (c) with (a), we would have again found

$$a_{k+\frac{1}{2}} = \frac{1}{2}$$

In the original formulation of the staggered-grid version of the model, we indeed wanted to obtain symmetric matrices in the vertical (maintaining a property of the regular-grid version of the model) and commutation occurred naturally (only one mean being explicit in the code, the second one occurring only in the elimination process). With the new coordinate we lost the symmetric property due to the presence of a first derivative in the analytic problem. But the requirement that half-levels be exactly in the middle of full levels is good for the accuracy of the hydrostatic relation and the commutation requirement, besides simplifying the code, may serve in improving the conservative properties of the scheme.

So far we have dealt with the difference and average operators away from the boundaries. Let us now look at them near the boundaries. The equations defined on half levels apply to the top and bottom where difference and average operators operate on some variables, namely ϕ' and q. But their values are required at one of the boundaries [$\phi'_S = \phi_S$ at the surface and q_T=0 at the top] while their values at the other can be obtained by numerical integration provided the difference operator leading to them is defined which it has been (it is by construction an off-centered difference though). This is why we consider the top and bottom to be full levels as far as ϕ' and q are concerned, respectively labeled 0 and N+1. The averaging operator then simply selects the corresponding value.

One last item remains to be explained referring to **Figure 1** describing Charney-Phillips grid. In the preceding paragraph, we said that some equations applied to the boundaries. With the presence of ϕ' and q, all required variables were also apparently defined there but, if so, the difference operators were off-centered and therefore only first order. Centered differences are recovered if we displace the thermodynamic and vertical momentum equations as well as the variables T and w to the middle of the half-layers nearing the boundaries [to levels ¾ and N+ ¼ as shown in the figure]. This is what we have done. We believe this is beneficial for temperature in particular which is shifted from the surface to a better place from the physical as well as numerical point of view. To better assess what we have done, here is a formal representation of the three linear equations affected by the change for the bottom (a similar change occurs at the top):

$$\left(L'_w\right)_{N+\frac{1}{4}} = \frac{w_{N+\frac{1}{4}}}{\tau} - g\frac{q_{N+\frac{1}{2}} - q_N}{\Delta\zeta_{N+\frac{1}{4}}}$$

$$\left(L'_\theta\right)_{N+\frac{1}{4}} = \frac{1}{\tau}\frac{q_{N+\frac{1}{2}} - q_N}{\Delta\zeta_{N+\frac{1}{4}}} - \frac{1}{\tau R T_*}\frac{P_{N+\frac{1}{2}} - P_N}{\Delta\zeta_{N+\frac{1}{4}}} - \kappa\left[\dot{\zeta} + \frac{Bs+q}{\tau}\right]_{N+\frac{1}{2}}$$

$$\left(L_\phi\right)_{N+\frac{1}{2}} = \frac{P_{N+\frac{1}{2}}}{\tau} - RT_*\dot{\zeta}_{N+\frac{1}{2}} - gw_{N+\frac{1}{4}}$$

In the thermodynamic equation defined at level N+¼ the term in brackets remains evaluated at the boundary, level N+½. In the geopotential equation defined at level N+½, w is taken at level N+¼. This can be interpreted, in the first case as an interpolation, in the second case as an extrapolation, constant in both cases [$f_{N+\frac{1}{4}} = \alpha f_{N-\frac{1}{2}} + (1-\alpha) f_{N+\frac{1}{2}}$ with α=0].

Appendix 6. The Dynamic Core Code and vertical discretization: *A brief description*

The dynamic core code is essentially organized as follows:

set_zeta, set_dync, set_oprz, preverln: *compute* constants *and* parameters *of the vertical discretization*

***Timestep** Loop*

tstpdyn: *performs a* dynamical time step *calling* **rhs, adw, pre, nli, sol, bac**

- **rhs:** compute the 6 basic Right-Hand-Side terms: $\mathbf{R}_h$, R_w, R_θ, R_C, R_ϕ

(**section 14**)

$$\mathbf{R}_h = \frac{\mathbf{V}_h}{\tau} \qquad -\beta\left(f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{h\zeta}\nabla_\zeta(Bs+q) + \left(1+\overline{\mu}^{h\zeta}\right)\nabla_\zeta\phi'\right)$$

$$R_w = \frac{w}{\tau} \qquad -\beta(-g\mu)$$

$$R_\theta = \frac{1}{\tau}\left[\ln\left(\frac{T}{T_*}\right) - \kappa\overline{(Bs+q)}^\zeta\right] \qquad -\beta\left(-\kappa\dot{\zeta}\right)$$

$$R_C = \frac{1}{\tau}\left[Bs + \ln\left(1+\delta_\zeta\overline{B}^\zeta s\right)\right] \qquad -\beta\left(\nabla_\zeta\cdot\mathbf{V}_h + \delta_\zeta\dot{\zeta} + \overline{\dot{\zeta}}^\zeta\right)$$

$$R_\phi = \frac{\overline{\phi'}^\zeta}{\tau} \qquad -\beta\left(-RT_*\dot{\zeta} - gw\right)$$

***Departure** Outer-Loop*

- **adw:** **adw_pos**: Compute the next estimate of the departure points.
  adw_int: Evaluate Right-Hand-Side terms at departure points.

- **pre:** combine R_θ, R_w, R_ϕ into R''_θ, R''_ϕ, combine $\mathbf{R}_h, R_w, R_C$ into R''_C and finally $R''_C, R''_\theta, R''_\phi$ into R_P:

(**section 16**)

$$\frac{\gamma}{\kappa\tau}\left(R_\theta + \frac{\varepsilon\tau}{H_*}R_w + \frac{\varepsilon}{RT_*}R_\phi\right) \equiv R''_\theta$$

$$\frac{\gamma}{\kappa\tau}\left(R_\theta + \frac{\varepsilon\tau}{H_*}R_w - \frac{\kappa}{RT_*}R_\phi\right) \equiv R''_\phi$$

$$\nabla_\zeta \cdot \mathbf{R}_h - \frac{1}{\tau}\left(R_C - \frac{\varepsilon\tau}{H_*}\overline{R_w}^{\zeta} \right) \equiv R''_C$$
$$R''_C - \left(\delta_\zeta R''_\theta + \overline{R''_\theta}^{\zeta}\right) - \varepsilon \overline{R''_\phi}^{\zeta} \equiv R_P$$

The final version of the Right-Hand sides are: $\mathbf{R}_h$, R_w, R''_θ, R''_ϕ, R_P

Non-linear *Inner_Loop*

- **nli:** compute non-linear Left-Hand sides: $\mathbf{N}_h, N'_w, N'_T, N_C, N_\phi$

(section 12)

$$\mathbf{N}_h = f\mathbf{k} \times \mathbf{V}_h + R\overline{T'}^{h\zeta}\nabla_\zeta (Bs + q) + \overline{\mu}^{h\zeta}\nabla_\zeta \phi'$$
$$N'_w = -g(\mu - \delta_\zeta q)$$
$$N'_\theta = \frac{1}{\tau}\left[\ln\left(\frac{T}{T_*}\right) + \frac{\delta_\zeta(\phi' + RT_* Bs)}{RT_*} \right]$$
$$N_C = \frac{1}{\tau}\left[Bs + \ln\left(1 + \delta_\zeta \overline{B}^{\zeta} s\right) - \overline{B}^{\zeta\zeta} s - \delta_\zeta \overline{B}^{\zeta} s \right]$$
$$N_\phi = 0$$

and combine them into $N''_T, N''_\phi, N''_C, N_P$

(section 16)

$$\frac{\gamma}{\kappa\tau}\left(N'_\theta + \frac{\varepsilon\tau}{H_*}N'_w + \frac{\varepsilon}{RT_*}N_\phi \right) \equiv N''_\theta$$
$$\frac{\gamma}{\kappa\tau}\left(N'_\theta + \frac{\varepsilon\tau}{H_*}N'_w - \frac{\kappa}{RT_*}N_\phi \right) \equiv N''_\phi$$
$$\nabla_\zeta \cdot \mathbf{N}_h - \frac{1}{\tau}\left(N_C - \frac{\varepsilon\tau}{H_*}\overline{N'_w}^{\zeta} \right) \equiv N''_C$$
$$N''_C - \left(\delta_\zeta N''_\theta + \overline{N''_\theta}^{\zeta}\right) - \varepsilon \overline{N''_\phi}^{\zeta} \equiv N_P$$

and obtain final Right-Hand Side of the Elliptic Problem $L_P = R_P - N_P$, including modifications imposed by boundary conditions $(L'_P)_{k_0}$ and $(L'_P)_N$.

(appendix 4b)

- **sol:** solve the Elliptic Problem

(**section 15 & appendices 4a and 4b**)

$$\mathbf{P}(L'_P) = \mathbf{P}\nabla^2_\zeta P + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\mathbf{P}_{\delta\delta} + \mathbf{P}_{\delta\mu} - \varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)P$$

- **bac:** back substitution: compute variables for next iteration/time step

(**section 17**)

$$\mathbf{V}_h: \quad \frac{\mathbf{V}_h}{\tau} = \left[\mathbf{R}_h - \mathbf{N}_h - \nabla_\zeta P\right]$$

$$w: \quad \frac{w}{\tau H_*} = \left[R''_\phi - N''_\phi + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa\overline{P}^\zeta\right)\right]$$

$$q: \quad \delta_\zeta q + \overline{q}^\zeta = -\frac{\varepsilon\tau^2}{H_*}\left[R_w - N'_w - \frac{w}{\tau}\right]; \qquad q_T = 0$$

$$s: \quad s = \frac{P_S - \phi_S}{RT_*} - q_S$$

$$\dot\zeta: \quad \frac{\dot\zeta}{\tau} = -\left[R''_\theta - N''_\theta + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon\overline{P}^\zeta\right)\right] - \frac{\overline{(Bs+q)}^\zeta}{\tau^2}; \quad \dot\zeta_T = \dot\zeta_S = 0$$

$$\phi': \quad \phi' = P - RT_*(q + Bs)$$

$$\mu: \quad 1 + \mu = e^{\overline{q}^\zeta}\left[1 + \frac{\delta_\zeta q}{\delta_\zeta(\zeta + Bs)}\right]$$

$$T: \quad \frac{T}{T_*} = e^{\overline{q}^\zeta}\left[1 - \frac{\delta_\zeta(\phi'/RT_* + Bs)}{\delta_\zeta(\zeta + Bs)}\right]$$

end inner loop

end outer loop

end timestep loop

N.B. It is sometimes necessary to be aware of the two horizontal averages applied on T or T' and μ in $\mathbf{R}_h$ and $\mathbf{N}_h$. Hence the above and following **bar *h*** indications:

$$\mathbf{R}_h = \frac{\mathbf{V}_h}{\tau} - \beta\left(f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{h\zeta}\nabla_\zeta(Bs+q) + \left(1+\overline{\mu}^{h\zeta}\right)\nabla_\zeta\phi'\right)$$

$$\mathbf{N}_h = \qquad f\mathbf{k}\times\mathbf{V}_h + R\overline{T}^{h\zeta}\nabla_\zeta(Bs+q) + \overline{\mu}^{h\zeta}\nabla_\zeta\phi'$$

More details in **Appendix 10**.

Appendix 7. The hydrostatic option

We start with the final form of the equations given in **section 5**:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\mu)\nabla_\zeta\phi' = 0$$

$$\frac{dw}{dt} - g\mu = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa(Bs+q)\right] - \kappa\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1+\frac{\partial B}{\partial\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \left(\frac{\partial}{\partial\zeta}+1\right)\dot{\zeta} = 0$$

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$1+\mu - e^q\left[1+\frac{\partial q}{\partial(\zeta+Bs)}+1\right] = 0$$

$$\frac{T}{T_*} - e^q\left[1-\frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta+Bs)}\right] = 0$$

The hydrostatic approximation may be considered to consist in neglecting non-hydrostatic pressure effects, therefore assuming q=0. Then μ=0 also and the vertical acceleration *dw/dt* is neglected. In fact, the vertical motion *w* becomes irrelevant. Neither the vertical momentum nor the geopotential tendency equations are required in the solution system although we may still solve the geopotential tendency equation to diagnose *w*. Therefore, we only need to solve:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta\phi' = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa Bs\right] - \kappa\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1+\frac{\partial B}{\partial\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \left(\frac{\partial}{\partial\zeta}+1\right)\dot{\zeta} = 0$$

$$\frac{T}{T_*} - 1 + \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta+Bs)} = 0$$

All the terms involving the prognostic vertical momentum and diagnostic μ equations which were not already equal to zero are set to vanish: F_w, G_w, L_w, N_w, R_w, L_μ, N_μ, L'_w, N'_w . The parameter ε=0, hence γ=1. In the code, set the switch Schm_hydro_L=.T. .

Appendix 8. The autobarotropic model (to be modified to work with truncated top boundary)

We build an auto-barotropic model (Dutton, *The Ceaseless Wind*, pp 186-7) from the three-dimensional code of GEM in order to simulate a barotropic model. We do that in

i) eliminating the physical effects,
ii) making the hydrostatic hypothesis,
iii) introducing a key $\delta_{autobarot}=0$ to eliminate the pressure tendency $d(Bs)/dt$ in both the thermodynamic and continuity equations,
iv) initializing with barotropic conditions :

$$\mathbf{V}_h \neq \mathbf{V}_h(\zeta);\ T = T_* = const;\ \dot{\zeta} = 0;\ \phi' + RT_* Bs = \phi'_T = \phi_S + RT_* s\ ,$$

conditions which will be maintained afterwards, hence the name autobarotropic model.

From the complete equations:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\mu)\nabla_\zeta\phi' = \mathbf{F}_h$$

$$\delta_H \frac{dw}{dt} - g\mu = F_w$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}(Bs+q) + \dot{\zeta}\right] = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial \zeta}s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$1 + \mu - e^q\left[1 + \frac{\partial q}{\partial(\zeta + Bs)}\right] = 0$$

$$\frac{T}{T_*} - e^q\left[1 - \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta + Bs)}\right] = 0$$

with B defined simply as $B = \dfrac{\zeta - \zeta_T}{\zeta_S - \zeta_T}$, we eliminate sources and sinks of momentum and heat and we make the hydrostatic approximation, reducing the number of equations and variables to (see **Appendix 7**):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta \phi' = 0$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}(Bs) + \dot{\zeta}\right] = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial \zeta}s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

$$\frac{T}{T_*} - 1 + \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta + Bs)} = 0$$

Considering barotropic initial conditions ($\mathbf{V}_h \neq \mathbf{V}_h(\zeta)$; $T = T_* = const$; $\dot{\zeta} = 0$), we derive from the hydrostatic equation that P is uniform in the vertical:

$$P = \phi' + RT_* Bs = \phi'_T = \phi_S + RT_* s \neq P(\zeta)$$

and we note that

$$s = \frac{\phi'_T - \phi_S}{RT_*}$$

Indeed, $\frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta + Bs)} = \frac{\partial P}{\partial \zeta}\frac{\partial \zeta}{\partial(\zeta + Bs)} = 0$, hence $\frac{\partial P}{\partial \zeta} = 0$.

We therefore have in the momentum equation:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + \nabla_\zeta \phi_T = 0$$

and since $P = \phi_T \neq P(\zeta)$, then $\mathbf{V}_h$ stays $\mathbf{V}_h \neq \mathbf{V}_h(\zeta)$.

Now, even though $\dot{\zeta} = 0$ and $T = T_* = const$ initially, temperature will change since the thermodynamic equation still says:

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) = \kappa \frac{d}{dt}(Bs) \neq 0$$

However if we write

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa \delta_{autobarot} \frac{d}{dt}(Bs) = 0$$

making $\delta_{autobarot} = 0$, then $\frac{d}{dt}\ln\left(\frac{T}{T_*}\right)$=0 et T will remain constant and equal to T_*. Similarly, in the continuity equation, $\dot{\zeta} = 0$ initially and introducing $\delta_{autobarot} = 0$:

$$\frac{d}{dt}\left[\delta_{autobarot} Bs + \ln\left(1 + \frac{\partial B}{\partial \zeta} s\right)\right] + \nabla_{\zeta} \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

we get

$$\frac{d}{dt}\ln\left(1 + \frac{\partial B}{\partial \zeta} s\right) + \nabla_{\zeta} \cdot \mathbf{V}_h = 0$$

$$\frac{d}{dt}\ln\left(1 + \frac{\phi'_T - \phi_S}{RT_*(\zeta_S - \zeta_T)}\right) + \nabla_{\zeta} \cdot \mathbf{V}_h = 0$$

$$\frac{d}{dt}\ln\left(\frac{\phi_{*T} + \phi'_T - \phi_S}{\phi_{*T}}\right) + \nabla_{\zeta} \cdot \mathbf{V}_h = 0$$

$$\frac{d}{dt}\ln(\phi_T - \phi_S) + \nabla_{\zeta} \cdot \mathbf{V}_h = 0$$

And this relation is invariant in the vertical, hence $\dot{\zeta} \neq \dot{\zeta}(\zeta)$ and $\dot{\zeta}$ =0 is maintained Hence, the model equations:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_{\zeta} Bs + \nabla_{\zeta}\phi' = 0$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\delta_{autobarot}\frac{d}{dt}(Bs) + \dot{\zeta}\right] = 0$$

$$\frac{d}{dt}\left[\delta_{autobarot} Bs + \ln\left(1 + \frac{\partial B}{\partial \zeta} s\right)\right] + \nabla_{\zeta} \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

$$\frac{T}{T_*} - \left[1 - \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta + Bs)}\right] = 0$$

with a vertical structure (many levels, at least 3: e.g. hyb = 0.583333, 0.75, 0.9166666 with p_{top}=50000., to satisfy the operations), but starting with barotropic conditions, simulates the barotropic equations:

$$\boxed{\begin{aligned} \frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + \nabla_{\zeta}\phi_T = 0 \\ \frac{d}{dt}\ln(\phi_T - \phi_S) + \nabla_{\zeta} \cdot \mathbf{V}_h = 0 \end{aligned}}$$

It is autobarotropic.

Appendix 9. *Open top* boundary conditions

The goal is to develop an open boundary condition at the top, i.e. a condition with $X_{OT} = \left(\dot{\zeta} + \frac{Bs+q}{\tau}\right)_{OT} \neq 0$, not only $\dot{\zeta}_{OT} \neq 0$ but also $B_{OT} \neq 0$ (the top no more being necessarily a hydrostatic pressure level) and $q_{OT} \neq 0$ (in the non-hydrostatic case).

First, let us deal with the linear system (**Appendix 4a**):

$$(\mathbf{L}_h)_k = \frac{\mathbf{V}_{h\,k}}{\tau} + \nabla_{\zeta} P_k$$

$$(L'_w)_{k+\frac{1}{2}} = \frac{w_{k+\frac{1}{2}}}{\tau} - g\left(\frac{q_{k+1}-q_k}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi^+_{k+\frac{1}{2}} q_{k+1} + \varpi^-_{k+\frac{1}{2}} q_k\right)$$

$$(L_\theta)_{k+\frac{1}{2}} = \frac{1}{\tau}\frac{T'_{k+\frac{1}{2}}}{T_*} - \kappa X_{k+\frac{1}{2}}$$

$$(L_C)_k = -\frac{1}{\tau}\left[\varpi^+_k\left(\frac{q_{k+1}-q_k}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi^+_{k+\frac{1}{2}} q_{k+1} + \varpi^-_{k+\frac{1}{2}} q_k\right) + \varpi^-_k\left(\frac{q_k - q_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}} + \varpi^+_{k-\frac{1}{2}} q_k + \varpi^-_{k-\frac{1}{2}} q_{k-1}\right)\right]$$
$$+ \nabla_{\zeta}\cdot\mathbf{V}_{h\,k} + \frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\zeta_k} + \varpi^+_k X_{k+\frac{1}{2}} + \varpi^-_k X_{k-\frac{1}{2}}$$

$$(L_\phi)_{k+\frac{1}{2}} = \frac{1}{\tau}\left(\varpi^+_{k+\frac{1}{2}} P_{k+1} + \varpi^-_{k=\frac{1}{2}} P_k\right) - RT_* X_{k+\frac{1}{2}} - g w_{k+\frac{1}{2}}$$

We know we can combine these equations into a set of only N equations in the vertical for N+2 unknowns P_k (k=k_0,N):

$$(L_P)_k = \nabla^2_{\zeta} P_k + \frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{1}{\Delta\zeta_k}\left(\frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right) + \varpi^+_k \frac{P_{k+1}-P_k}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi^-_k \frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right]$$
$$- \frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2 RT_*}\left[\varpi^+_k\left(\varpi^+_{k+\frac{1}{2}} P_{k+1} + \varpi^-_{k+\frac{1}{2}} P_k\right) + \varpi^-_k\left(\varpi^+_{k-\frac{1}{2}} P_k + \varpi^-_{k-\frac{1}{2}} P_{k-1}\right)\right]$$

and therefore requiring two additional equations (top and bottom boundary conditions) for its solution. As we have seen (**Appendix 4b**), a closed top boundary condition occurring at k_0=1 ($\dot{\zeta}_T = 0; \mathrm{B}_T = 0; q_T = 0$) is satisfied by using

$$(L''_\theta)_{k_0-\frac{1}{2}} + \frac{X_T}{\tau} = -\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_{k_0} - P_{k_0-1}}{\Delta\zeta_{k_0-\frac{1}{2}}} - \varepsilon\left(\varpi^+_{k_0-\frac{1}{2}} P_k + \varpi^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right]$$

to obtain a boundary condition in terms of X (*generalized vertical motion* $\dot{\zeta}$) since $X_T = \left[\dot{\boldsymbol{\zeta}} + (\mathrm{B}s + q)/\boldsymbol{\tau}\right]_T = 0$. For an open top occurring at $k_0 \neq 1$, we have none of the above conditions ($\dot{\boldsymbol{\zeta}}_{OT} \neq 0; \mathrm{B}_{OT} \neq 0; q_{OT} \neq 0$). Another relation must be found. There are two possibilities:

- (i) using L''_{ϕ} to obtain a boundary condition in terms of *vertical motion w,* specifying w_{OpenT}:

$$\left(L''_{\phi}\right)_{k_0-\frac{1}{2}} - \frac{g w_{OpenT}}{\tau R T_*} = -\frac{\gamma}{\kappa\tau^2 R T_*}\left[\frac{P_{k_0} - P_{k_0-1}}{\Delta\zeta_{k_0-\frac{1}{2}}} + \kappa\left(\varpi^+_{k_0-\frac{1}{2}} P_{k_0} + \varpi^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right]$$

- (ii) combining L''_{θ} with L_{θ} as follows

$$\left(L''_{\theta}\right)_{k_0-\frac{1}{2}} - \frac{1}{\kappa\tau}\left(L_{\theta}\right)_{k_0-\frac{1}{2}} + \frac{1}{\kappa\tau^2}\frac{T'_{OpenT}}{T_*} = -\frac{\gamma}{\kappa\tau^2 R T_*}\left[\frac{P_{k_0} - P_{k_0-1}}{\Delta\zeta_{k_0-\frac{1}{2}}} - \varepsilon\left(\varpi^+_{k_0-\frac{1}{2}} P_{k_0} + \varpi^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right]$$

to obtain a boundary condition in terms of *temperature T*, specifying T'_{OpenT}:

$$-\frac{\gamma}{\kappa\tau^2 R T_*}\left(\frac{P_{k_0} - P_{k_0-1}}{\Delta\zeta_{k_0-\frac{1}{2}}} - \varepsilon\left(\varpi^+_{k_0-\frac{1}{2}} P_{k_0} + \varpi^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right) = L_B = R_B - N_B + \frac{T'_{OpenT}}{\kappa\tau^2 T_*}$$

Although vertical motion *w* seems the logical choice, there are two big objections: first, it is well known that vertical motion can be quite noisy and it could be difficult to get a suitably balanced field; second, in the hydrostatic case, *w* is not even a prognostic variable of the model.

The open top case (ii) in fact leads to equations for P_{k_0-1}, ($k_0 \neq 1$) formally identical to the closed top case with k_0=1. In effect, we write

$$P_{k_0-1} = \alpha_{OpenT} P_{k_0} + C_{OpenT} L_B$$

with

$$\alpha_{OpenT} = \frac{1/\Delta\zeta_{k_0-\frac{1}{2}} - \varepsilon\varpi^+_{k_0-\frac{1}{2}}}{1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon\varpi^-_{k_0-\frac{1}{2}}}; \qquad C_{OpenT} = \frac{\kappa\tau^2 R T_*}{\gamma}\frac{1}{1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon\varpi^-_{k_0-\frac{1}{2}}}$$

Therefore

$$\Delta\zeta_{k_0}\left(L_P\right)_{k_0} - C''_{OpenT}\,L_B = \Delta\zeta_{k_0}\left(\nabla_\zeta^2 P\right)_{k_0}$$

$$+\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}} - \frac{(1-\alpha_T)P_{k_0}}{\Delta\zeta_{k_0-\frac{1}{2}}} + \frac{\Delta\zeta_{k_0}\varpi_{k_0}^{+}}{\Delta\zeta_{k_0-\frac{1}{2}}}\left(P_{k_0+1}-P_{k_0}\right) + \frac{\Delta\zeta_{k_0}\varpi_{k}^{-}}{\Delta\zeta_{k_0-\frac{1}{2}}}(1-\alpha_T)P_{k_0}\right]$$

$$-\frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2 RT_*}\Delta\zeta_{k_0}\left\{\varpi_{k_0}^{+}\varpi_{k_0+\frac{1}{2}}^{+}P_{k_0+1} + \left[\varpi_{k_0}^{+}\varpi_{k_0+\frac{1}{2}}^{-} + \varpi_{k_0}^{-}\left(\varpi_{k_0-\frac{1}{2}}^{+} + \varpi_{k_0-\frac{1}{2}}^{-}\alpha_T\right)\right]P_{k_0}\right\}$$

with

$$C''_{OpenT} = \frac{1/\Delta\zeta_{k_0-\frac{1}{2}} - \Delta\zeta_{k_0}\varpi_{k_0}^{-}\left(1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon(1-\kappa)\varpi_{k_0-\frac{1}{2}}^{-}\right)}{1/\Delta\zeta_{k_0-\frac{1}{2}} + \varepsilon\varpi_{k_0-\frac{1}{2}}^{-}}$$

All of this is trivial then, except for the calculation of the right-hand sides corresponding to L_B, i.e. R_B and N_B:

$$R_B = \left(R''_\theta\right)_{k_0-\frac{1}{2}} - \frac{1}{\kappa\tau}\left(R_\theta\right)_{k_0-\frac{1}{2}}$$

$$N_B = \left(N''_\theta\right)_{k_0-\frac{1}{2}} - \frac{1}{\kappa\tau}\left(N_\theta\right)_{k_0-\frac{1}{2}}$$

More explicitly for N_B,

$$N_B = \left(N''_\theta\right)_{k_0-\frac{1}{2}} - \frac{1}{\kappa\tau^2}\left[\ln\left(\frac{T_{k_0-\frac{1}{2}}}{T_*}\right) - \frac{T'_{k_0-\frac{1}{2}}}{T_*}\right]$$

In the non-hydrostatic case, another condition is needed, namely the true pressure at the top, p_{OpenT}, from which we may calculate

$$q_{OpenT} = \ln(p_{OpenT}/\pi_{OpenT}) = \ln p_{OpenT} - \left(\zeta_{OpenT} + \mathrm{B}_{OpenT}s\right) = \ln p_{OpenT} - \left(\zeta_{k_0-\frac{1}{2}} + \mathrm{B}_{k_0-\frac{1}{2}}s\right)$$

Appendix 10. Aspects of horizontal discretization (removing *a* from *U* and *V* in code)

First of all, note that by 'horizontal' is meant a model 'quasi-spherical' constant ζ surface. In the horizontal then, the equations in spherical coordinates are discretized on an Arakawa C grid, with the wind image components $U_{i+\frac{1}{2},j}\ (i=0,N_i,\ j=1,N_j)$ and $V_{i,j+\frac{1}{2}}\ (i=1,N_i,\ j=0,N_j)$ staggered with respect to all the other variables $w_{i,j}, T_{i,j}, (\dot{\zeta}_{i,j}, s_{i,j}), \phi'_{i,j}, \mu_{i,j}\ (i=1,N_i,\ j=1,N_j)$, an Arakawa C grid with the *U* points with indices *i*=0 and $i=N_i$ coinciding by symmetry and with the *V* points with indices *j*=0 and $j=N_j$ respectively landing on the south and north pole and therefore vanishing. Looking at the equations (**section 5**), we find that only three equations require attention: the two *horizontal momentum* equations

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\mu)\nabla_\zeta\phi' = 0$$

with

$$\mathbf{V}_h = u\hat{\lambda} + v\hat{\theta} = \frac{1}{\cos\theta}\left[U\hat{\lambda} + V\hat{\theta}\right]$$

defined in terms of its longitudinal component *u* in the direction $\hat{\lambda}$ and its latitudinal component *v* in the direction $\hat{\theta}$, or the so-called corresponding wind images *U* and *V,* and with the gradient operator given by:

$$\nabla_\zeta = \frac{\hat{\lambda}}{a\cos\theta}\frac{\partial}{\partial\lambda} + \frac{\hat{\theta}}{a}\frac{\partial}{\partial\theta} = \frac{1}{\cos\theta}\left[\hat{\lambda}\frac{\partial}{\partial X} + \hat{\theta}\frac{\partial}{\partial Y}\right]$$

with $dX=ad\lambda$ and $dY=ad\theta/\cos\theta=a\sin\theta/\cos^2\theta$, and the *continuity* equation written as follows:

$$\frac{d}{dt}\ln\left(\pi\frac{\partial\ln\pi}{\partial\zeta}\right) + D + \frac{\partial\dot{\zeta}}{\partial\zeta} = 0$$

where

$$D = \nabla_\zeta\cdot\mathbf{V}_h = \frac{1}{a\cos\theta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\theta)}{\partial\theta}\right] = \frac{1}{\cos^2\theta}\left[\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right]$$

is the horizontal divergence discretized very simply as follows:

$$D_{ij} = \frac{1}{\cos^2\theta_j}\left[(\delta_X U)_{ij} + (\delta_Y V)_{ij}\right]$$

$$(\delta_X U)_{ij} = \frac{U_{i+\frac{1}{2}j} - U_{i-\frac{1}{2}j}}{\Delta X_i}; \qquad (\delta_Y V)_{ij} = \frac{V_{ij+\frac{1}{2}} - V_{ij-\frac{1}{2}}}{\Delta Y_j}$$

Note that in a global integration we have periodicity in the λ-(X-)direction, so $U_{\frac{1}{2}j} = U_{N_i+\frac{1}{2}j}$, and, since V vanishes at both poles, $V_{i\frac{1}{2}} = V_{iN+\frac{1}{2}} = 0$, the problem is closed in the horizontal.

The 'horizontal' vector momentum equation is modified to be solved as a three-dimensional vector equation in Cartesian coordinates subject however to the constraint that the wind keeps parallel to the earth's surface (Côté, MWR 1988):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_{\zeta}(Bs+q) + (1+\mu)\nabla_{\zeta}\phi' + m\mathbf{r} = 0$$

The constraint, $m\mathbf{r}$, where $\mathbf{r}$ is the earth's radius and m a Lagrange multiplier, acts as a supplementary force normal to the surface. We then introduce the semi-Lagrangian implicit discretization (**section 8**) directly on the vector equation:

$$\frac{\mathbf{V}_h^A - \mathbf{V}_h^D}{\Delta t} + b^A\left(\mathbf{G}_h^A + m\mathbf{r}^A\right) + \left(1-b^A\right)\left(\mathbf{G}_h^D + m\mathbf{r}^D\right) = 0$$

$$\frac{\mathbf{V}_h^A}{\tau} + \mathbf{G}_h^A + m\mathbf{c} = \frac{\mathbf{V}_h^D}{\tau} - \beta\mathbf{G}_h^D \equiv \mathbf{R}_h^D$$

$$\frac{\mathbf{V}_h^A}{\tau} + \mathbf{G}_h^A = \mathbf{R}_h^D - m\mathbf{c} \equiv \mathbf{R}_h^C$$

with $\mathbf{c} = \mathbf{r}^A + \beta\,\mathbf{r}^D$. Multiplying through scalarly by $\mathbf{r}^A$

$$\mathbf{r}^A \cdot \left[\frac{\mathbf{V}_h^A}{\tau} + \mathbf{G}_h^A + m\mathbf{c}\right] = \mathbf{r}^A \cdot \mathbf{R}_h^D$$

gives $m = \dfrac{\mathbf{r}^A \cdot \mathbf{R}_h^D}{\mathbf{r}^A \cdot \mathbf{c}}$, since by construction $\mathbf{r}^A$ is $\perp$ to both $\mathbf{V}_h^A$ and $\mathbf{G}_h^A$. In Cartesian coordinates

$$m = \frac{x^A R_x^D + y^A R_y^D + z^A R_z^D}{x^A c_x + y^A c_y + z^A c_z}$$

Therefore the metric correction to be applied to $\mathbf{R}_h^D$, in order for the result to remain on the sphere, is:

$$\begin{Bmatrix} R_x^C \\ R_y^C \\ R_z^C \end{Bmatrix} = \begin{Bmatrix} R_x^D - m\mathrm{c}_\mathrm{x} \\ R_y^D - m\mathrm{c}_\mathrm{y} \\ R_z^D - m\mathrm{c}_\mathrm{z} \end{Bmatrix} = \begin{Bmatrix} R_x^D - m\left(x^A + \beta\, x^D\right) \\ R_y^D - m\left(y^A + \beta\, y^D\right) \\ R_z^D - m\left(z^A + \beta\, z^D\right) \end{Bmatrix}$$

However, $\mathbf{R}_h$ is given in spherical coordinates in terms of wind images:

$$\mathbf{R}_h = \frac{1}{\cos\theta}\left[R_U\,\hat{\boldsymbol{\lambda}} + R_V\,\hat{\boldsymbol{\theta}}\right]$$

with

$$R_U = \frac{U}{\boldsymbol{\tau}} - \boldsymbol{\beta}\left[-fV + RT\frac{\partial(Bs+q)}{\partial X} + (1+\boldsymbol{\mu})\frac{\partial \boldsymbol{\phi}'}{\partial X} \right]$$
$$R_V = \frac{V}{\boldsymbol{\tau}} - \boldsymbol{\beta}\left[+fU + RT\frac{\partial(Bs+q)}{\partial Y} + (1+\boldsymbol{\mu})\frac{\partial \boldsymbol{\phi}'}{\partial Y} \right]$$

To obtain the Cartesian coordinates (R_x^D, R_y^D, R_z^D) of $\mathbf{R}_h^D$ from its spherical coordinates (R_U^D, R_V^D), we apply the coordinate transformation law at the departure point:

$$\begin{Bmatrix} R_x^D \\ R_y^D \\ R_z^D \end{Bmatrix} = \begin{Bmatrix} -\sin\lambda^D & -\sin\theta^D \cos\lambda^D & \cos\theta^D \cos\lambda^D \\ \cos\lambda^D & -\sin\theta^D \sin\lambda^D & \cos\theta^D \sin\lambda^D \\ 0 & \cos\theta^D & \sin\theta^D \end{Bmatrix} \begin{Bmatrix} R_U^D / \cos\theta^D \\ R_V^D / \cos\theta^D \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\sin\lambda^D / \cos\theta^D & -\tan\theta^D \cos\lambda^D \\ \cos\lambda^D / \cos\theta^D & -\tan\theta^D \sin\lambda^D \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} R_U^D \\ R_V^D \end{Bmatrix}$$

Hence

$$\begin{Bmatrix} R_x^D \\ R_y^D \\ R_z^D \end{Bmatrix} = \begin{Bmatrix} -y^D / \cos^2\theta^D & -x^D z^D / \cos^2\theta^D \\ x^D / \cos^2\theta^D & -y^D z^D / \cos^2\theta^D \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} R_U^D \\ R_V^D \end{Bmatrix}$$

using

$$\begin{Bmatrix} x^D \\ y^D \\ z^D \end{Bmatrix} = \begin{Bmatrix} -\sin\lambda^D & -\sin\theta^D \cos\lambda^D & \cos\theta^D \cos\lambda^D \\ \cos\lambda^D & -\sin\theta^D \sin\lambda^D & \cos\theta^D \sin\lambda^D \\ 0 & \cos\theta^D & \sin\theta^D \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \cos\theta^D \cos\lambda^D \\ \cos\theta^D \sin\lambda^D \\ \sin\theta^D \end{Bmatrix}$$

Finally, to obtain the spherical coordinates (R_U^C, R_V^C) of $\mathbf{R}_h^C$, from its Cartesian coordinates (R_x^C, R_y^C, R_z^C), we apply the inverse transformation at the arrival point:

$$\begin{Bmatrix} R_U^C / \cos\theta^A \\ R_V^C / \cos\theta^A \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\sin\lambda^A & \cos\lambda^A & 0 \\ -\sin\theta^A \cos\lambda^A & -\sin\theta^A \sin\lambda^A & \cos\theta^A \\ \cos\theta^A \cos\lambda^A & \cos\theta^A \sin\lambda^A & \sin\theta^A \end{Bmatrix} \begin{Bmatrix} R_x^C \\ R_y^C \\ R_z^C \end{Bmatrix}$$

hence

$$\begin{Bmatrix} R_U^C \\ R_V^C \\ 0 \end{Bmatrix} = \begin{Bmatrix} -y^A & x^A & 0 \\ -z^A x^A & -z^A y^A & \cos^2\theta^A \\ x^A & y^A & z^A \end{Bmatrix} \begin{Bmatrix} R_x^C \\ R_y^C \\ R_z^C \end{Bmatrix}$$

using

$$\begin{Bmatrix} x^A \\ y^A \\ z^A \end{Bmatrix} = \begin{Bmatrix} \cos\theta^A \cos\lambda^A \\ \cos\theta^A \sin\lambda^A \\ \sin\theta^A \end{Bmatrix}$$

The vanishing of the last row of $\mathbf{R}_h^C$ is true by construction, $\mathbf{r}^A \cdot \mathbf{R}_h^C = 0$. We use the information to simplify the middle row getting finally:

$$\begin{Bmatrix} R_U^D \\ R_V^D \end{Bmatrix} = \begin{Bmatrix} -y^A & x^A & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} R_x^C \\ R_y^C \\ R_z^C \end{Bmatrix}$$

In summary then, having (R_U^D, R_V^D), i.e. $\mathbf{R}_h^D$ in spherical coordinates, we

1) transform $\mathbf{R}_h^D$ to Cartesian coordinates, computing R_x^D, R_y^D, R_z^D,
2) compute **c**, m and $\mathbf{R}_h^C$ in Cartesian coordinates,
3) transform $\mathbf{R}_h^C$ back to spherical coordinates, i.e. compute R_U^C, R_V^C

In order to solve this semi-Lagrangian equation, in fact all of the other equations as well, we must first solve the equation for the displacements themselves. Consider

$$a\frac{d\mathbf{r}}{dt} = \mathbf{V} = |\mathbf{V}|\mathbf{t}$$

where $\mathbf{t}$ is a unit vector tangent to the spherical earth in the direction of **V** and $|\mathbf{V}|$ is the module of **V** *assumed constant* during the displacement. For simplicity, we have taken the radius of the earth **r** as a unit vector all along. Then, in the plane of the displacement, the trajectory is an arc of circle, a great circle displacement. If $\mathbf{r}^D$, $\mathbf{t}^D$ and $\mathbf{r}^A$, $\mathbf{t}^A$ are unit vectors respectively at the *departure* and *arrival* points, we have

$$\begin{Bmatrix} \mathbf{r}^A \\ \mathbf{t}^A \end{Bmatrix} = \begin{bmatrix} \cos\Delta & \sin\Delta \\ -\sin\Delta & \cos\Delta \end{bmatrix} \begin{Bmatrix} \mathbf{r}^D \\ \mathbf{t}^D \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \mathbf{r}^D \\ \mathbf{t}^D \end{Bmatrix} = \begin{bmatrix} \cos\Delta & -\sin\Delta \\ \sin\Delta & \cos\Delta \end{bmatrix} \begin{Bmatrix} \mathbf{r}^A \\ \mathbf{t}^A \end{Bmatrix}$$

where $a\Delta = |\mathbf{V}|\Delta t$. We therefore can write

$$\mathbf{r}^D = \mathbf{r}^A \cos\Delta - \sin\Delta\, \mathbf{t}^A \qquad \text{or} \qquad \mathbf{r}^D = \frac{\mathbf{r}^A - \sin\Delta\, \mathbf{t}^D}{\cos\Delta}$$

where $\mathbf{t}^D = \mathbf{V}^D / |\mathbf{V}|$.

Assuming that **V** is known in spherical coordinates (U,V) and having a first estimate of the location (λ^M,θ^M) of the *mid-point* $\mathbf{r}^M$ between the departure and arrival points, we first obtain $\mathbf{V}^M$ by interpolating **V** at that position. Then we proceed to **improve** the estimate of $\mathbf{r}^M$ by performing a great circle displacement solving the above equation. We may proceed as follows:

1) compute $|\mathbf{V}| = \sqrt{\dfrac{(U^M)^2 + (V^M)^2}{\cos^2\theta^M}}$ and $a\Delta = \dfrac{\Delta t}{2}|\mathbf{V}|$
2) compute the arrival position $\mathbf{r}^A$ in Cartesian coordinates (x^A, y^A, z^A)

3) compute $\mathbf{V}^M$ in Cartesian coordinates using old $\mathbf{r}^M$ position

$$\mathbf{V}^M = a\dot{\mathbf{r}}^M = a\begin{Bmatrix} \dot{x}^M \\ \dot{y}^M \\ \dot{z}^M \end{Bmatrix} = a\begin{Bmatrix} -\sin\lambda^M \cos\theta^M \dot{\lambda}^M - \cos\lambda^M \sin\theta^M \dot{\theta}^M \\ \cos\lambda^M \cos\theta^M \dot{\lambda}^M - \sin\lambda^M \sin\theta^M \dot{\theta}^M \\ \cos\theta^M \dot{\theta}^M \end{Bmatrix}$$

$$= \begin{Bmatrix} -\left(y^M U^M + x^M z^M V^M\right)/\cos^2\theta^M \\ \left(x^M U^M - y^M z^M V^M\right)/\cos^2\theta^M \\ V^M \end{Bmatrix}$$

4) compute new $\mathbf{r}^M$ in Cartesian coordinates: $\mathbf{r}^M \equiv \begin{Bmatrix} x^M \\ y^M \\ z^M \end{Bmatrix} = \dfrac{1}{\cos\Delta}\begin{Bmatrix} x^A \\ y^A \\ z^A \end{Bmatrix} - \dfrac{\tan\Delta}{|\mathbf{V}|} a \begin{Bmatrix} \dot{x}^M \\ \dot{y}^M \\ \dot{z}^M \end{Bmatrix}$

5) obtain $\mathbf{r}^M$ in spherical coordinates: $\begin{Bmatrix} \lambda^M \\ \theta^M \end{Bmatrix} = \begin{Bmatrix} \tan^{-1}\left(y^M / x^M\right) \\ \sin^{-1}\left(z^M\right) \end{Bmatrix}$

In the model, the process is an iterative one (**section 8**). So we repeat the procedure until convergence. Once the new mid-point position $\mathbf{r}^M$ valid at t-$\Delta t/2$ is found, the true departure position $\mathbf{r}^D$ valid at t-Δt is obtained by doubling the great circle displacement:

6) obtain $\mathbf{r}^D = 2\cos\Delta \mathbf{r}^M - \mathbf{r}^A = \begin{Bmatrix} x^D \\ y^D \\ z^D \end{Bmatrix} = 2\cos\Delta \begin{Bmatrix} x^M \\ y^M \\ z^M \end{Bmatrix} - \begin{Bmatrix} x^A \\ y^A \\ z^A \end{Bmatrix}$, first in Cartesian and

7) finally in spherical coordinates: $\begin{Bmatrix} \lambda^D \\ \theta^D \end{Bmatrix} = \begin{Bmatrix} \tan^{-1}\left(y^D / x^D\right) \\ \sin^{-1}\left(z^D\right) \end{Bmatrix}$

We are now ready for the discretization in the horizontal. The equation

$$\frac{\mathbf{V}_h^A}{\boldsymbol{\tau}} + \mathbf{G}_h^A = \mathbf{L}_h + \mathbf{N}_h = \mathbf{R}_h^C$$

is decomposed into its components (**section 10**):

$$L_U + N_U = \left[\frac{U}{\boldsymbol{\tau}} - fV + RT\frac{\partial(Bs+q)}{\partial X} + (1+\boldsymbol{\mu})\frac{\partial\boldsymbol{\phi}'}{\partial X}\right]^A = R_U^C$$

$$L_V + N_V = \left[\frac{V}{\boldsymbol{\tau}} + fU + RT\frac{\partial(Bs+q)}{\partial Y} + (1+\boldsymbol{\mu})\frac{\partial\boldsymbol{\phi}'}{\partial Y}\right]^A = R_V^C$$

and horizontally discretized as follows

$$\left(L_U + N_U\right)_{i+\frac{1}{2}j} = \left[\frac{U}{\tau} - f\langle V\rangle^{YX} + R\overline{T}^X \boldsymbol{\delta}_X\left(Bs+q\right) + \left(1+\overline{\mu}^X\right)\boldsymbol{\delta}_X\phi'\right]^A_{i+\frac{1}{2}j} = \left(R_U^C\right)_{i+\frac{1}{2}j}$$

$$\left(L_V + N_V\right)_{ij+\frac{1}{2}} = \left[\frac{V}{\tau} + f\langle U\rangle^{XY} + R\overline{T}^Y \boldsymbol{\delta}_Y\left(Bs+q\right) + \left(1+\overline{\mu}^Y\right)\boldsymbol{\delta}_Y\phi'\right]^A_{ij+\frac{1}{2}} = \left(R_V^C\right)_{ij+\frac{1}{2}}$$

using the following simple two-point *difference* and *mean* operators:

$$\left(\boldsymbol{\delta}_X A\right)_{i+\frac{1}{2}j} = \frac{A_{i+1j} - A_{ij}}{\Delta X_{i+\frac{1}{2}}}; \qquad \left(\overline{A}^X\right)_{i+\frac{1}{2}j} = \varpi^X{}_{i+\frac{1}{2}} A_{i+1j} + \left(1-\varpi^X{}_{i+\frac{1}{2}}\right)A_{ij}$$

$$\left(\boldsymbol{\delta}_Y A\right)_{ij+\frac{1}{2}} = \frac{A_{ij+1} - A_{ij}}{\Delta Y_{j+\frac{1}{2}}}; \qquad \left(\overline{A}^Y\right)_{ij+\frac{1}{2}} = \varpi^Y{}_{j+\frac{1}{2}} A_{ij+1} + \left(1-\varpi^Y{}_{j+\frac{1}{2}}\right)A_{ij}$$

as well as the four-point (cubic interpolation) mean operators:

$$\langle V\rangle^Y_{ij} = \alpha_j \quad V_{ij-\frac{3}{2}} + \beta_j \quad V_{ij-\frac{1}{2}} + \gamma_j \quad V_{ij+\frac{1}{2}} + \delta_j \quad V_{ij+\frac{3}{2}}$$

$$\langle V\rangle^{YX}_{i+\frac{1}{2}j} = \left\langle\langle V\rangle^Y\right\rangle^X_{i+\frac{1}{2}j} = \alpha_{i+\frac{1}{2}}\ \langle V\rangle^Y_{i-1j} + \beta_{i+\frac{1}{2}}\ \langle V\rangle^Y_{ij} + \gamma_{i+\frac{1}{2}}\ \langle V\rangle^Y_{i+1j} + \delta_{i+\frac{1}{2}}\ \langle V\rangle^Y_{i+2j}$$

$$\langle U\rangle^X_{ij} = \alpha_i \quad U_{i-\frac{3}{2}j} + \beta_i \quad U_{i-\frac{1}{2}j} + \gamma_i \quad U_{i+\frac{1}{2}j} + \delta_i \quad U_{i+\frac{3}{2}j}$$

$$\langle U\rangle^{XY}_{ij+\frac{1}{2}} = \left\langle\langle U\rangle^X\right\rangle^Y_{ij+\frac{1}{2}} = \alpha_{j+\frac{1}{2}}\langle U\rangle^X_{ij-1} + \beta_{j+\frac{1}{2}}\langle U\rangle^X_{ij} + \gamma_{j+\frac{1}{2}}\langle U\rangle^X_{ij+1} + \delta_{j+\frac{1}{2}}\langle U\rangle^X_{ij+2}$$

The left-hand-sides (dropping the superscript ^{A}) are linearized separately (**section 11**):

$$\left(L_U\right)_{i+\frac{1}{2}j} = \frac{U_{i+\frac{1}{2}j}}{\tau} + \left(\boldsymbol{\delta}_X P\right)_{i+\frac{1}{2}j} \qquad P = \phi' + RT_*\left(Bs+q\right)$$

$$\left(L_V\right)_{ij+\frac{1}{2}} = \frac{V_{ij+\frac{1}{2}}}{\tau} + \left(\boldsymbol{\delta}_Y P\right)_{ij+\frac{1}{2}}$$

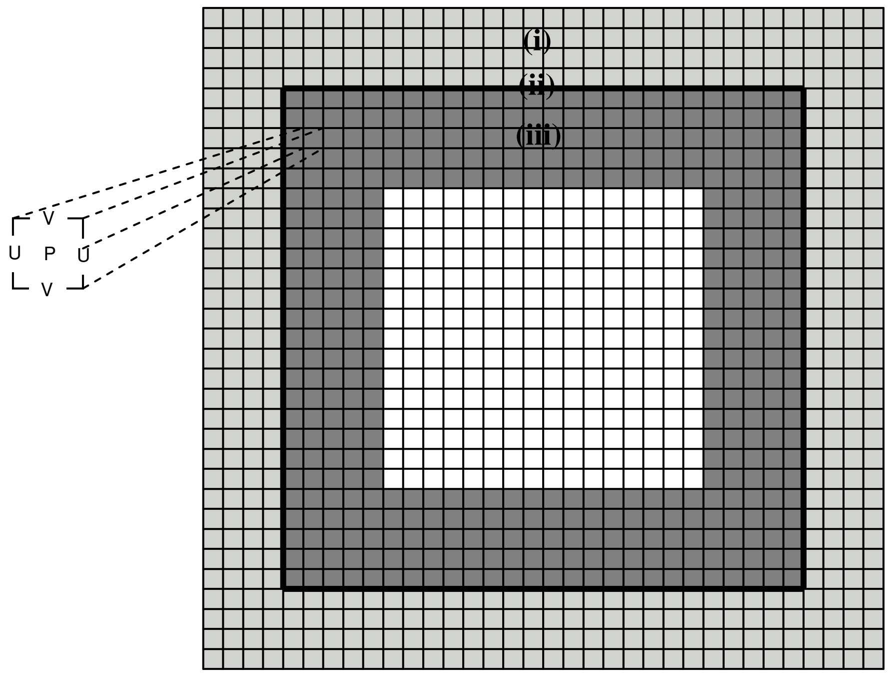
leaving as non-linear terms (**section 12**):

$$\left(N_U\right)_{i+\frac{1}{2}j} = -f\langle V\rangle^{YX}_{i+\frac{1}{2}j} + R\overline{T'}^X_{i+\frac{1}{2}j}\left[\boldsymbol{\delta}_X\left(Bs+q\right)\right]_{i+\frac{1}{2}j} + \overline{\mu}^X_{i+\frac{1}{2}j}\left(\boldsymbol{\delta}_X\phi'\right)_{i+\frac{1}{2}j}$$

$$\left(N_V\right)_{ij+\frac{1}{2}} = +f\langle U\rangle^{XY}_{ij+\frac{1}{2}} + R\overline{T'}^Y_{ij+\frac{1}{2}}\left[\boldsymbol{\delta}_Y\left(Bs+q\right)\right]_{ij+\frac{1}{2}} + \overline{\mu}^Y_{ij+\frac{1}{2}}\left(\boldsymbol{\delta}_Y\phi'\right)_{ij+\frac{1}{2}}$$

Appendix 11. Lateral boundary conditions

A limited area (LAM) version of GEM exists. It requires lateral boundary conditions. These are provided by three sets of grid point values:



(i) The *first* set is external to the LAM domain and allows the semi-Lagrangian scheme to function as if no boundary existed, i.e. a sufficient number of points exists outside of the domain so that the upwind values of all relevant fields can be obtained by interpolation provided a predetermined Courant number is not exceeded. The relevant fields are the R_i's, the Right-Hand Sides terms calculated from the previous timestep history carrying model variables. If the values provided to the LAM come from a global host-model identical to the LAM in all respects (space and time resolutions, physical parameterizations, etc) then the host-model results for the R_i's are reproduced.

(ii) The *second* set is the boundary set proper: it comprises exclusively the *wind component normal to the boundary and at the boundary* itself. These grid point values serve to close the elliptic problem in the horizontal. In effect, the so-called elliptic equation will contain in particular (see **section 15**) the following terms:

$$\left(\nabla_{\boldsymbol{\zeta}} \cdot \mathbf{L}_h\right)_{1jk} - \frac{L_{C1jk}}{\boldsymbol{\tau}} + \ldots \equiv \left(L_P\right)_{1jk} = \left(\nabla^2_{\boldsymbol{\zeta}} P\right)_{1jk} + \ldots$$

To the left, the L's must be known quantities. To the right, there is only the unknown P. Here we consider, as an example, the grid points with the i label equal to 1. This is the X-direction and we assume that i=1 is the first internal model cell on its left-hand side. Developing the operators, we obtain successively

$$\frac{1}{\cos^2\boldsymbol{\theta}_j}\left[\left(\boldsymbol{\delta}_X L_U\right)_{1jk}+\left(\boldsymbol{\delta}_Y L_V\right)_{1jk}\right]-\frac{L_{C1jk}}{\boldsymbol{\tau}}=\frac{1}{\cos^2\boldsymbol{\theta}_j}\left[\left(\boldsymbol{\delta}_X^2 P\right)_{1jk}+\left(\boldsymbol{\delta}_Y^2 P\right)_{1jk}\right]$$

$$\frac{1}{\cos^2\boldsymbol{\theta}_j}\left[\frac{\left(L_U\right)_{\frac{3}{2}jk}-\left(L_U\right)_{\frac{1}{2}jk}}{\Delta X_1}+\left(\boldsymbol{\delta}_Y L_V\right)_{1jk}\right]-\frac{L_{C1jk}}{\boldsymbol{\tau}}=\frac{1}{\cos^2\boldsymbol{\theta}_j}\left[\frac{\left(\boldsymbol{\delta}_X P\right)_{\frac{3}{2}jk}-\left(\boldsymbol{\delta}_X P\right)_{\frac{1}{2}jk}}{\Delta X_1}+\left(\boldsymbol{\delta}_Y^2 P\right)_{1jk}\right]$$

But note, the equation

$$\left(L_U\right)_{\frac{1}{2}jk}=\frac{U_{\frac{1}{2}jk}}{\boldsymbol{\tau}}+\left(\boldsymbol{\delta}_X P\right)_{\frac{1}{2}jk}$$

which has served to eliminate $U_{\frac{1}{2}jk}$ from the continuity equation does not exists. $\left(L_U\right)_{\frac{1}{2}jk}$ is an unknown quantity. Let us restore $U_{\frac{1}{2}jk}$ in the previous equation:

$$\frac{1}{\cos^2\boldsymbol{\theta}_j}\left[\frac{\left(L_U\right)_{\frac{3}{2}jk}-U_{\frac{1}{2}jk}/\boldsymbol{\tau}}{\Delta X_1}+\left(\boldsymbol{\delta}_Y L_V\right)_{1jk}\right]-\frac{L_{C1jk}}{\boldsymbol{\tau}}=\frac{1}{\cos^2\boldsymbol{\theta}_j}\left[\frac{\left(\boldsymbol{\delta}_X P\right)_{\frac{3}{2}jk}-0}{\Delta X_1}+\left(\boldsymbol{\delta}_Y^2 P\right)_{1jk}\right]$$

Thus the elliptic problem may be solved if we provide the normal wind component on the boundary $U_{\frac{1}{2}jk}$ as a boundary condition. The elliptic problem though appears as if we had set $\left(\boldsymbol{\delta}_X P\right)_{\frac{1}{2}jk}=0$ as a boundary condition on *P* to the left while replacing the unknown $\left(L_U\right)_{\frac{1}{2}jk}$ by the known value $U_{\frac{1}{2}jk}/\boldsymbol{\tau}$ to the right. The same procedure is applied to the normal wind components on all the boundaries of the LAM. Again, if the normal wind components provided to the LAM come from an identical global host-model, then the host-model results are reproduced. Since the solution of the elliptic problem corresponds to a future timestep, the set of boundary winds must come from the timestep following that from which came the external set.

(iii) Finally, a *third* set of grid point values are internal to the LAM domain. They allow for a gradual relaxation of LAM-fields to the HOST-fields as we approach the boundary. All history carrying variables are relaxed this way. Of course, if the host-model is identical (the *acid test*), this third step of the procedure is redundant.

In GEM presently, physical parameterization is added (split mode) after the dynamics, i.e. after the relaxation step just mentioned. Thus for the LAM to reproduce the host-model results, the *future* values provided in steps (ii) and (iii) must come from the host-model after the dynamics *prior* physical parameterization while the *past* values provided in step (i) must come from the host-model *after* physical parameterization.

N.B. As soon as horizontal winds are modified by space and time interpolation, i.e. when not performing the acid test, the vertical motion field $\dot{\boldsymbol{\zeta}}$ should be diagnosed (see **Appendix 15**)

Appendix 12. Time varying topography

The initial conditions as well as the lateral conditions (see **Appendix 11**) of a LAM are frequently provided by a host-model or by an analysis made on the host-model grid with much coarser horizontal resolution, typically at least a factor of three coarser. And the information usually comes in terrain-following vertical coordinates. Then the bottom surfaces, the topography, of the host and LAM may differ considerably. Straightforward interpolation-extrapolation often results in poorly balanced fields: a point fairly high in the host may have relatively strong winds which may find themselves near the surface in the LAM; vice versa a surface point with light winds in the host may find itself fairly high in the LAM. For the first two sets of lateral conditions, i.e. outside and on the boundary of the LAM domain, the host topography may be kept, but for the third set, the relaxation zone, the problem cannot be avoided. One may only attenuate the problem by relaxing the topography in essentially the same way that the other model fields are relaxed and then interpolating-extrapolating the variables. As for the initial imbalances, it has been found desirable to initialize the LAM with the coarser host topography, gradually modifying it to reach the finer LAM topography after a suitable interval of integration time: the LAM then having a so-called time-varying topography field. Artificial though it may be for the atmosphere, this is a perfectly acceptable mathematical procedure and, provided the induced vertical motions remain small, the meteorological consequences *may* remain acceptable (a 10 cm/s topography velocity is able to lift the terrain by more than 1 km in 3 hours).

Examining the equations, we find that a local tendency of geopotential is provided and calculated implicitly by the equation:

$$\frac{d\phi'}{dt} - RT_* \dot{\zeta} - gw = 0$$

A surface level is present in the vertical discretization ($\dot{\zeta}_S = 0$):

$$\frac{d\phi_S}{dt} - gw_S = 0$$

After time discretization, we have:

$$\frac{(\phi_S)^A}{\tau} - g(w_S)^A = \frac{(\phi_S)^D}{\tau} + g\beta(w_S)^D$$

$(\phi_S)^A$ is the surface geopotential at the arrival point, i.e. at the grid point at the future time. It is an external parameter which must be externally specified. In hydrostatic-pressure coordinate, the **time varying topography option** is just and *only just* that: modifying ϕ_S at the appropriate place in the model code.

Appendix 13. Numerically truncated top boundary condition

On the regular grid of GEM3, temperature as well as all other model variables were present at the model top. A thermodynamic equation along with momentum equations were therefore needed at the top. In developing GEM4 on the Charney-Phillips grid, there arose the question as to whether or not to include a top thermodynamic level. Three arguments militated in favor of its inclusion: the fact that such a level already existed in GEM3, a level used by the analysis; a symmetry argument since we were planning a bottom thermodynamic level to better accommodate the surface layer parameterization; finally the fact that a similar surface layer parameterization would be needed at the top of an ocean model. Now, we have much stronger arguments in favor of its exclusion. First, the *level is dynamically disconnected* from the rest of the model: in effect, the equation is highly simplified due to the boundary condition: $d\ln T/dt=Q/c_pT$ since by construction $d\ln p/dt=0$. Considering that vertical advection also vanishes since $\dot{\zeta}=0$, the resulting predicted temperature becomes dynamically very poorly connected with the rest of the model. Second, the disconnectedness is further amplified by the typical *lack of vertical resolution near the top* of operational forecast models. Third, *the boundary condition is very artificial*. (Note that none of these arguments would apply to the top of an ocean model.) Finally and foremost, experiments have shown that we obtain better results without than with this top thermodynamic level.

In this perspective, it is very interesting to examine the equations with the hydrostatic option **(Appendix 7**) once discretized:

$$\frac{d\mathbf{V}_h}{dt}+f\mathbf{k}\mathrm{x}\mathbf{V}_h+R\overline{T}^{\zeta}\nabla_{\zeta}Bs+\nabla_{\zeta}\phi'=0 \qquad (k=1,N)$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right)-\kappa\overline{B}^{\zeta}s\right]-\kappa\dot{\zeta}=0 \qquad (k=\tfrac{3}{4},\tfrac{3}{2},...,N+\tfrac{1}{4})$$

$$\frac{d}{dt}\left[Bs+\ln\left(1+\delta_{\zeta}\overline{B}^{\zeta}s\right)\right]+\nabla_{\zeta}\cdot\mathbf{V}_h+\delta_{\zeta}\dot{\zeta}+\overline{\dot{\zeta}}^{\zeta}=0 \qquad (k=1,N)$$

$$\frac{T'}{T_*}+\frac{\delta_{\zeta}\left(\phi'/RT_*+Bs\right)}{\delta_{\zeta}\left(\zeta+Bs\right)}=0 \qquad (k=\tfrac{3}{4},\tfrac{3}{2},...,N+\tfrac{1}{4})$$

Eliminating the two equations at k=3/4 will force the elimination of two degrees of freedom, two variables: $T_{3/4}$ and $\phi'_0\equiv\phi'_T$ but ϕ'_T does not serve elsewhere and $T_{3/4}$ only serves in the pressure gradient term at level 1 of the momentum equation. But the pressure gradient term at level 1 only subsists if the metric parameter $B\neq 0$. It is in fact desirable to have this parameter as close to zero as possible near the model top. Thus the top thermodynamic level can be eliminated in the hydrostatic case with little consequences. Of course, the semi-Lagrangian scheme has to be notified of the absence of this degree of freedom in the temperature field. But this is in fact where most of the benefices will come from, as too large differences between the top and adjacent levels tend to generate noise (in particular, kinks in the vertical).

We now examine the full equations with the same perspective:

$$\frac{d\mathbf{V}_h}{dt}+f\mathbf{k}\mathrm{x}\mathbf{V}_h+R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q)+\left(1+\overline{\mu}^{\zeta}\right)\nabla_{\zeta}\phi'=0 \qquad (k=1,N)$$

$$\frac{dw}{dt}-g\mu=0 \qquad (k=\tfrac{3}{4},\tfrac{3}{2},...,N+\tfrac{1}{4})$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right)-\kappa\left(\overline{Bs+q}^{\zeta}\right)\right]-\kappa\dot{\zeta}=0 \qquad (k=\tfrac{3}{4},\tfrac{3}{2},...,N+\tfrac{1}{4})$$

$$\frac{d}{dt}\left[Bs+\ln\left(1+\delta_{\zeta}\overline{B}^{\zeta}s\right)\right]+\nabla_{\zeta}\cdot\mathbf{V}_h+\delta_{\zeta}\dot{\zeta}+\overline{\dot{\zeta}}^{\zeta}=0 \qquad (k=1,N)$$

$$\frac{d\overline{\phi'}^{\zeta}}{dt}-RT_*\dot{\zeta}-gw=0 \qquad (k=\tfrac{1}{2},...,N+\tfrac{1}{2})$$

$$1+\mu-e^{\overline{q}^{\zeta}}\left[1+\frac{\delta_{\zeta}q}{\delta_{\zeta}(\zeta+Bs)}\right]=0 \qquad (k=\tfrac{3}{4},\tfrac{3}{2},...,N+\tfrac{1}{4})$$

$$\frac{T}{T_*}-e^{\overline{q}^{\zeta}}\left[1-\frac{\delta_{\zeta}(\phi'/RT_*+Bs)}{\delta_{\zeta}(\zeta+Bs)}\right]=0 \qquad (k=\tfrac{3}{4},\tfrac{3}{2},...,N+\tfrac{1}{4})$$

We find three new equations at k=3/4 which should logically be eliminated along with three new variables that must disappear from the list of degrees of freedom: $w_{3/4}$ which is no more referred to, q_0 and $\mu_{3/4}$. And we must estimate q_1 and μ_1. Note that the presence of q_1 restores the pressure gradient term in the momentum equations. However, the simple extension of the top boundary condition on q downward to level 1, i.e. setting $q_1=0$, essentially retires again the pressure gradient term from the top level momentum equations. For μ, setting $\overline{\mu}_1^{\zeta}=0$ was chosen, again avoiding extrapolation.

Less elegant perhaps but not more arbitrary and factually less problematic than the original procedure which for the temperature in particular added an apparently purely numerical degree of freedom in the vertical, this procedure will be called the *numerically truncated top boundary condition*. In the absence of certain degrees of freedom, we had to estimate for the top level momentum equations the non-hydrostatic contribution to both the pressure gradient and geopotential gradient terms. We decided to make this contribution vanish, setting both $q_1=0$ and $\overline{\mu}_1^{\zeta}=0$.

As seen before (Appendix 9), changing the top boundary condition affects the details of the matrix problem as specified in **Appendix 4b**. Here we start with the original discretized linear continuity equation (**section 11**):

$$\left(L_C\right)_{k_0} = \nabla_{\boldsymbol{\zeta}} \cdot \mathbf{V}_{h\,k_0} + \left[\boldsymbol{\delta}_{\boldsymbol{\zeta}} \left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right) + \overline{\left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right)}^{\boldsymbol{\zeta}} \right]_{k_0}$$

$$\left(L_C\right)_{k_0} = \nabla_{\boldsymbol{\zeta}} \cdot \mathbf{V}_{h\,k_0} + \frac{\left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right)_{k_0+\frac{1}{2}} - \left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right)_{k_0-\frac{1}{2}}}{\Delta \boldsymbol{\zeta}_{k_0}} + \boldsymbol{\varpi}^{+}_{k_0} \left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right)_{k_0+\frac{1}{2}} + \boldsymbol{\varpi}^{-}_{k_0} \left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right)_{k_0-\frac{1}{2}}$$

For k_0=1, we have the boundary condition $\left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right)_{k_0-\frac{1}{2}}$=0. We may therefore immediately write:

$$\left(L_C\right)_{k_0} = \nabla_{\boldsymbol{\zeta}} \cdot \mathbf{V}_{h\,k_0} + \left(\frac{1}{\Delta \boldsymbol{\zeta}_{k_0}} + \boldsymbol{\varpi}^{+}_{k_0} \right) \left(\dot{\boldsymbol{\zeta}} + \overline{B}^{\zeta} s / \boldsymbol{\tau} \right)_{k_0+\frac{1}{2}}$$

$$\left(L_C\right)_{k_0} = \nabla_{\boldsymbol{\zeta}} \cdot \mathbf{V}_{h\,k_0} + \left(\frac{1}{\Delta \boldsymbol{\zeta}_{k_0}} + \boldsymbol{\varpi}^{+}_{k_0} \right) \left(X_{k_0+\frac{1}{2}} - \frac{\boldsymbol{\varpi}^{+}_{k_0+\frac{1}{2}} q_{k_0+1} + \boldsymbol{\varpi}^{-}_{k_0+\frac{1}{2}} q_{k_0}}{\boldsymbol{\tau}} \right)$$

and proceed with the elimination process toward the elliptic equation using only the equations for $\left(\mathbf{L}_h\right)_{k_0}, \left(L'_w\right)_{k_0+\frac{1}{2}}, \left(L'_T\right)_{k_0+\frac{1}{2}}, \left(L_{\boldsymbol{\phi}}\right)_{k_0+\frac{1}{2}}$, never in fact requiring the top level equations $\left(L'_w\right)_{k_0-\frac{1}{2}}, \left(L'_T\right)_{k_0-\frac{1}{2}}, \left(L_{\boldsymbol{\phi}}\right)_{k_0-\frac{1}{2}}$. In effect, we compute

$$\nabla_{\boldsymbol{\zeta}} \cdot \left(\mathbf{L}_h\right)_{k_0} - \frac{1}{\boldsymbol{\tau}} \left\{ \left(L_C\right)_{k_0} - \frac{\boldsymbol{\varepsilon}}{w_*} \boldsymbol{\varpi}^{+}_{k_0} \left(L'_w\right)_{k_0+\frac{1}{2}} \right\} \equiv \left(L''_C\right)_{k_0}$$

getting

$$\left(L''_C\right)_{k_0} = \nabla^2_{\zeta} P_{k_0} - \frac{1}{\boldsymbol{\tau}} \left(\frac{1}{\Delta \boldsymbol{\zeta}_{k_0}} + \boldsymbol{\varpi}^{+}_{k_0} \right) X_{k_0+\frac{1}{2}} + \frac{\boldsymbol{\varepsilon}}{\boldsymbol{\tau}^2 w_*} \boldsymbol{\varpi}^{+}_{k_0} w_{k_0+\frac{1}{2}}$$
$$+ \frac{1}{\boldsymbol{\tau}^2} \left(\frac{\boldsymbol{\varpi}^{+}_{k_0+\frac{1}{2}}}{\Delta \boldsymbol{\zeta}_{k_0}} - \frac{\boldsymbol{\varpi}^{+}_{k_0}}{\Delta \boldsymbol{\zeta}_{k+\frac{1}{2}}} \right) q_{k_0+1} + \frac{1}{\boldsymbol{\tau}^2} \left(\frac{\boldsymbol{\varpi}^{-}_{k_0+\frac{1}{2}}}{\Delta \boldsymbol{\zeta}_{k_0}} + \frac{\boldsymbol{\varpi}^{+}_{k_0}}{\Delta \boldsymbol{\zeta}_{k_0+\frac{1}{2}}} \right) q_{k_0}$$

and since $\frac{\boldsymbol{\varpi}^{+}_{k_0+\frac{1}{2}}}{\Delta \boldsymbol{\zeta}_{k_0}} - \frac{\boldsymbol{\varpi}^{+}_{k_0}}{\Delta \boldsymbol{\zeta}_{k_0+\frac{1}{2}}} = 0$ (definition of $\boldsymbol{\varpi}^{+}_{k_0}$) while $q_{k_0} = 0$ (by construction), we get

$$\left(L''_C\right)_{k_0} = \nabla^2_{\zeta} P_{k_0} - \left(\frac{1}{\Delta \boldsymbol{\zeta}_{k_0}} + \boldsymbol{\varpi}^{+}_{k_0} \right) \frac{X_{k_0+\frac{1}{2}}}{\boldsymbol{\tau}} + \boldsymbol{\varepsilon} \boldsymbol{\varpi}^{+}_{k_0} \frac{w_{k_0+\frac{1}{2}}}{\boldsymbol{\tau}^2 w_*}$$

Then we compute

$$\frac{\gamma}{\kappa\tau}\left[\left(L'_\theta\right)_{k_0+\frac{1}{2}}+\frac{\varepsilon}{w_*}\left(L'_w\right)_{k_0+\frac{1}{2}}+\frac{\varepsilon}{RT_*}\left(L_\phi\right)_{k_0+\frac{1}{2}}\right]\equiv\left(L''_\theta\right)_{k_0+\frac{1}{2}}$$

$$\frac{\gamma}{\kappa\tau}\left[\left(L'_\theta\right)_{k_0+\frac{1}{2}}+\frac{\varepsilon}{w_*}\left(L'_w\right)_{k_0+\frac{1}{2}}-\frac{\kappa}{RT_*}\left(L_\phi\right)_{k_0+\frac{1}{2}}\right]\equiv\left(L''_\phi\right)_{k_0+\frac{1}{2}}$$

$$\left(L''_\theta\right)_{k_0+\frac{1}{2}}=-\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}-\varepsilon\left(\varpi^+_{k_0+\frac{1}{2}}P_{k_0+1}+\varpi^-_{k_0+\frac{1}{2}}P_{k_0}\right)\right]-\frac{X_{k_0+\frac{1}{2}}}{\tau}$$

$$\left(L''_\phi\right)_{k_0+\frac{1}{2}}=-\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}+\kappa\left(\varpi^+_{k_0+\frac{1}{2}}P_{k_0+1}+\varpi^-_{k_0+\frac{1}{2}}P_{k_0}\right)\right]+\frac{w_{k_0+\frac{1}{2}}}{\tau^2 w_*}$$

and combine

$$\left(L''_C\right)_{k_0}-\left(\frac{1}{\Delta\zeta_{k_0}}+\varpi^+_{k_0}\right)\left(L''_\theta\right)_{k_0+\frac{1}{2}}-\varepsilon\varpi^+_{k_0}\left(L''_\phi\right)_{k_0+\frac{1}{2}}=\left(L_P\right)_{k_0}$$

getting

$$\begin{aligned}\left(L''_C\right)_{k_0}=\nabla^2_\zeta P_{k_0}+\frac{\gamma}{\kappa\tau^2RT_*}&\left\{\left(\frac{1}{\Delta\zeta_{k_0}}+\left(1+\varepsilon\right)\varpi^+_{k_0}\right)\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}-\frac{1}{\Delta\zeta_{k_0}}\varepsilon\left(\varpi^+_{k_0+\frac{1}{2}}P_{k_0+1}+\varpi^-_{k_0+\frac{1}{2}}P_{k_0}\right)\right\}\\&-\frac{\gamma\varepsilon\left(1-\kappa\right)}{\kappa\tau^2RT_*}\varpi^+_{k_0}\left(\varpi^+_{k_0+\frac{1}{2}}P_{k_0+1}+\varpi^-_{k_0+\frac{1}{2}}P_{k_0}\right)\end{aligned}$$

$$\begin{aligned}\left(L''_C\right)_{k_0}=\nabla^2_\zeta P_{k_0}+\frac{\gamma}{\kappa\tau^2RT_*}&\left\{\left(\frac{1}{\Delta\zeta_{k_0}}+\varpi^+_{k_0}\right)\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}+\varepsilon\left[\left(\frac{\varpi^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}-\frac{\varpi^+_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0}}\right)P_{k_0+1}-\left(\frac{\varpi^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}+\frac{\varpi^-_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0}}\right)P_{k_0}\right]\right\}\\&-\frac{\gamma\varepsilon\left(1-\kappa\right)}{\kappa\tau^2RT_*}\varpi^+_{k_0}\left(\varpi^+_{k_0+\frac{1}{2}}P_{k_0+1}+\varpi^-_{k_0+\frac{1}{2}}P_{k_0}\right)\end{aligned}$$

$$\begin{aligned}\left(L''_C\right)_{k_0}=\nabla^2_\zeta P_{k_0}+\frac{\gamma}{\kappa\tau^2RT_*}&\left\{\left(\frac{1}{\Delta\zeta_{k_0}}+\varpi^+_{k_0}\right)\frac{P_{k_0+1}-P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}-\varepsilon\frac{P_{k_0}}{\Delta\zeta_{k_0}}\right\}\\&-\frac{\gamma\varepsilon\left(1-\kappa\right)}{\kappa\tau^2RT_*}\varpi^+_{k_0}\left(\varpi^+_{k_0+\frac{1}{2}}P_{k_0+1}+\varpi^-_{k_0+\frac{1}{2}}P_{k_0}\right)\end{aligned}$$

since $\frac{\varpi^+_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}}=\frac{\varpi^+_{k_0+\frac{1}{2}}}{\Delta\zeta_{k_0}}$ and since $\varpi^+_{k_0+\frac{1}{2}}$=1/2

Considering the original expressions (**Appendix 4b**)

$$\left(L'_P\right)_{k_0} = \left(L_P\right)_{k_0} - C''_T \, L_B / \Delta\zeta_1$$

$$\left(L'_P\right)_{k_0} = \left(\nabla^2_\zeta P\right)_{k_0} + \frac{\gamma}{\kappa\tau^2 RT_*}\left[\left(\frac{1}{\Delta\zeta_{k_0}} + \varpi^+_{k_0}\right)\frac{P_{k_0+1} - P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}} - \frac{(1-\alpha_T)P_{k_0}}{\Delta\zeta_{k_0}\Delta\zeta_{k_0-\frac{1}{2}}} + \frac{\varpi^-_{k_0}}{\Delta\zeta_{k_0-\frac{1}{2}}}(1-\alpha_T)P_{k_0}\right]$$
$$- \frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2 RT_*}\left\{\varpi^+_{k_0}\left(\varpi^+_{k_0+\frac{1}{2}} P_{k_0+1} + \varpi^-_{k_0+\frac{1}{2}} P_{k_0}\right) + \varpi^-_{k_0}\left(\varpi^+_{k_0-\frac{1}{2}} + \varpi^-_{k_0-\frac{1}{2}}\alpha_T\right)P_{k_0}\right\}$$

and setting $L_B = 0$ and $\varpi^-_{k_0} = 0$, we get

$$\left(L'_P\right)_{k_0} = \left(\nabla^2_\zeta P\right)_{k_0} + \frac{\gamma}{\kappa\tau^2 RT_*}\left[\left(\frac{1}{\Delta\zeta_{k_0}} + \varpi^+_{k_0}\right)\frac{P_{k_0+1} - P_{k_0}}{\Delta\zeta_{k_0+\frac{1}{2}}} - \frac{(1-\alpha_T)P_{k_0}}{\Delta\zeta_{k_0}\Delta\zeta_{k_0-\frac{1}{2}}}\right]$$
$$- \frac{\gamma\varepsilon(1-\kappa)}{\kappa\tau^2 RT_*}\varpi^+_{k_0}\left(\varpi^+_{k_0+\frac{1}{2}} P_{k_0+1} + \varpi^-_{k_0+\frac{1}{2}} P_{k_0}\right)$$

and we note that further setting $\alpha_T = 1 - \varepsilon\Delta\zeta_{k_0-\frac{1}{2}}$ takes care of the ε-term.

Appendix 14. Trapezoidal rule for trajectory calculations

Here we compare **mid-point rule** and **trapezoidal rule** for the calculation of displacements $\Delta\mathbf{r}$ in the semi-Lagrangian scheme.

The mid-point rule (a time mean followed by a space interpolation) can be described as follows:

$$\Delta\mathbf{r}^{i} = \Delta t \frac{\mathbf{V}(t) + \mathbf{V}(t-\Delta t)}{2}\left(\mathbf{r} - \Delta\mathbf{r}^{i-1}/2\right) = \Delta t \mathbf{V}_M$$

where i is for iterations being made due to the non-linear nature of the process, while the trapezoidal rule (a space interpolation followed by a space-time mean) can be written:

$$\Delta\mathbf{r}^{i} = \Delta t \frac{\mathbf{V}(t,\mathbf{r}) + \mathbf{V}\left(t-\Delta t, \mathbf{r} - \Delta\mathbf{r}^{i-1}\right)}{2} = \Delta t \frac{\mathbf{V}_A + \mathbf{V}_D}{2}$$

Changing rule is fairly straightforward except for the 'horizontal' on the sphere. In effect, consider the equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{V} = |\mathbf{V}|\mathbf{t}$$

where $\mathbf{t}$ is a unit vector tangent to the spherical earth in the direction of $\mathbf{V}$ and $|\mathbf{V}|$ is the module of $\mathbf{V}$ *assumed constant* during the displacement. For convenience here, we take the radius of the earth $\mathbf{r}$ to be a unit vector, normalizing the winds accordingly. Then, in the plane of the displacement, the trajectory is an arc of circle, a great circle displacement. If $\mathbf{r}_D$, $\mathbf{t}_D$ and $\mathbf{r}_A$, $\mathbf{t}_A$ are unit vectors respectively at the *departure* and *arrival* points, we have

$$\begin{Bmatrix} \mathbf{r}_A \\ \mathbf{t}_A \end{Bmatrix} = \begin{bmatrix} \cos\Delta_D & \sin\Delta_D \\ -\sin\Delta_D & \cos\Delta_D \end{bmatrix} \begin{Bmatrix} \mathbf{r}_D \\ \mathbf{t}_D \end{Bmatrix} \qquad (1\text{a})$$

or

$$\begin{Bmatrix} \mathbf{r}_D \\ \mathbf{t}_D \end{Bmatrix} = \begin{bmatrix} \cos\Delta_A & -\sin\Delta_A \\ \sin\Delta_A & \cos\Delta_A \end{bmatrix} \begin{Bmatrix} \mathbf{r}_A \\ \mathbf{t}_A \end{Bmatrix} \qquad (1\text{b})$$

where $a\Delta_D = |\mathbf{V}_D|\Delta t$ and $a\Delta_A = |\mathbf{V}_A|\Delta t$ are the *modules* of the displacements and where $\mathbf{t}_D = \mathbf{V}_D / |\mathbf{V}_D|$ and $\mathbf{t}_A = \mathbf{V}_A / |\mathbf{V}_A|$ are their *directions*. We therefore can write

$$\mathbf{r}_D = \frac{\mathbf{r}_A - \sin\Delta_D \, \mathbf{t}_D}{\cos\Delta_D} \qquad (2\text{a})$$

or

$$\mathbf{r}_D = \mathbf{r}_A \cos\Delta_A - \sin\Delta_A \, \mathbf{t}_A \qquad (2\text{b})$$

Now, these are equivalent exact relations for a *constant* wind on the sphere. For the semi-Lagrangian calculations though, we know the wind at some positions and estimate displacements *assuming* constant wind. For the **mid-point rule**, we take the wind $\mathbf{V}_M$ as known at the mid-point of the trajectory and we calculate a half-displacement from the *mid-point* $\mathbf{r}_M$ to the *arrival point* $\mathbf{r}_A$, therefore using formula (2a) with subscript $_M$ replacing $_D$:

$$\mathbf{r}_M = \frac{\mathbf{r}_A - \sin\Delta_M \; \mathbf{t}_M}{\cos\Delta_M}$$

We then double this displacement backward to find the *departure point* $\mathbf{r}_D$ using formula (2b) with subscript $_M$ this time replacing subscript $_A$:

$$\mathbf{r}_D = \mathbf{r}_M \cos\Delta_M - \sin\Delta_M \; \mathbf{t}_M$$

Combining the above two equations gives the geometrical formula for the mid-point:

$$\mathbf{r}_M = \frac{\mathbf{r}_D + \mathbf{r}_A}{2\cos\Delta_M}$$

For the **trapezoidal rule**, having the wind known at both ends of the trajectory, $\mathbf{V}_D$ and $\mathbf{V}_A$, we perform two half-displacements using formula (2a) for the first

$$\mathbf{r}_{d1} = \frac{\mathbf{r}_{a1} - \sin\Delta_{d1} \; \mathbf{t}_{d1}}{\cos\Delta_{d1}},$$

with $a\Delta_{d1} = \Delta t \left|\mathbf{V}_D\right|/2$ and $\mathbf{t}_{d1} = \mathbf{V}_D / \left|\mathbf{V}_D\right|$, and (2b) for the second

$$\mathbf{r}_{d2} = \mathbf{r}_{a2} \cos\Delta_{a2} - \sin\Delta_{a2} \; \mathbf{t}_{a2}$$

with $a\Delta_{a1} = \Delta t \left|\mathbf{V}_A\right|/2$ and $\mathbf{t}_{a2} = \mathbf{V}_A / \left|\mathbf{V}_A\right|$. In sequence, therefore index $_{d1}$ becomes $_D$, index $_{a2}$ becomes $_A$ and we set $\mathbf{r}_{a1} = \mathbf{r}_{d2}$, hence

$$\mathbf{r}_D = \frac{\mathbf{r}_A \cos\Delta_A - \sin\Delta_A \; \mathbf{t}_A - \sin\Delta_D \; \mathbf{t}_D}{\cos\Delta_D}$$

having combined the previous two equations.

A first order verification of this formula ($\cos\Delta_A \approx \cos\Delta_D \approx 1$; $\sin\Delta_D \approx \Delta_D$; $\sin\Delta_A \approx \Delta_A$) is:

$$\Delta\mathbf{r} = \mathbf{r}_A - \mathbf{r}_D \approx \Delta_A \; \mathbf{t}_A + \Delta_D \; \mathbf{t}_D = \frac{\Delta t}{2}\left(\mathbf{V}_A + \mathbf{V}_D\right)$$

Steps in the mid-point rule

The winds $\mathbf{V}(t, \mathbf{r}_A)$ and $\mathbf{V}(t\text{-}\Delta t, \mathbf{r}_A)$ are assumed known at two time levels on the grid points $\mathbf{r}_A$ (one Crank-Nicolson step). We compute:

1) the Spherical Coordinates (U,V) of $\mathbf{V}(t-\Delta t/2, \mathbf{r}_A) = \dfrac{\mathbf{V}(t,\mathbf{r}_A)+\mathbf{V}(t-\Delta t,\mathbf{r}_A)}{2}$
2) $\mathbf{V}_M^i = \mathbf{V}(t-\Delta t/2, \mathbf{r}_M^{i-1})$ by interpolation (Spherical Coordinates)
3) the Cartesian Coordinates $(\dot{x}_M^i, \dot{y}_M^i, \dot{z}_M^i)$ of $\mathbf{V}_M^i$
4) the module $|\mathbf{V}_M^i| = \sqrt{(\dot{x}_M^i)^2 + (\dot{y}_M^i)^2 + (\dot{z}_M^i)^2}$
5) the displacement $\Delta_M^i = \dfrac{\Delta t}{2}\left|\dfrac{\mathbf{V}_M^i}{a}\right|$
6) the Cartesian Coordinates of $\mathbf{r}_M^i = \dfrac{\mathbf{r}_A}{\cos\Delta_M^i} - \tan\Delta_M^i \dfrac{\mathbf{V}_M^i}{|\mathbf{V}_M^i|}$
7) the Spherical Coordinates $(\boldsymbol{\lambda}_M^i, \boldsymbol{\theta}_M^i)$ of $\mathbf{r}_M^i$
8) the Cartesian Coordinates of $\mathbf{r}_D^i = 2\cos\Delta_M^i\, \mathbf{r}_M^i - \mathbf{r}_A$
9) the Spherical Coordinates $(\boldsymbol{\lambda}_D^i, \boldsymbol{\theta}_D^i)$ of $\mathbf{r}_D^i$

Only steps 2 to 7 need be repeated for the required number of iterations. Note that the module is computed using Cartesian coordinates.

The Cartesian Coordinates $(\dot{x}, \dot{y}, \dot{z})$ of $\mathbf{V}$ at position $\mathbf{r} = (x, y, z)$ are obtained from the values (U, V) in Spherical Coordinates as follows

$$\mathbf{V} = a\dot{\mathbf{r}} = a\begin{Bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{Bmatrix} = a\begin{Bmatrix}-\sin\lambda\cos\theta\dot{\lambda} - \cos\lambda\sin\theta\dot{\theta}\\ \cos\lambda\cos\theta\dot{\lambda} - \sin\lambda\sin\theta\dot{\theta}\\ \cos\theta\ \dot{\theta}\end{Bmatrix} = \begin{Bmatrix}-(yU + xzV)/\cos^2\theta\\ (xU - yzV)/\cos^2\theta\\ V\end{Bmatrix}$$

The position vector transforms as follows

$$\begin{Bmatrix}x\\ y\\ z\end{Bmatrix} = \begin{Bmatrix}\cos\boldsymbol{\lambda}\cos\boldsymbol{\theta}\\ \sin\boldsymbol{\lambda}\cos\boldsymbol{\theta}\\ \sin\boldsymbol{\theta}\end{Bmatrix} \qquad \begin{Bmatrix}\lambda\\ \theta\end{Bmatrix} = \begin{Bmatrix}\tan^{-1}(y/x)\\ \sin^{-1}(z)\end{Bmatrix}$$

Steps in the trapezoïdal rule

The winds $\mathbf{V}(t, \mathbf{r}_A)$ and $\mathbf{V}(t\text{-}\Delta t, \mathbf{r}_A)$ are assumed known at two time levels on the grid points $\mathbf{r}_A$ (one Crank-Nicolson step). We compute:

1) the module $\left|\mathbf{V}_A\right| = \sqrt{\dfrac{(U_A)^2 + (V_A)^2}{\cos^2 \boldsymbol{\theta}_A}}$

2) the angular displacement $\Delta_A = \dfrac{\Delta t}{2}\left|\dfrac{\mathbf{V}_A}{a}\right|$

3) the Cartesian Coordinates $(\dot{x}_A, \dot{y}_A, \dot{z}_A)$ of $\mathbf{V}_A = \mathbf{V}(t, \mathbf{r}_A)$

4) the Cartesian Coordinates of $\mathbf{r}_{DA} = \mathbf{r}_A \cos\Delta_A - \sin\Delta_A \mathbf{V}_A / \left|\mathbf{V}_A\right|$

5) $\mathbf{V}_D^i = \mathbf{V}\left(t - \Delta t, \mathbf{r}_D^{i-1}\right)$ by interpolation (Spherical Coordinates)

6) the Cartesian Coordinates of $\left(\dot{x}_D^i, \dot{y}_D^i, \dot{z}_D^i\right)$ of $\mathbf{V}_D^i$

7) the module $\left|\mathbf{V}_D^i\right| = \sqrt{\left(\dot{x}_D^i\right)^2 + \left(\dot{y}_D^i\right)^2 + \left(\dot{z}_D^i\right)^2}$

8) the angular displacement $\Delta_D^i = \dfrac{\Delta t}{2}\left|\dfrac{\mathbf{V}_D^i}{a}\right|$

9) the Cartesian Coordinates of $\mathbf{r}_D^i = \dfrac{\mathbf{r}_{DA}}{\cos\Delta_D^i} - \tan\Delta_D^i \dfrac{\mathbf{V}_D^i}{\left|\mathbf{V}_D^i\right|}$

10) the Spherical Coordinates $\left(\boldsymbol{\lambda}_D^i, \boldsymbol{\theta}_D^i\right)$ of $\mathbf{r}_D^i$

Only steps 5 to 10 need be repeated for the required number of iterations. *Remarkably*, these calculations to obtain $\mathbf{r}_D^i$ from $\mathbf{r}_D^{i-1}$ and $\mathbf{r}_{DA}$ using $\mathbf{V}_D^i$ with the trapezoidal rule are identical to those required to obtain $\mathbf{r}_M^i$ from $\mathbf{r}_M^{i-1}$ and $\mathbf{r}_A$ using $\mathbf{V}_M^i$ with the mid-point rule. Note that the module at departure is computed using Cartesian coordinates.

Appendix 15. Diagnostic calculation of vertical motion at initial time

There are two vertical motion field required at initial time. The first, $\dot{\zeta}$, is truly a diagnostic field. The second, *w*, is a diagnostic field only when the hydrostatic approximation is made; in the non-hydrostatic case, *w* could become an analyzed field.

1. Diagnostic calculation of $\dot{\zeta}$

From the continuity equation (**section 4**):

$$\frac{d}{dt}\ln\left(\pi\frac{\partial\ln\pi}{\partial\zeta}\right)+\nabla_\zeta\cdot\mathbf{V}_h+\frac{\partial\dot{\zeta}}{\partial\zeta}=0$$

transformed as follows

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial\pi}{\partial t}+\dot{\zeta}\frac{\partial\pi}{\partial\zeta}\right)+\nabla_\zeta\cdot(\pi\mathbf{V}_h)=0$$

and integrated

$$\frac{\partial\pi}{\partial t}+\dot{\zeta}\frac{\partial\pi}{\partial\zeta}+\int_{\zeta_T}^{\zeta}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta=0 \qquad (1)$$

$$\frac{\partial\pi_S}{\partial t}+\int_{\zeta_T}^{\zeta_S}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta=0$$

we derive an explicit relation for $\dot{\zeta}$ **:**

$$\dot{\zeta}\frac{\partial\ln\pi}{\partial\zeta}=\frac{B}{\pi_S}\int_{\zeta_T}^{\zeta_S}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta-\frac{1}{\pi}\int_{\zeta_T}^{\zeta}\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)d\zeta$$

In effect, $\ln\pi=\zeta+Bs$, hence

$$\frac{\partial\ln\pi}{\partial t}=B\frac{\partial s}{\partial t}=B\frac{\partial\ln\pi_S}{\partial t}$$

and

$$\frac{\partial\ln\pi}{\partial\zeta}=1+\frac{\partial B}{\partial\zeta}s$$

In discrete form we have

$$\dot{\zeta}_{k-\frac{1}{2}}\left(\frac{\partial\ln\pi}{\partial\zeta}\right)_{k-\frac{1}{2}}=\frac{B_{k-\frac{1}{2}}\mathcal{J}_{N+\frac{1}{2}}}{\pi_S}-\frac{\mathcal{J}_{k-\frac{1}{2}}}{\pi_{k-\frac{1}{2}}}; \qquad \mathcal{J}_{k-\frac{1}{2}}=\sum_{l=1}^{k-1}\left[\nabla_\zeta\cdot\left(\frac{\partial\pi}{\partial\zeta}\mathbf{V}_h\right)\right]_l\Delta\zeta_l$$

2. Diagnostic calculation of *w*

We use the approximation:

$$w \approx -\frac{\dot{\pi}}{g\boldsymbol{\rho}} \approx -\frac{RT}{g}\frac{\dot{\pi}}{\pi}$$

The approximation seems acceptable in general but note: at the model top $\dot{\boldsymbol{\pi}} = 0$ by construction while $w \neq 0$; similarly at the bottom, when the terrain is flat, $w = 0$ while $\dot{\boldsymbol{\pi}} \neq 0$ in general. We obtain an explicit relation for $\dot{\pi}$ again from the integrated continuity equation as follows:

$$\dot{\pi} = \frac{\partial \pi}{\partial t} + \mathbf{V}_h \cdot \nabla_{\boldsymbol{\zeta}} \pi + \dot{\boldsymbol{\zeta}} \frac{\partial \pi}{\partial \boldsymbol{\zeta}}$$

$$\dot{\pi} = \mathbf{V}_h \cdot \nabla_{\boldsymbol{\zeta}} \pi - \int_{\zeta_T}^{\zeta} \nabla_{\boldsymbol{\zeta}} \cdot \left(\frac{\partial \pi}{\partial \boldsymbol{\zeta}} \mathbf{V}_h \right) d\boldsymbol{\zeta}$$

$$\dot{\pi} = \pi B \mathbf{V}_h \cdot \nabla_{\boldsymbol{\zeta}} s - \int_{\zeta_T}^{\zeta} \nabla_{\boldsymbol{\zeta}} \cdot \left(\frac{\partial \pi}{\partial \boldsymbol{\zeta}} \mathbf{V}_h \right) d\boldsymbol{\zeta}$$

and it is convenient to replace the advection term by the difference of two divergences:

$$\dot{\pi} = \pi B \left[\nabla_{\boldsymbol{\zeta}} \cdot s\mathbf{V}_h - s\nabla_{\boldsymbol{\zeta}} \cdot \mathbf{V}_h \right] - \int_{\zeta_T}^{\zeta} \nabla_{\boldsymbol{\zeta}} \cdot \left(\frac{\partial \pi}{\partial \boldsymbol{\zeta}} \mathbf{V}_h \right) d\boldsymbol{\zeta}$$

In discrete form, we have:

$$\dot{\pi}_{ijk-\frac{1}{2}} = \pi_{ijk-\frac{1}{2}} B_{k-\frac{1}{2}} \left[\nabla_{\boldsymbol{\zeta}} \cdot s\overline{\mathbf{V}_h}^{\boldsymbol{\zeta}} - s\nabla_{\boldsymbol{\zeta}} \cdot \overline{\mathbf{V}_h}^{\boldsymbol{\zeta}} \right]_{ijk-\frac{1}{2}} - \mathcal{J}_{ijk-\frac{1}{2}} \qquad \begin{array}{c} (\mathbf{V}_h)_0 = (\mathbf{V}_h)_1 \\ (\mathbf{V}_h)_{N+1} = (\mathbf{V}_h)_N \end{array}$$

$$\text{N.B.} \quad \dot{\pi} = -g\boldsymbol{\rho}w + \underbrace{\left(\frac{\partial \pi}{\partial t} + \mathbf{V}_h \cdot \nabla \pi \right)}_{\text{(neglected tern)}} + \underbrace{\boldsymbol{\rho}\left(\frac{\partial \boldsymbol{\phi}}{\partial t} + \mathbf{V}_h \cdot \nabla \boldsymbol{\phi} \right)}_{\text{(neglected tern)}}$$

Table 1. The equations of GEM in 4 transformations

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$
$$\frac{d \ln T}{dt} - \kappa \frac{d \ln p}{dt} = \frac{Q}{c_p T}$$
$$\frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{V} = 0$$
$$\rho = \frac{p}{RT}$$

Vertical coordinate transformation: z to ζ (unspecified)

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}$$
$$\frac{\partial}{\partial z} \equiv \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}$$

1 →

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta}\right) = \mathbf{F}_h$$
$$\frac{dw}{dt} + RT\frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta} + g = F_w$$
$$\frac{d \ln T}{dt} - \kappa \frac{d \ln p}{dt} = \frac{Q}{c_p T}$$
$$\frac{d}{dt}\ln\left(\rho \frac{\partial z}{\partial \zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0$$
$$\frac{dz}{dt} - w = 0$$
$$\rho = \frac{p}{RT}$$

Vertical coordinate transformation: z to ζ (specified)

$$RT = -\frac{p}{\pi}\frac{\partial \phi}{\partial \ln \pi}$$
$$\phi = gz$$
$$\mu = \frac{p}{\pi}\frac{\partial \ln p}{\partial \ln \pi} - 1$$

2 →

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\mu)\nabla_\zeta \phi = \mathbf{F}_h$$
$$\frac{dw}{dt} - g\mu = F_w$$
$$\frac{d \ln T}{dt} - \kappa \frac{d \ln p}{dt} = \frac{Q}{c_p T}$$
$$\frac{d}{dt}\ln\left(\pi \frac{\partial \ln \pi}{\partial \zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0$$
$$\frac{d\phi}{dt} - gw = 0$$
$$1 + \mu - \frac{p}{\pi}\frac{\partial \ln p}{\partial \ln \pi} = 0$$
$$RT + \frac{p}{\pi}\frac{\partial \phi}{\partial \ln \pi} = 0$$

Going to model thermodynamic variables $\phi', q, s, \dot{\zeta}$

$$\phi' = \phi - \phi_*$$
$$\ln p = \ln \pi + q$$
$$\ln \pi = \zeta + Bs$$

③ →

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta (Bs+q) + (1+\mu)\nabla_\zeta \phi' = \mathbf{F}_h$$
$$\frac{dw}{dt} - g\mu = F_w$$
$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}(Bs+q) + \dot{\zeta}\right] = \frac{Q}{c_p T}$$
$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial \zeta}s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$
$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$
$$1 + \mu - e^q\left(1 + \frac{\partial q}{\partial(\zeta + Bs)}\right) = 0$$
$$\frac{T}{T_*} - e^q\left(1 - \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta + Bs)}\right) = 0$$

Discretizing in the vertical

$$\overline{(\ )}^\zeta$$
$$\delta_\zeta(\ )$$

④ →

—— $w, T, \dot{\zeta}, \mu$

- - - - $\mathbf{V}_h, q, \phi'$

—— $w, T, \dot{\zeta}, \mu$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^\zeta\nabla_\zeta (Bs+q) + \left(1+\overline{\mu}^\zeta\right)\nabla_\zeta \phi' = \mathbf{F}_h$$
$$\frac{dw}{dt} - g\mu = F_w$$
$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa\left(\overline{Bs+q}^\zeta\right)\right] - \kappa\dot{\zeta} = \frac{Q}{c_p T}$$
$$\frac{d}{dt}\left[Bs + \ln\left(1 + \delta_\zeta \overline{B}^\zeta s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^\zeta = 0$$
$$\frac{d\overline{\phi'}^\zeta}{dt} - RT_*\dot{\zeta} - gw = 0$$
$$1 + \mu - e^{\bar{q}^\zeta}\left[1 + \frac{\delta_\zeta q}{\delta_\zeta(\zeta + Bs)}\right] = 0$$
$$\frac{T}{T_*} - e^{\bar{q}^\zeta}\left[1 - \frac{\delta_\zeta(\phi'/RT_* + Bs)}{\delta_\zeta(\zeta + Bs)}\right] = 0$$

Table 2. The Equations of GEM *vertically discretized* on **Charney-Phillips grid**

$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\mu}^{\zeta}\right)\nabla_{\zeta}\phi' = \mathbf{F}_h$	----
$\frac{dw}{dt} - g\mu = F_w$	____
$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa\left(\overline{Bs+q}^{\zeta}\right)\right] - \kappa\dot{\zeta} = \frac{Q}{c_p T}$	____
$\frac{d}{dt}\left[Bs + \ln\left(1+\delta_{\zeta}\overline{B}^{\zeta}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \delta_{\zeta}\dot{\zeta} + \overline{\dot{\zeta}}^{\zeta} = 0$	----
$\frac{d\overline{\phi'}^{\zeta}}{dt} - RT_*\dot{\zeta} - gw = 0$	____
$1+\mu - e^{\overline{q}^{\zeta}}\left[1+\frac{\delta_{\zeta}q}{\delta_{\zeta}(\zeta+Bs)}\right] = 0$	____
$\frac{T}{T_*} - e^{\overline{q}^{\zeta}}\left[1-\frac{\delta_{\zeta}(\phi'/RT_* + Bs)}{\delta_{\zeta}(\zeta+Bs)}\right] = 0$	____

____	$w, T, \dot{\zeta}, \mu$
----	$\mathbf{V}_h, \phi', q$
____	$w, T, \dot{\zeta}, \mu$

$\mathbf{V}_h$: *horizontal wind*; f : *Coriolis parameter*

w : *vertical velicity*; $\kappa = R/c_p$

T : *temperature*; $T' = T\text{-}T_*$; $T_* = const$

ϕ : *geopotential*; $\phi' = \phi - \phi_*$; $\phi_* = -RT_*(\zeta - \zeta_S)$

$q = \ln(p/\pi)$: *non-hydrostatic log-pressure deviation*

p : *pressure*; π : *hydrostatic pressure*; $\partial\phi/\partial\pi = -RT/p$

$\mu = \partial p/\partial\pi - 1$: *ratio of vertical acceleration to gravitational acceleration*

$s = \ln(\pi_S / p_{ref})$: *log-surface-pressure*; B, $\mathrm{B} = \overline{B}^{\zeta}$: *metric parameter*

$\dot{\zeta} = d\zeta/dt$; ζ : *model vertical coordinate*

$\overline{(\ )}^{\zeta}$: *averaging operator*; δ_{ζ} : *differencing operator*

$p_T / p_{ref} = \eta_T < \eta < 1$: *specified* π*-like model* levels; $p_{ref} = 10^5\,Pa$

$$\zeta = \zeta_S + \ln(\eta)$$

$\ln p_T = \zeta_T \le \zeta \le \zeta_S = \ln p_{ref}$: *calculated* $\ln\pi$-like *model* levels

$$\ln\pi = A + Bs$$

$$A = \zeta;\quad B = \lambda^r;\quad 0 < r = r_{\max} - (r_{\max} - r_{\min})\lambda < 200;\quad \lambda = \max\left[\frac{\zeta - \zeta_U}{\zeta_S - \zeta_U}, 0\right];\quad \zeta_U \ge \zeta_T$$

Boundary Conditions: $\dot{\zeta}_S = \dot{\zeta}_T = 0$ $[q_T = \ln(p_{top}/\pi_T) = 0; \phi_S = gz_{topo}; p_{top} = \mathrm{const}]$