GEM4.1: A non-hydrostatic atmospheric model (Euler equations)

in terrain-following vertical coordinate ($\zeta = \zeta_S + \ln\eta$; $\zeta_S = \ln p_{ref}$)
of the **log-hydrostatic-pressure** type ($\ln\pi = \zeta + Bs$; $s = \ln(\pi_S/p_{ref})$)

vertically discretized on a Charney-Phillips grid
with simple differences and means

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other significant contributions:

Sylvie Gravel: Semi-Lagrangian Scheme
Abdessamad Qaddouri: Non-symmetric Elliptic Solver
Stéphane Chamberland: Physics Interface
Lubos Spacek: Physics
Vivian Lee: Input/Output, Cascade, Acid test
Michel Desgagné: Coordination

Revision 1

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*N.B. In this document, only **vertical** discretization is discussed in depth. **Time** discretization is sketched for convenience. Aspects of **horizontal** discretization are presented in **Appendix** 10.*

PREFACE

GEM4.1: work in progress …

Reduction of noise in GEM was the main motivation for the present project consisting in *the introduction of vertical staggering* (Charney-Phillips grid). It was deemed the first and primary ingredient to achieve this goal. In effect, there are numerical modes which were theoretically diagnosed on the previous un-staggered grid which are absent from the new one. As a first step therefore in this project, only the grid was changed. Everything else, the equations, the independent as well as the dependent variables, were kept unchanged. Very *positive results* were obtained *with respect to noise*. But there remain problems, in particular an accuracy problem in the hydrostatic relation at upper levels when the true resolution (in terms of height) is insufficient.

Improving the accuracy of the hydrostatic relation using *logarithmic differencing* wherever appropriate was therefore the goal of a second step. The *results* from this modification of the code were very satisfying with *improved scores in the stratosphere*.

With this incentive, it was tempting to try and implement *a full log-hydrostatic-pressure coordinate*, ζ. A theoretical advantage of ζ is its linear relationship with ln*p*, [$\ln p = \ln(p/\boldsymbol{\pi}) + \ln(\boldsymbol{\pi}/\boldsymbol{\pi}_*) + \ln\boldsymbol{\pi}_* = q + Bs + \boldsymbol{\zeta}$]. Along with the fact that $q = \ln(p/\boldsymbol{\pi})$ and $s = \ln(\boldsymbol{\pi}_S / p_{ref})$ are already model variables, this greatly simplifies the linearization of model equations. Again the accuracy of the hydrostatic equation is improved since the finite differences not only are calculated logarithmically but also become defined at logarithmic mid-points. This third step though has *little impact* on model performance.

A recent and important development: it was discovered that the initial staggered version of the semi-Lagrangian scheme, averaging the departure positions for variables arriving on thermodynamic levels, resulted in significant loss of kinetic energy. *Calculating independent departure positions* is rather the thing to do.

Finally, a word concerning a secondary motivation for the project, namely the resolution of accuracy and noise problems encountered in the simulation of non-hydrostatic mountain waves, specifically what we call Schär's case: considerable understanding was achieved but no fully satisfactory solution is available.

For the sake of clarity, a lot of the details of the progressive model transformation from GEM3 to GEM4, contained in the original version of this document, **GEM4.0,** which remains available for consultation, have been removed from the present one.

1) The meteorological equations in height coordinate

- 4 independent variables: t, $\mathbf{r}=(\mathbf{r}_h, z)$
- 6 dependent variables: $\mathbf{V}=(\mathbf{V}_h, w)$, T, ρ, p
- 6 scalar equations: $\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$ $\frac{d \ln T}{dt} - \frac{R}{c_p}\frac{d \ln p}{dt} = \frac{Q}{c_p T}$ $\frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{V} = 0$ $\rho = \frac{p}{RT}$

- There are: 5 prognostic equations (momentum + energy + mass conservation),
  1 diagnostic equation (perfect gas law).

N.B. The Coriolis force is approximated (traditional meteorological approximations apply).

N.B. Many more approximations are implied if we consider that the atmospheric substance contains, in addition to dry air, not only a variable quantity of water vapor but also condensed water and precipitations. The above equations are valid under the assumptions of *dynamic* (precipitations falling at terminal velocity) and *thermodynamic* (neglecting temperature differences between air and hydrometeors) *equilibrium* and neglecting precipitation fluxes. Equations for the displacement and evolution of the hydrometeors are required to complete the system.

N.B. In the above equations the coefficients R and c_p are variable. The introduction of virtual temperature (replacing RT by $R_d T_v$ where R_d is now constant) and approximating the ratio $\kappa=R/c_p$ by the constant ratio $\kappa_d=R_d/c_{pd}$ lead to further simplifications (see **Appendix 1.** Virtual temperature).

2) The equations transformed to generalized η-coordinate

- Note the necessary decomposition of vector equations into their horizontal/vertical components due to the different horizontal/vertical transformation rules.

2 transformation rules: $\nabla_z \equiv \nabla_\eta - \nabla_\eta z \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$; $\qquad \frac{\partial}{\partial z} \equiv \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$

- 4 independent variables : t, $\mathbf{r}_h$, η

- 8 dependent variables: $\mathbf{V}_h$, w, T, ρ, p, $\dot{\eta}$, z

- 8 equations (6 prognostic and 2 diagnostic):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_{\boldsymbol{\eta}} \ln p - \nabla_\eta z \frac{\partial \boldsymbol{\eta}}{\partial z}\frac{\partial \ln p}{\partial \boldsymbol{\eta}}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial \boldsymbol{\eta}}{\partial z}\frac{\partial \ln p}{\partial \boldsymbol{\eta}} + g = F_w$$

$$\frac{d \ln T}{dt} - \boldsymbol{\kappa}\frac{d \ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\ln\left(\boldsymbol{\rho}\frac{\partial z}{\partial \boldsymbol{\eta}}\right) + \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\boldsymbol{\eta}}}{\partial \boldsymbol{\eta}} = 0$$

$$\frac{dz}{dt} - w = 0$$

$$\boldsymbol{\rho} = \frac{p}{RT}$$

$$z \equiv z(\boldsymbol{\eta}, \mathbf{r}_h, t)$$

- Were added then: 1 prognostic equation (***dz/dt*=w**) for varying height in space and time,
1 diagnostic equation (yet to be specified) defining the coordinate $\boldsymbol{\eta}$.

- the continuity equation is the only one requiring more than simple manipulation:

$$w = \frac{dz}{dt} = \frac{\partial z}{\partial t} + \mathbf{V}_h \cdot \nabla_\eta z + \dot{\eta}\frac{\partial z}{\partial \eta}$$

$$\frac{\partial \eta}{\partial z}\frac{\partial w}{\partial \eta} = \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial t} + \mathbf{V}_h \cdot \nabla_\eta z + \dot{\eta}\frac{\partial z}{\partial \eta}\right) = \frac{\partial \eta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \eta}\cdot \nabla_\eta z + \frac{\partial \dot{\eta}}{\partial \eta} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \eta}\right)$$

hence

$$\nabla_z \cdot \mathbf{V}_h + \frac{\partial w}{\partial z} = \nabla_\eta \cdot \mathbf{V}_h - \frac{\partial \eta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \eta}\cdot \nabla_\eta z + \frac{\partial \eta}{\partial z}\frac{\partial w}{\partial \eta} = \nabla_\eta \cdot \mathbf{V}_h + \frac{\partial \dot{\eta}}{\partial \eta} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \eta}\right)$$

- See **Appendix 2** for some details on transformation rules.

3) Eliminating $\boldsymbol{\rho}$ introducing $\ln\pi$, log-hydrostatic pressure, eliminating z defining the geopotential ϕ and adding μ (ratio of vertical acceleration to gravitational acceleration)

$$\frac{\partial \boldsymbol{\pi}}{\partial z} = -g\boldsymbol{\rho}\,; \qquad RT = -\frac{p}{\boldsymbol{\pi}}\frac{\partial \boldsymbol{\phi}}{\partial \ln \boldsymbol{\pi}}\,; \quad \boldsymbol{\phi} = gz\,; \qquad \boldsymbol{\mu} = \frac{p}{\boldsymbol{\pi}}\frac{\partial \ln p}{\partial \ln \boldsymbol{\pi}} - 1$$

- 9 dependent variables: $\mathbf{V}_h$, w, T, p, $\dot{\eta}$, ϕ, μ, π

- 9 equations (added diagnostic equation for μ):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_{\boldsymbol{\eta}} \ln p + (1+\boldsymbol{\mu})\nabla_{\boldsymbol{\eta}}\boldsymbol{\phi} = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$

$$\frac{d\ln T}{dt} - \boldsymbol{\kappa}\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\ln\left(\boldsymbol{\pi}\frac{\partial \ln \boldsymbol{\pi}}{\partial \boldsymbol{\eta}}\right) + \nabla_{\boldsymbol{\eta}}\cdot\mathbf{V}_h + \frac{\partial \dot{\boldsymbol{\eta}}}{\partial \boldsymbol{\eta}} = 0$$

$$\frac{d\boldsymbol{\phi}}{dt} - gw = 0$$

$$1 + \boldsymbol{\mu} - \frac{p}{\boldsymbol{\pi}}\frac{\partial \ln p}{\partial \ln \boldsymbol{\pi}} = 0$$

$$RT + \frac{p}{\boldsymbol{\pi}}\frac{\partial \boldsymbol{\phi}}{\partial \ln \boldsymbol{\pi}} = 0$$

$$\ln \boldsymbol{\pi} \equiv \ln \boldsymbol{\pi}(\boldsymbol{\eta}, \mathbf{r}_h, t)$$

- N.B. For the rest of the presentation, the ***physical forcings*** $\mathbf{F}_h$, F_w, Q will be excluded and the ***parameters*** R and $\boldsymbol{\kappa}$ will be treated as constants.

4) The new ζ-coordinate for GEM4.1 is lnπ-like

$$\zeta = \zeta_S + \ln\eta; \qquad \zeta_S = \ln p_{ref}; \quad p_{ref} = 10^5$$

$$\ln\pi = A(\zeta) + B(\zeta)s; \qquad s = \ln\pi_S - \zeta_S = \ln(\pi_S / p_{ref})$$

$$A = \zeta; \qquad B = \left(\frac{\zeta - \zeta_T}{\zeta_S - \zeta_T}\right)^r \qquad \begin{cases} \zeta_T = \ln p_{top} \\ 0 \le r = r_{\max} - (r_{\max} - r_{\min})\left(\frac{\zeta - \zeta_T}{\zeta_S - \zeta_T}\right) \le 30 \end{cases}$$

$$\ln\pi = \zeta + B(\zeta)s$$

transformation rules: $\nabla_\eta \equiv \nabla_\zeta - \nabla_\zeta \boldsymbol{\eta} \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{\eta}} \frac{\partial}{\partial \boldsymbol{\zeta}} = \nabla_\zeta;$ $\qquad \frac{\partial}{\partial \boldsymbol{\eta}} \equiv \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{\eta}} \frac{\partial}{\partial \boldsymbol{\zeta}} = \frac{1}{\boldsymbol{\eta}} \frac{\partial}{\partial \boldsymbol{\zeta}}$

- 9 dependent variables: $\mathbf{V}_h, w, T, p, \dot{\zeta}, \phi, \mu, \pi$

- 9 equations:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1 + \boldsymbol{\mu})\nabla_\zeta \boldsymbol{\phi} = 0$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = 0$$

$$\frac{d\ln T}{dt} - \boldsymbol{\kappa}\frac{d\ln p}{dt} = 0$$

$$\frac{d}{dt}\ln\left(\boldsymbol{\pi}\frac{\partial \ln\boldsymbol{\pi}}{\partial\boldsymbol{\zeta}}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial\dot{\boldsymbol{\zeta}}}{\partial\boldsymbol{\zeta}} = 0$$

$$\frac{d\boldsymbol{\phi}}{dt} - gw = 0$$

$$1 + \boldsymbol{\mu} - \frac{p}{\boldsymbol{\pi}}\frac{\partial \ln p}{\partial \ln\boldsymbol{\pi}} = 0$$

$$RT + \frac{p}{\boldsymbol{\pi}}\frac{\partial\boldsymbol{\phi}}{\partial \ln\boldsymbol{\pi}} = 0$$

$$\ln\boldsymbol{\pi} \equiv \boldsymbol{\zeta} + Bs$$

- **obviously,** at this point, the form of the equations in ζ and η coordinates is identical

N.B. $p_{top} / p_{ref} < \eta < 1$ is now but a label to characterize the model levels. Another way to characterize the levels would be to use a number, H, having the units of height and corresponding approximately to *model level height* (above ground):

$$\zeta = \zeta_S - H / H_{ref}; \qquad H_{ref} = \left(\frac{RT}{g}\right)_{ref} = \frac{16000}{\ln 10} \approx 6950m$$

See **Appendix 3** for more information on the metric parameter B.

5) Perturbation thermodynamic variables, T', ϕ', q, and simplifications

Introducing the logarithm of the non-hydrostatic pressure perturbation $q=\ln(p/\pi)$ and perturbation variables T' and ϕ'. Eliminating p, ϕ and π. We keep T for convenience.

$$T' = T - T_*; \qquad T_* = const$$
$$\phi' = \phi - \phi_*; \qquad \phi_*(\zeta) = -RT_*(\zeta - \zeta_S)$$
$$\ln p = \ln \pi + q = \zeta + Bs + q$$

- **8 variables**: $\mathbf{V}_h, w, T$ or $T', q, \left(\dot{\zeta}, s\right), \phi', \mu$, **final number**

- **8 equations** [6 prognostic & 2 diagnostic], **final form** ready for linearization:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\boldsymbol{\mu})\nabla_\zeta \boldsymbol{\phi}' = 0$$
$$\frac{dw}{dt} - g\boldsymbol{\mu} = 0$$
$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}(Bs+q)\right] - \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}} = 0$$
$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial \boldsymbol{\zeta}} s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \left(\frac{\partial}{\partial \boldsymbol{\zeta}} + 1\right)\dot{\boldsymbol{\zeta}} = 0$$
$$\frac{d\boldsymbol{\phi}'}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$
$$1 + \boldsymbol{\mu} - e^q\left(\frac{\partial q}{\partial(\boldsymbol{\zeta} + Bs)} + 1\right) = 0$$
$$\frac{T}{T_*} - e^q \frac{\partial(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*)}{\partial(\boldsymbol{\zeta} + Bs)} = 0$$

N. B. S is a 2-D variable and $\dot{\zeta}$ vanishes at the surface. The combination ($\dot{\zeta}$, S) may therefore be considered to constitute a single 3-D variable.

N.B. $\phi' + RT_*(Bs+q) = \phi' + RT_*(\ln p - \zeta) = \phi + RT_* \ln(p/p_{ref}) = P$, which may be called *generalized pressure*, a variable which will be convenient to invoke later on.

6) Boundary Conditions

The model top (subscript $_T$) and bottom (subscript $_S$ for earth's surface when talking of the bottom of the atmosphere), are defined to be material surfaces. Therefore we have the following **two** boundary conditions:

$$\dot{\zeta}_T = \dot{\zeta}(\zeta_T) = 0$$
$$\dot{\zeta}_S = \dot{\zeta}(\zeta_S) = 0$$

In addition, the behavior of theses surfaces must be specified and this will lead to **one additional** condition in the non-hydrostatic case. The bottom surface is assumed to be corrugated but immobile. In effect, the bottom geopotential ϕ_S is usually taken to vary with the geographical position (terrain-following coordinate system) but to remain fixed in time: $\partial\phi_S / \partial t = 0$. This though does not imply that the vertical velocity at the surface vanishes and therefore $gw_S = [d\phi / dt]_S \neq 0$ generally. At the top, we consider a *flexible surface* whereby a constant top pressure:

$$p_{top} = \pi_T$$

is assumed to be maintained. This is automatic in the hydrostatic case since the top surface pressure cannot be anything other than a material hydrostatic pressure surface.

In the non-hydrostatic case, to impose a constant top pressure equal to the constant top hydrostatic pressure surface provides a boundary condition for the top pressure. In terms of the non-hydrostatic pressure variable *q*, this becomes:

$$q_T = \ln\left(p_{top} / \pi_T\right) = 0$$

The surface is nevertheless free to move in response to this artificially imposed pressure p_{top}.

N.B. Open boundary conditions are of course a possibility: see **Appendix 9.** *Open top* boundary conditions.

7) Vertical discretization with staggering

For vertical discretization, the following choice is made:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\boldsymbol{\mu}}^{\zeta}\right)\nabla_{\zeta}\boldsymbol{\phi}' = 0$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left(\mathrm{B}s + \overline{q}^{\zeta}\right)\right] - \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}} = 0$$

$$\frac{d}{dt}\left[\overline{\mathrm{B}}^{\zeta}s + \ln\left(1+\boldsymbol{\delta}_{\zeta}\mathrm{B}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \boldsymbol{\delta}_{\zeta}\dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta} = 0$$

$$\frac{d\overline{\boldsymbol{\phi}'}^{\zeta}}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$

$$1 + \boldsymbol{\mu} - e^{\overline{q}^{\zeta}}\left[\frac{\boldsymbol{\delta}_{\zeta}q}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} + 1\right] = 0$$

$$\frac{T}{T_*} + e^{\overline{q}^{\zeta}}\left[\frac{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\phi}'/RT_* + Bs)}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} - 1\right] = 0$$

In other words, the derivatives are replaced by simple finite differences represented by the operator δ_{ζ} and averaging operators represented by over bars are introduced where required. From the notation, it may be gathered that $\mathbf{V}_h, q, \boldsymbol{\phi}', B$ are defined on the same levels to be called *full* or **momentum** levels. They are staggered with respect to $w, T, \boldsymbol{\mu}, \dot{\boldsymbol{\zeta}}, \mathrm{B}$ placed on *half* or **thermodynamic** levels. Note the distinction made between the known metric parameter defined on full (B) and half (B) levels, discrete operations only being invoked when required by later manipulations. With this staggering, double operations on dependent variables are severely reduced. No difference is calculated over more than two levels. The number of averaging operators is minimized. In the horizontal momentum equations, they occur on non-linear terms only; in the hydrostatic case (with q=μ=0 and w dropping out of the system), only one averaging operator remains on linear terms, namely on $\dot{\zeta}$ in the continuity equation. *Details of the discretization* are given in **Appendices 4a, 4b, 5** and **6** (but read these after the rest of the main document). Taking into account the boundary conditions, it is natural to have half levels rather than full levels coincide with the top and bottom. This essentially, though not fully, completes the description of the *vertical grid.* See **Figure 1**, next page.

N.B. For B we have chosen the numerical definition: $\mathrm{B} = \overline{B}^{\zeta}$, instead of the analytical definition: $\mathrm{B} = B(\overline{\boldsymbol{\zeta}}^{\zeta})$. This is a convenient but arbitrary choice.

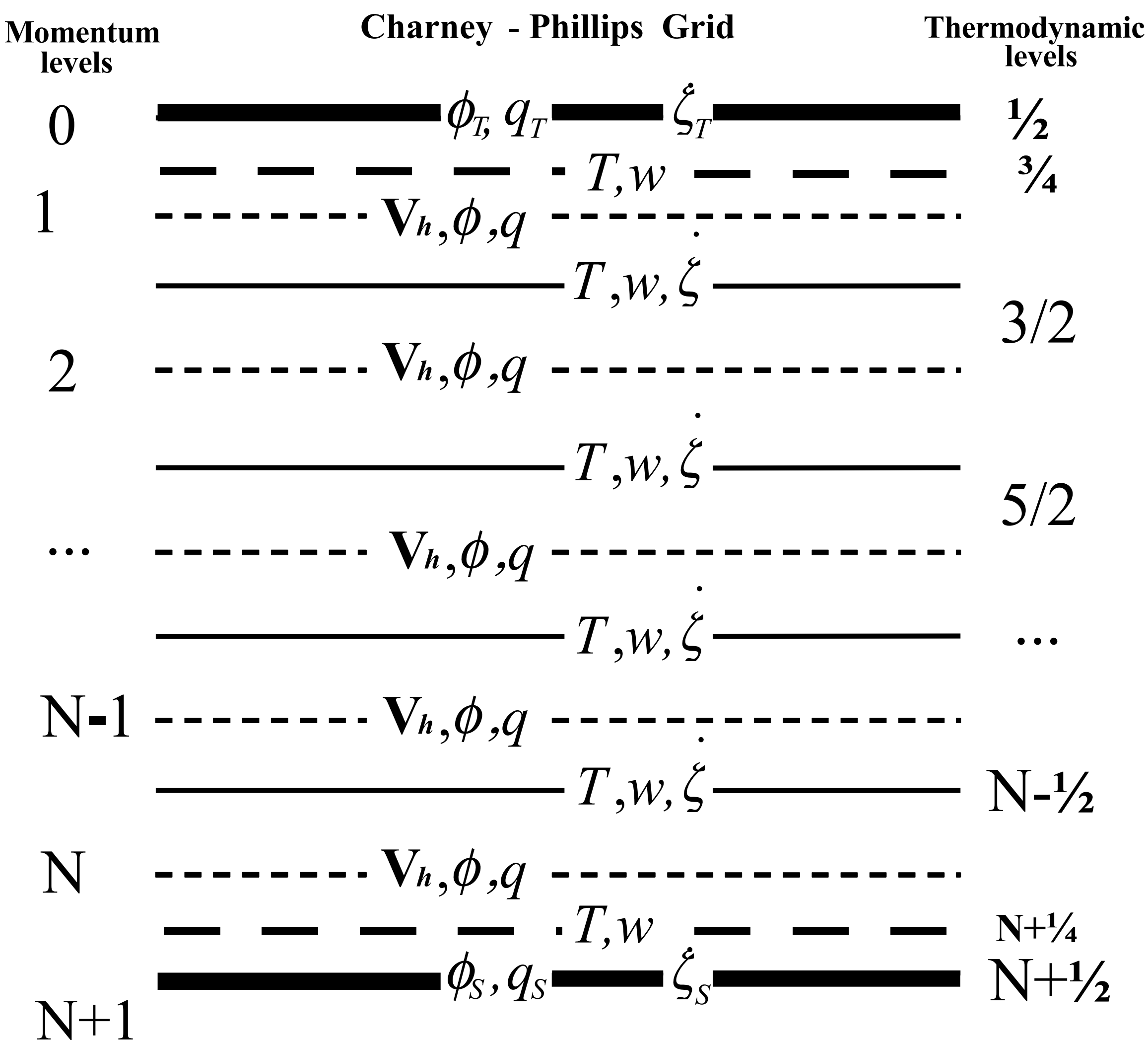

Figure 1. The Charney-Phillips grid, giving the position occupied by each variable in the vertical domain. The model is composed of N layers, inside of which (in the middle of which only if the layers are equal) are the momentum levels [1,2,…,N] where the wind components *U* and *V*, the geopotential ϕ and q are positioned. ϕ and q are also defined on the boundaries (top level 0 and surface level N+1). These N layers are delimited by N-1 interfaces corresponding to N-1 so-called thermodynamic levels [3/2,…,N-1/2] where are positioned the temperature T and the two vertical motion fields w and $\dot{\zeta}$, exactly in the middle of the momentum levels. $\dot{\zeta}$ also has 2 additional levels [½ and N+½] corresponding to the top and bottom surfaces. T and w also have 2 additional levels [¾ and N+¼] positioned exactly in between, respectively, the top surface and first momentum level and the last momentum level and bottom surface.

See **Table 1**, page 57, for a summary of the model equations and transformations.

8) Semi-Lagrangian **Implicit** time discretization (n.b. not Semi-Implicit)

- Approximating the ***substantial derivatives*** and averaging the ***dynamical forcings***, each of the equations (index *i*) may be formally written as follows:

$$\boxed{\frac{dF_i}{dt} + G_i = 0}$$

$$\frac{dF_i}{dt} \approx \frac{F_i^A - F_i^D}{\Delta t}; \quad G_i \approx b^A G_i^A + \left(1 - b^A\right) G_i^D; \qquad b^A \approx 0.5:0.6 \ \text{(off-centering)}$$

$A: \ (\mathbf{r}, t)$ Arrival

$D: \ (\mathbf{r} - \Delta\mathbf{r}, t - \Delta t)$ Departure

- Separating the time levels ($\boldsymbol{\tau} = \Delta t b^A; \boldsymbol{\beta} = \left(1 - b^A\right)/b^A$)

$$\frac{F_i^A - F_i^D}{\Delta t} + b^A G_i^A + \left(1 - b^A\right) G_i^D = 0$$

$$\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A = \frac{F_i^D}{\boldsymbol{\tau}} - \boldsymbol{\beta} G_i^D \equiv R_i$$

- Decomposing the left-hand side into linear and residual non-linear parts

$$\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A = L_i + N_i = R_i$$

$$L_i \equiv \left(\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A\right)_{lin}$$

$$N_i \equiv \frac{F_i^A}{\boldsymbol{\tau}} + G_i^A - \left(\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A\right)_{lin}$$

- Defining the solution method (a Crank-Nicholson scheme)

$$L_i = R_i - N_i$$

Iterating (*jter*: *departure loop, iter*: *non-linear loop*) :

Do jter=1,2
Do iter=1,2

$$\left(L_i\right)^{iter,jter} = \left(R_i\right)^{jter} - \left(N_i\right)^{iter-1,jter}; \quad \left(N_i\right)^{0,1} = N_i\left(\mathbf{r}, t - \Delta t\right)$$

end do
end do

$$\left(R_i\right)^{jter} = R_i\left(t - \Delta t, \mathbf{r} - \Delta\mathbf{r}^{jter}\right); \quad \Delta\mathbf{r}^{jter} = \tfrac{\Delta t}{2}\left(\mathbf{v}(t - \Delta t) + \mathbf{v}(t)^{jter}\right)\left(\mathbf{r} - \tfrac{\Delta\mathbf{r}^{jter-1}}{2}\right)$$

$$\mathbf{v}(t)^1 = \mathbf{v}(t - \Delta t); \ \Delta\mathbf{r}^0 \ \text{from previous timestep}$$

9) The *F's* and the *G's*

$$\mathbf{F}_h \equiv \mathbf{V}_h \qquad \mathbf{G}_h \equiv f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\boldsymbol{\mu}}^{\zeta}\right)\nabla_{\zeta}\boldsymbol{\phi}'$$

$$F_w \equiv w \qquad G_w \equiv -g\boldsymbol{\mu}$$

$$F_T \equiv \ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left(\mathrm{B}s + \overline{q}^{\zeta}\right) \qquad G_T \equiv -\boldsymbol{\kappa}\dot{\boldsymbol{\zeta}}$$

$$F_C \equiv \overline{\mathrm{B}}^{\zeta} s + \ln\left(1+\boldsymbol{\delta}_{\zeta}\mathrm{B}s\right) \qquad G_C \equiv \nabla_{\zeta}\cdot\mathbf{V}_h + \boldsymbol{\delta}_{\zeta}\dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta}$$

$$F_{\phi} \equiv \overline{\boldsymbol{\phi}'}^{\zeta} \qquad G_{\phi} \equiv -RT_*\dot{\boldsymbol{\zeta}} - gw$$

$$F_{\mu} \equiv 0 \qquad G_{\mu} \equiv 1 + \boldsymbol{\mu} - e^{\overline{q}^{\zeta}}\left[\frac{\boldsymbol{\delta}_{\zeta} q}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} + 1\right] = 0$$

$$F_H \equiv 0 \qquad G_H \equiv \frac{T}{T_*} + e^{\overline{q}^{\zeta}}\left[\frac{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\phi}'/RT_* + Bs)}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} - 1\right] = 0$$

N.B. Since $F_{\mu}=F_H=0$ and $G_{\mu}=G_H=0$, then of course $R_{\mu}=R_H=0$.

The role of diagnostic equations is to abbreviate other equations. If, in the 6 prognostic equations, we replace the symbols μ and T by their definitions, the diagnostic equations as well as the associated variables vanish.

10) The Left-Hand Side terms: $L_i + N_i$

$$\boxed{L_i + N_i \equiv \frac{F_i^A}{\boldsymbol{\tau}} + G_i^A}$$

Prognostic (dropping the superscript *A*):

$$\mathbf{L}_h + \mathbf{N}_h = \frac{\mathbf{V}_h}{\boldsymbol{\tau}} + f\mathbf{k}\times\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\boldsymbol{\mu}}^{\zeta}\right)\nabla_{\zeta}\boldsymbol{\phi}'$$

$$L_w + N_w = \frac{w}{\boldsymbol{\tau}} - g\boldsymbol{\mu}$$

$$L_T + N_T = \frac{1}{\boldsymbol{\tau}}\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left(\dot{\boldsymbol{\zeta}} + \frac{\mathrm{B}s + \overline{q}^{\zeta}}{\boldsymbol{\tau}}\right)$$

$$L_C + N_C = \frac{1}{\boldsymbol{\tau}}\left[\overline{\mathrm{B}}^{\zeta}s + \ln\left(1+\boldsymbol{\delta}_{\zeta}\mathrm{B}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \boldsymbol{\delta}_{\zeta}\dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta}$$

$$L_{\phi} + N_{\phi} = \frac{\overline{\boldsymbol{\phi}'}^{\zeta}}{\boldsymbol{\tau}} - RT_*\dot{\boldsymbol{\zeta}} - gw$$

Diagnostic:

$$L_{\mu} + N_{\mu} = 1 + \boldsymbol{\mu} - e^{\overline{q}^{\zeta}}\left[\frac{\boldsymbol{\delta}_{\zeta}q}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} + 1\right] = 0$$

$$L_H + N_H = \frac{T}{T_*} + e^{\overline{q}^{\zeta}}\left[\frac{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\phi}'/RT_* + Bs)}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} - 1\right] = 0$$

11) The linear Left-Hand Side terms: L_i

$$L_i \equiv \left[\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A\right]_{linear}$$

Linearizing (approximating the logarithms [$\ln(1+\alpha) \approx \alpha$], the exponentials [$e^{\boldsymbol{\alpha}} \approx 1 + \boldsymbol{\alpha}$] and the products [$(1+\alpha)(1+\beta)^{\pm 1} \approx 1 + \alpha \pm \beta$]; note the Coriolis term $f\mathbf{k} \times \mathbf{V}_h$ is treated as if it was a non-linear term) yields:

$$\mathbf{L}_h = \frac{\mathbf{V}_h}{\boldsymbol{\tau}} + \nabla_\zeta \left[\boldsymbol{\phi}' + RT_*(Bs + q)\right]$$

$$L_w = \frac{w}{\boldsymbol{\tau}} - g\boldsymbol{\mu}$$

$$L_T = \frac{T'}{\boldsymbol{\tau} T_*} - \boldsymbol{\kappa}\left(\dot{\boldsymbol{\zeta}} + \frac{\mathrm{B}s + \overline{q}^{\zeta}}{\boldsymbol{\tau}}\right)$$

$$L_C = \frac{1}{\boldsymbol{\tau}}\left[\overline{\mathrm{B}}^{\zeta} s + \boldsymbol{\delta}_\zeta \mathrm{B}s\right] + \nabla_\zeta \cdot \mathbf{V}_h + \boldsymbol{\delta}_\zeta \dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta}$$

$$L_\phi = \frac{\overline{\boldsymbol{\phi}'}^{\zeta}}{\boldsymbol{\tau}} - RT_* \dot{\boldsymbol{\zeta}} - gw$$

$$L_\mu = \boldsymbol{\mu} - \left(\boldsymbol{\delta}_\zeta q + \overline{q}^{\zeta}\right) \neq 0$$

$$L_H = \frac{T'}{T_*} - \overline{q}^{\zeta} + \frac{\boldsymbol{\delta}_\zeta(\boldsymbol{\phi}' + RT_* Bs)}{RT_*} \neq 0$$

12) The non-linear Left-Hand side terms, N_i, are the left-over differences

$$N_i = \left[\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A\right] - \left[\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A\right]_{linear} = \left[\frac{F_i^A}{\boldsymbol{\tau}} + G_i^A\right] - L_i$$

and therefore:

$$\begin{aligned}
\mathbf{N}_h &= f\mathbf{k}\times\mathbf{V}_h + R\overline{T'}^{\zeta}\nabla_{\zeta}\left(Bs+q\right) + \overline{\boldsymbol{\mu}}^{\zeta}\nabla_{\zeta}\boldsymbol{\phi}' \\
N_w &= 0 \\
N_T &= \frac{1}{\boldsymbol{\tau}}\left[\ln\left(\frac{T}{T_*}\right) - \frac{T'}{T_*}\right] \\
N_C &= \frac{1}{\boldsymbol{\tau}}\left[\ln\left(1+\boldsymbol{\delta}_{\zeta}\mathrm{B}s\right) - \boldsymbol{\delta}_{\zeta}\mathrm{B}s\right] \\
N_{\phi} &= 0 \\
N_{\mu} &= -\left(\boldsymbol{\mu} - \boldsymbol{\delta}_{\zeta}q - \overline{q}^{\zeta}\right) = -L_{\mu} \\
N_H &= -\left(\frac{T'}{T_*} - \overline{q}^{\zeta} + \frac{\boldsymbol{\delta}_{\zeta}\left(\boldsymbol{\phi}' + RT_*Bs\right)}{RT_*}\right) = -L_H
\end{aligned}$$

13) Elimination of the diagnostic equations from the solution system

As noted above, $R_\mu = R_H = 0$. It is then convenient to immediately eliminate the two diagnostic equations, involving the diagnostic variables μ and T', from the Left-Hand side terms, i.e. to eliminate L_μ, L_H and N_μ, N_H. We are left with 6 *basic* equations for the linear system:

$$\mathbf{L}_h = \frac{\mathbf{V}_h}{\boldsymbol{\tau}} + \nabla_\zeta \left[\boldsymbol{\phi}' + RT_* \left(Bs + q \right) \right]$$

$$L_w + gL_{\boldsymbol{\mu}} \equiv L'_w = \frac{w}{\boldsymbol{\tau}} - g\left(\boldsymbol{\delta}_\zeta q + \overline{q}^\zeta \right)$$

$$L_T - \frac{L_H}{\boldsymbol{\tau}} \equiv L'_T = \frac{\overline{q}^\zeta}{\boldsymbol{\tau}} - \frac{\boldsymbol{\delta}_\zeta \left(\boldsymbol{\phi}' + RT_* Bs \right)}{\boldsymbol{\tau} RT_*} - \boldsymbol{\kappa} \left(\dot{\boldsymbol{\zeta}} + \frac{\mathrm{B}s + \overline{q}^\zeta}{\boldsymbol{\tau}} \right)$$

$$L_C = \frac{1}{\boldsymbol{\tau}} \left[\overline{\mathrm{B}}^\zeta s + \boldsymbol{\delta}_\zeta \mathrm{B}s \right] + \nabla_\zeta \cdot \mathbf{V}_h + \boldsymbol{\delta}_\zeta \dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^\zeta$$

$$L_{\boldsymbol{\phi}} = \frac{\overline{\boldsymbol{\phi}'}^\zeta}{\boldsymbol{\tau}} - RT_* \dot{\boldsymbol{\zeta}} - gw$$

Similarly for the non-linear system we have

$$\mathbf{N}_h = f\mathbf{k} \times \mathbf{V}_h + R\overline{T'}^\zeta \nabla_\zeta \left(Bs + q \right) + \overline{\boldsymbol{\mu}}^\zeta \nabla_\zeta \boldsymbol{\phi}'$$

$$N_w + gN_{\boldsymbol{\mu}} \equiv N'_w = -g\left(\boldsymbol{\mu} - \boldsymbol{\delta}_\zeta q - \overline{q}^\zeta \right)$$

$$N_T - \frac{N_H}{\boldsymbol{\tau}} \equiv N'_T = \frac{1}{\boldsymbol{\tau}} \left[\ln\left(\frac{T}{T_*} \right) - \overline{q}^\zeta + \frac{\boldsymbol{\delta}_\zeta \left(\boldsymbol{\phi}' + RT_* Bs \right)}{RT_*} \right]$$

$$N_C = \frac{1}{\boldsymbol{\tau}} \left[\ln\left(1 + \boldsymbol{\delta}_\zeta \mathrm{B}s \right) - \boldsymbol{\delta}_\zeta \mathrm{B}s \right]$$

$$N_{\boldsymbol{\phi}} = 0$$

14) The Previous time step on the Right-Hand Sides: R_i

$$R_i \equiv \frac{F_i^D}{\boldsymbol{\tau}} - \boldsymbol{\beta} G_i^D$$

(dropping the superscript *D*)

$$\mathbf{R}_h = \frac{\mathbf{V}_h}{\boldsymbol{\tau}} \qquad - \boldsymbol{\beta}\left[f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta} \nabla_{\zeta} (Bs + q) + \left(1 + \overline{\boldsymbol{\mu}}^{\zeta}\right) \nabla_{\zeta} \boldsymbol{\phi}' \right]$$

$$R_w = \frac{w}{\boldsymbol{\tau}} \qquad - \boldsymbol{\beta}(- g\boldsymbol{\mu})$$

$$R_T = \frac{1}{\boldsymbol{\tau}}\left[\ln\left(\frac{T}{T_*} \right) - \boldsymbol{\kappa}\left(\mathrm{B}s + \overline{q}^{\zeta}\right)\right] \qquad - \boldsymbol{\beta}(- \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}})$$

$$R_C = \frac{1}{\boldsymbol{\tau}}\left[\overline{\mathrm{B}}^{\zeta} s + \ln\left(1 + \boldsymbol{\delta}_{\zeta} \mathrm{B}s\right)\right] \qquad - \boldsymbol{\beta}\left(\nabla_{\zeta} \cdot \mathbf{V}_h + \boldsymbol{\delta}_{\zeta} \dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta} \right)$$

$$R_{\phi} = \frac{\overline{\boldsymbol{\phi}'}^{\zeta}}{\boldsymbol{\tau}} \qquad - \boldsymbol{\beta}(- RT_* \dot{\boldsymbol{\zeta}} - gw)$$

15) The elliptic problem

Introducing $P \equiv \boldsymbol{\phi}' + RT_*(Bs+q)$ and $X = \dot{\boldsymbol{\zeta}} + \dfrac{\mathrm{B}s + \overline{q}^{\zeta}}{\boldsymbol{\tau}}$, the linear system takes the form:

$$\begin{aligned}
\mathbf{L}_h &= \frac{\mathbf{V}_h}{\boldsymbol{\tau}} + \nabla_\zeta P \\
L'_w &= \frac{w}{\boldsymbol{\tau}} - g\left(\boldsymbol{\delta}_\zeta q + \overline{q}^{\zeta}\right) \\
L'_T &= \frac{1}{\boldsymbol{\tau}}\left(\boldsymbol{\delta}_\zeta q + \overline{q}^{\zeta}\right) - \frac{\boldsymbol{\delta}_\zeta P}{\boldsymbol{\tau} RT_*} - \boldsymbol{\kappa} X \\
L_C &= -\frac{1}{\boldsymbol{\tau}}\overline{\left(\boldsymbol{\delta}_\zeta q + \overline{q}^{\zeta}\right)}^{\zeta} + \nabla_\zeta \cdot \mathbf{V}_h + \boldsymbol{\delta}_\zeta X + \overline{X}^{\zeta} \\
L_\phi &= \frac{\overline{P}^{\zeta}}{\boldsymbol{\tau}} - RT_* X - gw
\end{aligned}$$

The number of equations and dependent variables, $\mathbf{V}_h$, w, P, q, X, is easily reduced to 3 thus (variables left: P, w, X):

$$\begin{aligned}
\nabla_\zeta \cdot \mathbf{L}_h - \frac{1}{\boldsymbol{\tau}}\left(L_C - \frac{\overline{L'_w}^{\zeta}}{g\boldsymbol{\tau}}\right) &\equiv L''_C = \nabla_\zeta^2 P - \frac{1}{\boldsymbol{\tau}}\left(\boldsymbol{\delta}_\zeta X + \overline{X}^{\zeta}\right) + \frac{g\boldsymbol{\varepsilon}}{\boldsymbol{\tau} RT_*}\overline{w}^{\zeta} \\
\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(L'_T + \frac{L'_w}{g\boldsymbol{\tau}} + \frac{\boldsymbol{\varepsilon}}{RT_*}L_\phi\right) &\equiv L''_T = -\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}_\zeta P - \boldsymbol{\varepsilon}\overline{P}^{\zeta}\right) - \frac{X}{\boldsymbol{\tau}} \\
\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(L'_T + \frac{L'_w}{g\boldsymbol{\tau}} - \frac{\boldsymbol{\kappa}}{RT_*}L_\phi\right) &\equiv L''_\phi = -\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}_\zeta P + \boldsymbol{\kappa}\overline{P}^{\zeta}\right) + \frac{gw}{\boldsymbol{\tau} RT_*}
\end{aligned}$$

with $\varepsilon = \dfrac{RT_*}{g^2\tau^2}$ and $\gamma = \dfrac{\kappa}{\kappa + \varepsilon}$.

Here note: we have assumed $\boldsymbol{\delta}_\zeta \overline{q}^{\zeta} \equiv \overline{\boldsymbol{\delta}_\zeta q}^{\zeta}$, i.e. we have assumed commutation of the mean and difference operators. See **Appendix 5** for further details on averaging operators and commutation.

Finally, these three equations are combined to give:

$$L''_C - \left(\boldsymbol{\delta}_\zeta L''_T + \overline{L''_T}^{\zeta}\right) - \boldsymbol{\varepsilon}\overline{L''_\phi}^{\zeta} \equiv L_P = \nabla_\zeta^2 P + \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}_\zeta^2 P + \overline{\boldsymbol{\delta}_\zeta P}^{\zeta} - \boldsymbol{\varepsilon}(1-\boldsymbol{\kappa})\overline{P}^{\zeta\zeta}\right)$$

again provided commutation holds.

This is the elliptic problem to be solved with boundary conditions (on P) given by

$$L''_T = -\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}_\zeta P - \boldsymbol{\varepsilon}\overline{P}^\zeta\right) - \frac{1}{\boldsymbol{\tau}}\left(\dot{\boldsymbol{\zeta}} + \frac{\mathrm{B}s + \overline{q}^\zeta}{\boldsymbol{\tau}}\right)$$

applied at both top and bottom as follows:

$$\left[\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}_\zeta P - \boldsymbol{\varepsilon}\overline{P}^\zeta\right)\right]_T = -\left(L''_T\right)_T$$

$$\left[\frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa\overline{P}^\zeta\right)\right]_S = -\left(L''_T\right)_S + \frac{\phi_S}{\tau^2 RT_*} = -\left(L'''_T\right)_S$$

In effect

$$\left(\dot{\boldsymbol{\zeta}} + \frac{\mathrm{B}s + \overline{q}^\zeta}{\boldsymbol{\tau}}\right)_T = 0$$

since $\dot{\zeta}_T = 0$, $\mathrm{B}_T = 0$ and $q_T = 0$ at the top and (noting that $X \equiv \dot{\zeta} + \frac{\overline{P - \phi'}^\zeta}{\tau RT_*}$)

$$\left(\dot{\boldsymbol{\zeta}} + \frac{\mathrm{B}s + \overline{q}^\zeta}{\boldsymbol{\tau}}\right)_S = \frac{1}{\boldsymbol{\tau}}\left(q_S + s\right) = \frac{P_S - \boldsymbol{\phi}_S}{\boldsymbol{\tau} RT_*}$$

since $\dot{\zeta}_S = 0$ and $\mathrm{B}_S = 1$ at the bottom. ϕ_S is a known quantity.

N.B. These are closed boundary conditions. Open top boundary conditions are considered in **Appendix 9**.

16) The non-linear problem

To find the solution to the non-linear problem we need to perform the following operations iteratively

$$
\begin{aligned}
\left(\mathbf{L}'_h\right)^{1+iter,jter} &= \left(\mathbf{R}_h\right)^{jter} - \left(\mathbf{N}'_h\right)^{iter,jter} \\
\left(L'_w\right)^{1+iter,jter} &= \left(R_w\right)^{jter} - \left(N'_w\right)^{iter,jter} \\
\left(L''_T\right)^{1+iter,jter} &= \left(R''_T\right)^{jter} - \left(N''_T\right)^{iter,jter} \\
\left(L''_\phi\right)^{1+iter,jter} &= \left(R''_\phi\right)^{jter} - \left(N''_\phi\right)^{iter,jter} \\
\left(L_P\right)^{1+iter,jter} &= \left(R_P\right)^{jter} - \left(N_P\right)^{iter,jter}
\end{aligned}
$$

In order to obtain R_P, R''_T, R''_ϕ and N_P, N''_T, N''_ϕ, we transform the *R*'s and *N*'s, like was done for the *L*'s to obtain L_P, L''_T, L''_ϕ, i.e.:

$$
\begin{aligned}
\nabla_\zeta \cdot \mathbf{R}_h - \frac{1}{\boldsymbol{\tau}}\left(R_C - \frac{\overline{R_w}^{\zeta}}{g\boldsymbol{\tau}}\right) &\equiv R''_C \\
\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(R_T + \frac{R_w}{g\boldsymbol{\tau}} + \frac{\boldsymbol{\varepsilon}}{RT_*}R_\phi\right) &\equiv R''_T \\
\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(R_T + \frac{R_w}{g\boldsymbol{\tau}} - \frac{\boldsymbol{\kappa}}{RT_*}R_\phi\right) &\equiv R''_\phi \\
R''_C - \left(\boldsymbol{\delta}_\zeta R''_T + \overline{R''_T}^{\zeta}\right) - \boldsymbol{\varepsilon}\overline{R''_\phi}^{\zeta} &\equiv R_P
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\zeta \cdot \mathbf{N}_h - \frac{1}{\boldsymbol{\tau}}\left(N_C - \frac{\overline{N'_w}^{\zeta}}{g\boldsymbol{\tau}}\right) &\equiv N''_C \\
\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(N'_T + \frac{N'_w}{g\boldsymbol{\tau}} + \frac{\boldsymbol{\varepsilon}}{RT_*}N_\phi\right) &\equiv N''_T \\
\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(N'_T + \frac{N'_w}{g\boldsymbol{\tau}} - \frac{\boldsymbol{\kappa}}{RT_*}N_\phi\right) &\equiv N''_\phi \\
N''_C - \left(\boldsymbol{\delta}_\zeta N''_T + \overline{N''_T}^{\zeta}\right) - \boldsymbol{\varepsilon}\overline{N''_\phi}^{\zeta} &\equiv N_P
\end{aligned}
$$

Note that we have R_w, R_T on the left and N'_w, N'_T on the right and remember that $N_\phi = 0$.

17) Back substitution

The following equations give in a straight forward manner the 6 prognostic variables $\mathbf{V}_h, w, q, \left(s, \dot{\boldsymbol{\zeta}} \right)$ and ϕ':

$$\mathbf{V}_h : \qquad \frac{\mathbf{V}_h}{\boldsymbol{\tau}} = \left[\mathbf{R}_h - \mathbf{N}_h - \nabla_\zeta P \right]$$

$$\mathrm{w} : \qquad \frac{w}{\boldsymbol{\tau}} = -\frac{RT_*}{g} \left[R''_\phi - N''_\phi + \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*} \left(\boldsymbol{\delta}_\zeta P + \boldsymbol{\kappa} \overline{P}^\zeta \right) \right]$$

$$q : \quad \boldsymbol{\delta}_\zeta q + \overline{q}^\zeta = -\frac{1}{g} \left[R_w - N'_w - \frac{w}{\boldsymbol{\tau}} \right]; \qquad q_T = 0$$

$$s : \qquad s = \frac{P_S - \boldsymbol{\phi}_S}{RT_*} - q_S$$

$$\dot{\boldsymbol{\zeta}} : \qquad \frac{\dot{\boldsymbol{\zeta}}}{\boldsymbol{\tau}} = -\left[R''_T - N''_T + \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*} \left(\boldsymbol{\delta}_\zeta P - \boldsymbol{\varepsilon} \overline{P}^\zeta \right) \right] - \frac{\mathrm{B}s + \overline{q}^\zeta}{\boldsymbol{\tau}^2}; \quad \dot{\boldsymbol{\zeta}}_T = \dot{\boldsymbol{\zeta}}_S = 0$$

$$\boldsymbol{\phi}' : \qquad \boldsymbol{\phi}' = P - RT_* \left(q + Bs \right)$$

Finally we may compute μ and T diagnostically:

$$1 + \boldsymbol{\mu} = e^{\overline{q}^\zeta} \left[\frac{\boldsymbol{\delta}_\zeta q}{\boldsymbol{\delta}_\zeta \left(\boldsymbol{\zeta} + Bs \right)} + 1 \right]$$

$$\frac{T}{T_*} = e^{\overline{q}^\zeta} \frac{\boldsymbol{\delta}_\zeta \left(\boldsymbol{\zeta} - \boldsymbol{\phi}' / RT_* \right)}{\boldsymbol{\delta}_\zeta \left(\boldsymbol{\zeta} + Bs \right)}$$

For a brief description of *The Dynamic Core Code*, see **Appendix 6.**

There is THE HYDROSTATIC OPTION. For a description, see **Appendix 7.**

There is THE AUTOBAROTROPIC OPTION. For a description, see **Appendix 8**.

Aspects of HORIZONTAL DISCRETIZATION are given in **Appendix 10**.

See **Table 2**, page 59, for a summary of the equations, variables, etc.

THE END

Appendix 1. Virtual temperature

In presence of water vapor q_v and various types of hydrometeors q_i, the density of atmospheric substance is given by

$$\rho = \rho\left(q_d + q_v + \sum q_i\right)$$

where q_d is the dry air specific mass. The equation of state is given by

$$\begin{aligned} p &= \rho\left(R_d q_d + R_v q_v\right)T \\ &= \rho R_d\left(1 + \delta q_v - \sum q_i\right)T \end{aligned}$$

where $\delta = R_v / R_d - 1 \approx 0.6$ and we rewrite the equation of state as follows:

$$p = \rho R_d T_v$$

defining virtual temperature thus

$$T_v = T\left(1 + \delta q_v - \sum q_i\right)$$

Rewriting the equations to appear in terms of virtual temperature and approximating the ratio $\kappa = R/c_p$ by $\kappa_d = R_d/c_{pd}$, the equations of **section 1** may then be replaced by the following:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + R_d T_v \nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d\ln T_v}{dt} - \boldsymbol{\kappa}_d \frac{d\ln p}{dt} = \frac{Q_v}{c_{pd}T_v} = \frac{Q}{c_{pd}T_v} + \frac{T}{T_v}\left(\boldsymbol{\delta}\frac{dq_v}{dt} - \sum \frac{dq_i}{dt}\right)$$

$$\frac{d\ln \boldsymbol{\rho}}{dt} + \nabla \cdot \mathbf{V} = 0$$

$$\boldsymbol{\rho} = \frac{p}{R_d T_v}$$

From the point of view of the pure dynamics, these equations are formally identical to those in **section 1** in which R and c_p would take the dry air constant values, temperature be replaced by virtual temperature and appropriate source terms be added in the thermodynamic equation. The advantage of this formulation is of course the fact that the parameters R and c_p no longer varies while all of the virtual effects, including *water vapor buoyancy* and *condensed water loading* effects, are implicitly taken into account. The only approximation made here, the replacement of κ by κ_d in the thermodynamic equation, is facultative.

Appendix 2. Coordinate transformation rules

Appendix 2a. Invariance of the total derivative

By the *chain rule* we first verify the invariance of the total derivative *df/dt* under a general coordinate transformation. In effect, if we consider $f(x,y,z,t)$, then:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{x,y,z} + \left(\frac{\partial f}{\partial x}\right)_{y,z,t}\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{x,z,t}\frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{x,y,t}\frac{dz}{dt}$$

while for $f(x,y,\zeta,t)$, we naturally have:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{x,y,\zeta} + \left(\frac{\partial f}{\partial x}\right)_{y,\zeta,t}\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{x,\zeta,t}\frac{dy}{dt} + \left(\frac{\partial f}{\partial \zeta}\right)_{x,y,t}\frac{d\zeta}{dt}$$

Here we only have changed the vertical coordinate from z to ζ with the result that the horizontal components of the velocity $(dx/dt, dy/dt) = (U,V) = \mathbf{V}_h$ remain unchanged. The vertical motion though has transformed from $dz/dt = w$ into $d\zeta/dt = \dot{\zeta}$. Shortening the notation, we also write the above relations respectively as follows:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_z + U\left(\frac{\partial f}{\partial x}\right)_z + V\left(\frac{\partial f}{\partial y}\right)_z + w\frac{\partial f}{\partial z} = \frac{\partial f}{\partial t} + \mathbf{V}_h \cdot \nabla_z f + w\frac{\partial f}{\partial z}$$

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_\zeta + U\left(\frac{\partial f}{\partial x}\right)_\zeta + V\left(\frac{\partial f}{\partial y}\right)_\zeta + \dot{\zeta}\frac{\partial f}{\partial \zeta} = \frac{\partial f}{\partial t} + \mathbf{V}_h \cdot \nabla_\zeta f + \dot{\zeta}\frac{\partial f}{\partial \zeta}$$

Thus we minimize the indices. We also introduced the vector notation for the 'horizontal' part of the advection operator. Note though that the new coordinate ζ is generally curvilinear and non-orthogonal and the scalar product must be interpreted with care (see appendix 2c)

Appendix 2b. Transformation rules for derivatives.

It is remarkable that not only can all these rules *be recovered* from the invariance of the total derivative but also that these derivative transformation rules suffice to transform completely the Euler equations. In effect, the three velocity components may be treated as three independent scalars ('pseudo-scalars'), the velocity vector not being transformed (we are left though with a 'hybrid' system since maintaining two vertical velocities w and $\dot{\zeta}$ and therefore needing an additional predictive equation when $(\partial z/\partial t)_\zeta \neq 0$).

The transformation rules are obtained by equating the above two relations. In effect, we must have

$$0=\left(\frac{\partial f}{\partial t}\right)_z-\left(\frac{\partial f}{\partial t}\right)_\zeta+U\left[\left(\frac{\partial f}{\partial x}\right)_z-\left(\frac{\partial f}{\partial x}\right)_\zeta\right]+V\left[\left(\frac{\partial f}{\partial y}\right)_z-\left(\frac{\partial f}{\partial y}\right)_\zeta\right]+w\frac{\partial f}{\partial z}-\dot{\zeta}\frac{\partial f}{\partial \zeta}$$

and since

$$w=\frac{dz}{dt}=\left(\frac{\partial z}{\partial t}\right)_\zeta+U\left(\frac{\partial z}{\partial x}\right)_\zeta+V\left(\frac{\partial z}{\partial y}\right)_\zeta+\dot{\zeta}\frac{\partial z}{\partial \zeta}$$

then

$$0=\left[\left(\frac{\partial f}{\partial t}\right)_z-\left(\frac{\partial f}{\partial t}\right)_\zeta+\left(\frac{\partial z}{\partial t}\right)_\zeta\frac{\partial f}{\partial z}\right]+U\left[\left(\frac{\partial f}{\partial x}\right)_z-\left(\frac{\partial f}{\partial x}\right)_\zeta+\left(\frac{\partial z}{\partial x}\right)_\zeta\frac{\partial f}{\partial z}\right]$$
$$+V\left[\left(\frac{\partial f}{\partial y}\right)_z-\left(\frac{\partial f}{\partial y}\right)_\zeta+\left(\frac{\partial z}{\partial y}\right)_\zeta\frac{\partial f}{\partial z}\right]+\dot{\zeta}\left[\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \zeta}\right]$$

Each bracket must vanish independently. Therefore the rules are:

$$\left(\frac{\partial f}{\partial t}\right)_z=\left(\frac{\partial f}{\partial t}\right)_\zeta-\left(\frac{\partial z}{\partial t}\right)_\zeta\frac{\partial f}{\partial z}$$
$$\left(\frac{\partial f}{\partial x}\right)_z=\left(\frac{\partial f}{\partial x}\right)_\zeta-\left(\frac{\partial z}{\partial x}\right)_\zeta\frac{\partial f}{\partial z}$$
$$\left(\frac{\partial f}{\partial y}\right)_z=\left(\frac{\partial f}{\partial y}\right)_\zeta-\left(\frac{\partial z}{\partial y}\right)_\zeta\frac{\partial f}{\partial z}$$
$$\frac{\partial f}{\partial z}=\frac{\partial \zeta}{\partial z}\frac{\partial f}{\partial \zeta}$$

Appendix 2c. Vectors in non-orthogonal curvilinear coordinates

In non-orthogonal curvilinear coordinates $\hat{\mathbf{x}}=\left(\hat{x}^1,\hat{x}^2,\hat{x}^3\right)$ (see Dutton, John A, *The Ceaseless Wind*, chapters 5 and 7), there appear two sets of basis vectors (usually not even of unit length) and two sets of vector components. Applying the chain rule, we obtain the following two expansions (summation convention):

$$d\mathbf{x}=\frac{\partial \mathbf{x}}{\partial \hat{x}^j}d\hat{x}^j \qquad \mathrm{d}\hat{x}^i=\frac{\partial \hat{x}^i}{\partial x^j}dx^j=\left(\nabla \hat{x}^i\right)\cdot d\mathbf{x}$$
$$=\boldsymbol{\tau}_j d\hat{x}^j \qquad\qquad =\boldsymbol{\eta}^{\mathrm{i}}\cdot d\mathbf{x}$$

where $\boldsymbol{\tau}_j$ is covariant: *tangent to the curve along which only* $\hat{x}^j$ *varies* and $\boldsymbol{\eta}^{\mathrm{i}}$ is contra-variant: *normal to the surface* $\hat{x}^i$ = const. and we have the orthogonality relation

$$\boldsymbol{\tau}_j \boldsymbol{\eta}^i = \delta^i_j$$

Representing a vector **A** as

$$\mathbf{A} = A^k \boldsymbol{\tau}_k = A_k \boldsymbol{\eta}^k$$

we may recover the components [A_k (A^k): covariant (contravariant) components] using the above orthogonality relation:

$$A^i = \mathbf{A} \cdot \boldsymbol{\eta}^i = A^j \boldsymbol{\tau}_j \cdot \boldsymbol{\eta}^i$$

$$A_j = \mathbf{A} \cdot \boldsymbol{\tau}_j = A_i \boldsymbol{\eta}^i \cdot \boldsymbol{\tau}_j$$

The scalar product is

$$\mathbf{A} \cdot \mathbf{B} = A^k B_k = A_k B^k$$

Therefore in generalized vertical coordinate $\hat{\mathbf{x}} = (x, y, \zeta)$ the basis vectors become [the original orthogonal Cartesian coordinate being $\mathbf{x} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$]

$$\boldsymbol{\eta}^1 = \nabla x = \mathbf{i} \qquad \boldsymbol{\tau}_1 = \frac{\partial \mathbf{x}}{\partial x} = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}$$

$$\boldsymbol{\eta}^2 = \nabla y = \mathbf{j} \qquad \boldsymbol{\tau}_2 = \frac{\partial \mathbf{x}}{\partial y} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}$$

$$\boldsymbol{\eta}^3 = \nabla \zeta \qquad \boldsymbol{\tau}_3 = \frac{\partial \mathbf{x}}{\partial \zeta} = \frac{\partial z}{\partial \zeta}\mathbf{k}$$

The contravariant components of the velocity vector $V^i = \mathbf{V} \cdot \boldsymbol{\eta}^i$ are found to be

$$U,\ V, \mathbf{V} \cdot \nabla \zeta = \dot{\zeta}$$

While the covariant components of the gradient $\partial f / \partial \hat{x}^j = \nabla f \cdot \boldsymbol{\tau}_i$ are found to be

$$\left(\frac{\partial f}{\partial x}\right)_\zeta, \left(\frac{\partial f}{\partial y}\right)_\zeta, \frac{\partial f}{\partial \zeta}$$

And the vector product $\mathbf{V} \cdot \nabla f$ may be computed as follows:

$$\mathbf{V} \cdot \nabla f = V^i \boldsymbol{\tau}_i \cdot \boldsymbol{\eta}^j \frac{\partial f}{\partial \hat{x}^j}$$

$$= \left[U\left(\mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}\right) + V\left(\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}\right) + \dot{\zeta}\frac{\partial z}{\partial \zeta}\mathbf{k} \right] \cdot \left[\left(\frac{\partial f}{\partial x}\right)_\zeta \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_\zeta \mathbf{j} + \frac{\partial f}{\partial \zeta}\nabla \zeta \right]$$

$$
= U\left[\left(\frac{\partial f}{\partial x}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\mathbf{i}\cdot\nabla\zeta + \frac{\partial z}{\partial x}\mathbf{k}\cdot\nabla\zeta\right)\right] + V\left[\left(\frac{\partial f}{\partial y}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\mathbf{j}\cdot\nabla\zeta + \frac{\partial z}{\partial y}\mathbf{k}\cdot\nabla\zeta\right)\right] + \dot{\zeta}\,\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial \zeta}\mathbf{k}\cdot\nabla\zeta
$$

$$
= U\left[\left(\frac{\partial f}{\partial x}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial \zeta}{\partial z}\right)\right] + V\left[\left(\frac{\partial f}{\partial y}\right)_\zeta + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial \zeta}{\partial z}\right)\right] + \dot{\zeta}\,\frac{\partial z}{\partial \zeta}\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial z}
$$

$$
= U\left(\frac{\partial f}{\partial x}\right)_\zeta + V\left(\frac{\partial f}{\partial y}\right)_\zeta + \dot{\zeta}\,\frac{\partial f}{\partial \zeta}
$$

since

$$
\left(\frac{\partial \zeta}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_z \frac{\partial \zeta}{\partial z} = \left(\frac{\partial \zeta}{\partial x}\right)_\zeta = 0
$$

$$
\left(\frac{\partial \zeta}{\partial y}\right)_z + \left(\frac{\partial z}{\partial y}\right)_z \frac{\partial \zeta}{\partial z} = \left(\frac{\partial \zeta}{\partial y}\right)_\zeta = 0
$$

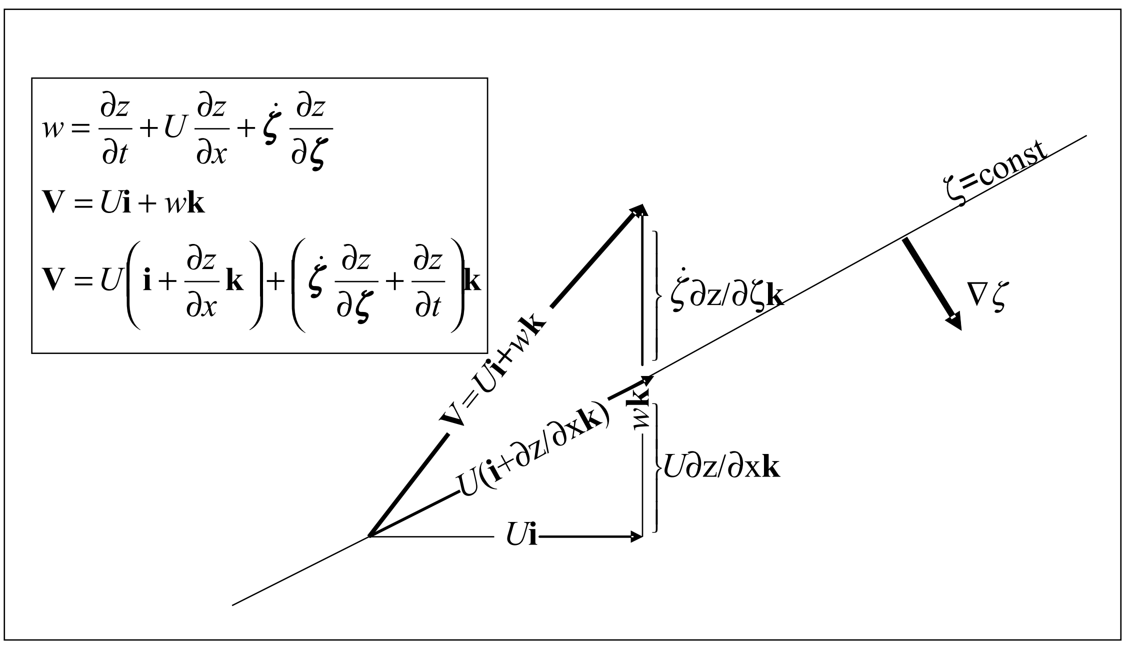

Figure 2. Representation of the wind vector in both orthogonal *z*-coordinate and oblique ζ-coordinate

Appendix 3. The metric parameter B and its effect

The log-hydrostatic-pressure is given by

$$\ln \pi = \zeta + B(\zeta)s; \qquad s = \ln \pi_S - \zeta_S = \ln(\pi_S / p_{ref})$$

$$\lambda = \frac{\zeta - \zeta_T}{\zeta_S - \zeta_T}$$

$$B = \lambda^r; \qquad r = r_{\max} - (r_{\max} - r_{\min})\lambda$$

$$\ln \pi = \zeta + B[\ln \pi_S - \zeta_S]$$

We want to investigate the behavior of this relation as a function of B and the surface pressure π_S. We have $0 \le B \le 1$; we calculate the derivative

$$\frac{\partial \ln B}{\partial \boldsymbol{\lambda}} = \frac{\partial r \ln \boldsymbol{\lambda}}{\partial \boldsymbol{\lambda}} = \frac{1}{\boldsymbol{\lambda}}[r - \Delta r \boldsymbol{\lambda} \ln \boldsymbol{\lambda}] \ge 0; \qquad \Delta r = r_{\max} - r_{\min}$$

Monotonicity requires that

$$\frac{\partial \ln \pi}{\partial \zeta} = 1 + \frac{\partial B}{\partial \lambda}[\ln \pi_S - \zeta_S]\frac{\partial \lambda}{\partial \zeta} > 0$$

$$\frac{\ln \pi_S}{\zeta_S} > K \ge 1 - \left(\frac{\partial B}{\partial \lambda}\right)^{-1}\left(1 - \frac{\zeta_T}{\zeta_S}\right\}; \qquad K = 1 - \left(\frac{\partial B}{\partial \lambda}\right)_{\max}^{-1}\left(1 - \frac{\zeta_T}{\zeta_S}\right)$$

$$\pi_S > p_{ref}\exp(K)$$

When r is constant (Δr=0) , $(\partial B / \partial \lambda)_{\max} = r$ at λ=1. K=1-1/$r\left(1 - \zeta_T / \zeta_S\right)$ and the monotonicity requirement is $\pi_S > p_{ref}^{\,1-\frac{1}{r}\left(1-\frac{\zeta_T}{\zeta_S}\right)}$. For $p_{top} = 10mb; r = 4$, $\pi_S > \left(10^5\right)^{1-\frac{0.4}{4}} \approx 316mb$. Not too restrictive. Nevertheless, we must worry about the amount of squeezing of the layer thicknesses even when π_S is larger:

Maximum squeezing ratio when $\dfrac{\Delta \ln \pi}{\Delta \zeta} = 1 - \left(\dfrac{\partial B}{\partial \lambda}\right)_{\max}\left[\dfrac{\zeta_S - \ln \pi_S}{\zeta_S - \zeta_T}\right]$

Surface squeezing ratio when $\dfrac{\Delta \ln \pi}{\Delta \zeta} = 1 - \left(\dfrac{\partial B}{\partial \lambda}\right)_{srf}\left[\dfrac{\ln(p_{ref} / \pi_S)}{\ln(p_{ref} / p_{top})}\right]$

Let's say π_S=500mb, then

$$\frac{\Delta \ln \pi}{\Delta \zeta} = 1 - 4 \frac{\ln 2}{\ln 100} \approx .4$$

With r variable, we may at the same time avoid too much squeezing near the surface where the layers are already quite thin, while achieving fast rectification. For example, we may choose r_{min}=2 (r at the surface) and r_{max}=15 (r at the top) with a maximum squeezing ratio of ~.45 but a surface squeezing ratio of ~.7. Two figures are shown below: The 57 *fitted* levels of the Meso-global s*taggered* version of GEM in the new ζ-coordinate (a) with r_{max}=r_{min}=1 (**Figure 3**), (b) with r_{max}=15 and r_{min}=2 (**Figure 4**).

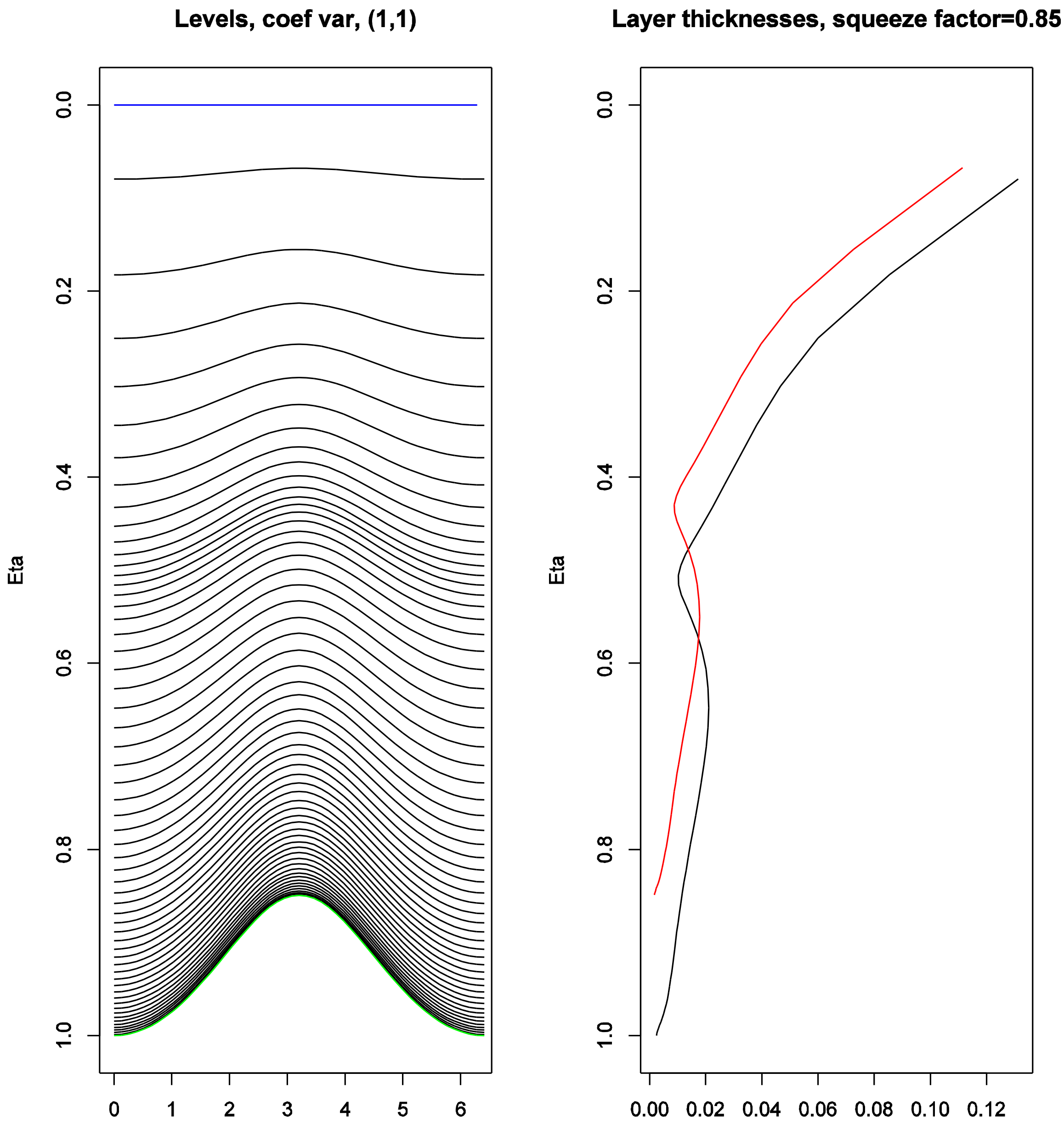

Figure 3. The 57 *fitted* levels of the Meso-global s*taggered* version of GEM in the new ζ-coordinate with r_{max}=r_{min}=1.

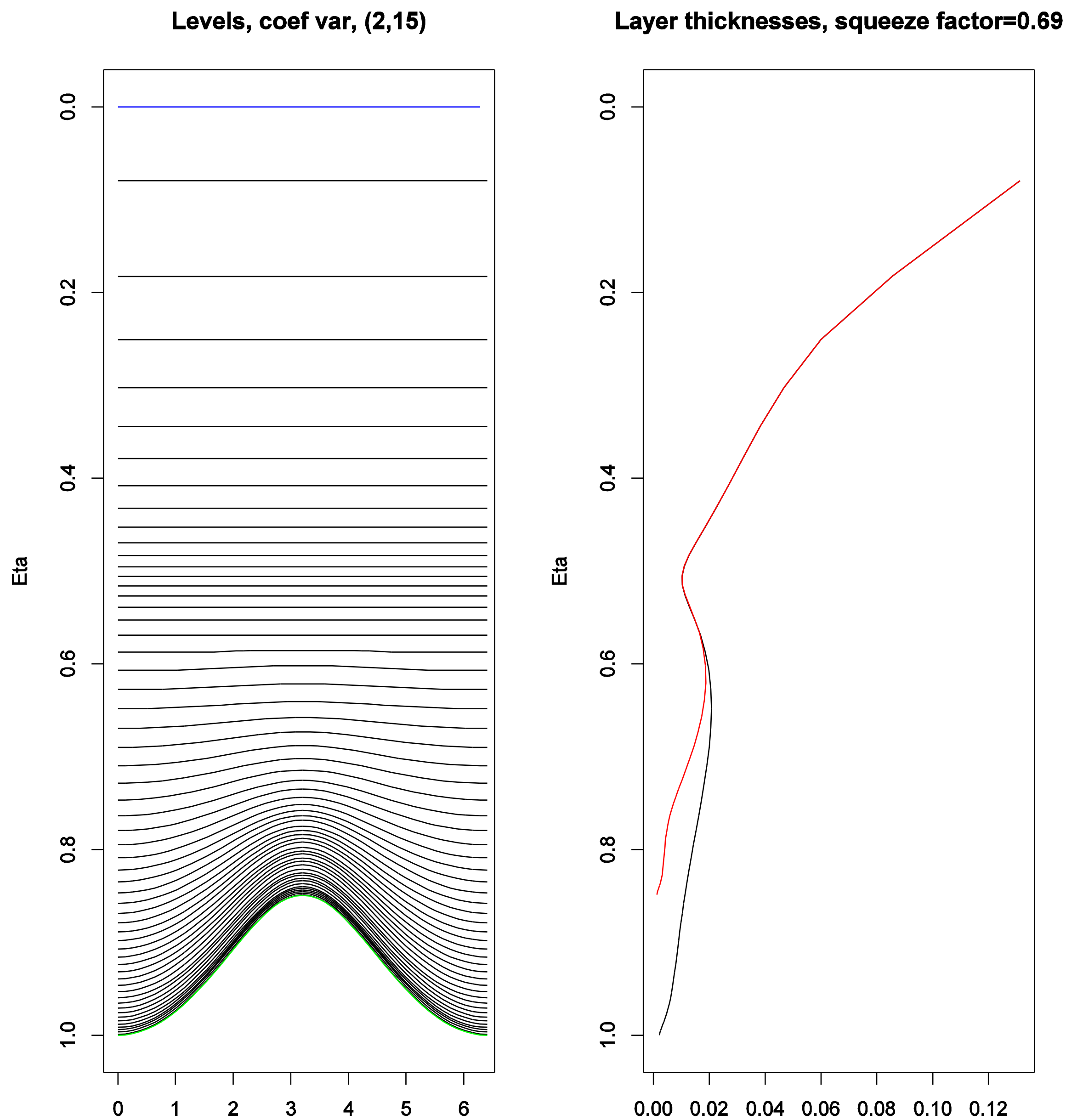

Figure 4. The 57 *fitted* levels of the Meso-global s*taggered* version of GEM in the new ζ-coordinate with r_{max}=15 and r_{min}=2.

The surface pressure at the top of the hill is 500*mb*. The 'squeeze factor' is the surface 'squeeze factor'. The red (black) curves correspond to layer thicknesses in Eta-units above (below) the hill [Eta=$\dfrac{\zeta-\zeta_T}{\zeta_S-\zeta_T}$].

Appendix 4a. Detailed spatial discretization with staggering: *the linear terms*

In **section 7**, we described the vertical discretization succinctly. In **section 15**, we examined the Elliptic Problem. We showed that all variables could be readily eliminated in favor of *P*. We now go back and examine the discrete linear system leading to the elliptic problem in full details. As mentioned earlier, the finite differences replacing the derivatives are made as simple as possible, i.e.

$$\left(\delta_\zeta F\right)_{k-\frac{1}{2}} = \frac{F_k - F_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}} \quad (k=1,N+1) \quad ; \quad \left(\delta_\zeta G\right)_k = \frac{G_{k+\frac{1}{2}} - G_{k-\frac{1}{2}}}{\Delta\zeta_k} \quad (k=1,N)$$

$$\Delta\zeta_{k-\frac{1}{2}} = \zeta_k - \zeta_{k-1} \qquad\qquad \Delta\zeta_k = \zeta_{k+\frac{1}{2}} - \zeta_{k-\frac{1}{2}}$$

with the top $\zeta_0 = \zeta_{\frac{1}{2}} = \zeta_T$, the surface $\zeta_{N+1} = \zeta_{N+\frac{1}{2}} = \zeta_S$ and the momentum levels $\zeta_k \ (1 \le k \le N)$ specified while the intermediate thermodynamic levels $\zeta_{k-\frac{1}{2}} = \frac{\zeta_k + \zeta_{k-1}}{2} \ (2 \le k \le N)$ are calculated. The use of averaging is minimized. Starting with $(L_C)_k$, $(\mathbf{V}_h)_k$ and $X_{k-\frac{1}{2}}$ are chosen and therefore $(\mathbf{L}_h)_k$ and P_k. The hydrostatic case (q being absent) suggests $(L'_T)_{k-\frac{1}{2}}$, therefore $(L_H)_{k-\frac{1}{2}}$ and $T'_{k-\frac{1}{2}}$ (**section 11**). In $(L'_T)_{k-\frac{1}{2}}$ and $(L_H)_{k-\frac{1}{2}}$, we now introduce q_k. This in turn suggests $(L'_w)_{k-\frac{1}{2}}$, $w_{k-\frac{1}{2}}$ and finally $(L_\phi)_{k-\frac{1}{2}}$. Hence

$$\left(\mathbf{L}_h\right)_k = \left(\frac{\mathbf{V}_h}{\tau} + \nabla_\zeta P\right)_k \qquad (k=1,N)$$

$$\left(L'_w\right)_{k-\frac{1}{2}} = \left(\frac{w}{\tau} - g\left(\delta_\zeta q + \overline{q}^\zeta\right)\right)_{k-\frac{1}{2}} \qquad (k=1,N+1)$$

$$\left(L'_T\right)_{k-\frac{1}{2}} = \left(\frac{\delta_\zeta q + \overline{q}^\zeta}{\tau} - \frac{\delta_\zeta P}{\tau R T_*} - \kappa X\right)_{k-\frac{1}{2}} \qquad (k=1,N+1)$$

$$\left(L_C\right)_k = \left(-\frac{\overline{\delta_\zeta q + \overline{q}^\zeta}^\zeta}{\tau} + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta X + \overline{X}^\zeta\right)_k \qquad (k=1,N)$$

$$\left(L_\phi\right)_{k-\frac{1}{2}} = \left(\frac{\overline{P}^\zeta}{\tau} - RT_* X - gw\right)_{k-\frac{1}{2}} \qquad (k=1,N+1)$$

In the vertical, this leads to Charney-Phillips grid (**Figure 1**, page 11). For the two vertical means, we formally write

$$\left(\overline{F}^{\zeta}\right)_{k-\frac{1}{2}} = \boldsymbol{\varpi}^{+}_{k-\frac{1}{2}} F_k + \boldsymbol{\varpi}^{-}_{k-\frac{1}{2}} F_{k-1}, \quad \boldsymbol{\varpi}^{-}_{k-\frac{1}{2}} = 1 - \boldsymbol{\varpi}^{+}_{k-\frac{1}{2}} \quad \text{and} \quad \left(\overline{G}^{\zeta}\right)_{k} = \boldsymbol{\varpi}^{+}_{k} G_{k+\frac{1}{2}} + \boldsymbol{\varpi}^{-}_{k} G_{k-\frac{1}{2}}, \quad \boldsymbol{\varpi}^{-}_{k} = 1 - \boldsymbol{\varpi}^{+}_{k}.$$

The first one, averaging variables from momentum or full levels toward thermodynamic or half-levels, follows the rule of calculation for the half-levels, i.e.

$$\boldsymbol{\varpi}^{+}_{\frac{1}{2}} = 0, \quad \boldsymbol{\varpi}^{+}_{k-\frac{1}{2}} = \frac{1}{2} \ (2 \le k \le N), \quad \boldsymbol{\varpi}^{+}_{N+\frac{1}{2}} = 1$$

This ensures optimal (second-order) accuracy for the hydrostatic equation L_H in particular. For the second one, averaging variables from thermodynamic levels toward momentum levels, three choices were considered: linear interpolation, simple average, average commuting with difference. Due to lack of sensibility, the last was adopted because it simplifies the code:

$$\boldsymbol{\varpi}^{+}_{k} = \frac{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}{2\Delta\boldsymbol{\zeta}_{k}} = \frac{\boldsymbol{\zeta}_{k+1} - \boldsymbol{\zeta}_{k}}{\boldsymbol{\zeta}_{k+1} - \boldsymbol{\zeta}_{k-1}} \ (1 \le k \le N-1)$$
$$\boldsymbol{\varpi}^{+}_{N} = \frac{\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_{N}} = \frac{\boldsymbol{\zeta}_{N+\frac{1}{2}} - \boldsymbol{\zeta}_{N}}{\boldsymbol{\zeta}_{N+\frac{1}{2}} - \boldsymbol{\zeta}_{N-\frac{1}{2}}}$$

More explicitly,

$$\left(\mathbf{L}_h\right)_k = \frac{\mathbf{V}_{h\,k}}{\boldsymbol{\tau}} + \nabla_{\zeta} P_k$$

$$\left(L'_w\right)_{k-\frac{1}{2}} = \frac{w_{k-\frac{1}{2}}}{\boldsymbol{\tau}} - g\left(\frac{q_k - q_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} + \boldsymbol{\varpi}^{+}_{k-\frac{1}{2}} q_k + \boldsymbol{\varpi}^{-}_{k-\frac{1}{2}} q_{k-1}\right)$$

$$\left(L'_T\right)_{k-\frac{1}{2}} = \frac{1}{\boldsymbol{\tau}}\left(\frac{q_k - q_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} + \boldsymbol{\varpi}^{+}_{k-\frac{1}{2}} q_k + \boldsymbol{\varpi}^{-}_{k-\frac{1}{2}} q_{k-1}\right) - \frac{1}{\boldsymbol{\tau} R T_*}\frac{P_k - P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} - \boldsymbol{\kappa} X_{k-\frac{1}{2}}$$

$$\left(L_C\right)_k = -\frac{1}{\boldsymbol{\tau}}\left[\boldsymbol{\varpi}^{+}_{k}\left(\frac{q_{k+1} - q_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} + \boldsymbol{\varpi}^{+}_{k+\frac{1}{2}} q_{k+1} + \boldsymbol{\varpi}^{-}_{k+\frac{1}{2}} q_k\right) + \boldsymbol{\varpi}^{-}_{k}\left(\frac{q_k - q_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} + \boldsymbol{\varpi}^{+}_{k-\frac{1}{2}} q_k + \boldsymbol{\varpi}^{-}_{k-\frac{1}{2}} q_{k-1}\right)\right]$$
$$+ \nabla_{\zeta} \cdot \mathbf{V}_{h\,k} + \frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_{k}} + \boldsymbol{\varpi}^{+}_{k} X_{k+\frac{1}{2}} + \boldsymbol{\varpi}^{-}_{k} X_{k-\frac{1}{2}}$$

$$\left(L_{\boldsymbol{\phi}}\right)_{k-\frac{1}{2}} = \frac{1}{\boldsymbol{\tau}}\left(\boldsymbol{\varpi}^{+}_{k-\frac{1}{2}} P_k + \boldsymbol{\varpi}^{-}_{k-\frac{1}{2}} P_{k-1}\right) - R T_* X_{k-\frac{1}{2}} - g w_{k-\frac{1}{2}}$$

We have in the vertical direction $3N$ [$\mathbf{L}_h, L_C$] + $3(N+1)$ [L'_w, L'_T, L_ϕ], i.e. $6N+3$ equations and $2N\,[U,V]_{1,N}$ + $2(N+1)\,[w,X,]_{1/2,N+1/2}$ + $2(N+2)\,[P,q]_{0,N+1}$, i.e. $6N+6$ variables. As expected (section 7), we will need 3 boundary conditions in the vertical to close the problem. Now, 3 variables ($\mathbf{V}_h$, q) can easily be eliminated by combining the equations as follows:

$$\nabla_\zeta \cdot (\mathbf{L}_h)_k - \frac{1}{\tau}\left\{(L_C)_k - \frac{1}{g\tau}\left[\varpi_k^+ (L'_w)_{k+\frac{1}{2}} + \varpi_k^- (L'_w)_{k-\frac{1}{2}}\right]\right\} \equiv (L''_C)_k$$

$$\frac{\gamma}{\kappa\tau}\left[(L'_T)_{k-\frac{1}{2}} + \frac{1}{g\tau}(L'_w)_{k-\frac{1}{2}} + \frac{\varepsilon}{RT_*}(L_\phi)_{k-\frac{1}{2}}\right] \equiv (L''_T)_{k-\frac{1}{2}}$$

$$\frac{\gamma}{\kappa\tau}\left[(L'_T)_{k-\frac{1}{2}} + \frac{1}{g\tau}(L'_w)_{k-\frac{1}{2}} - \frac{\kappa}{RT_*}(L_\phi)_{k-\frac{1}{2}}\right] \equiv (L''_\phi)_{k-\frac{1}{2}}$$

to give

$$(L''_C)_k = \nabla_\zeta^2 P_k - \frac{1}{\tau}\left(\frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\zeta_k} + \varpi_k^+ X_{k+\frac{1}{2}} + \varpi_k^- X_{k-\frac{1}{2}}\right) + \frac{g\varepsilon}{\tau RT_*}\left(\varpi_k^+ w_{k+\frac{1}{2}} + \varpi_k^- w_{k-\frac{1}{2}}\right)$$

$$(L''_T)_{k-\frac{1}{2}} = -\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}} - \varepsilon\left(\varpi_{k-\frac{1}{2}}^+ P_k + \varpi_{k-\frac{1}{2}}^- P_{k-1}\right)\right] - \frac{X_{k-\frac{1}{2}}}{\tau}$$

$$(L''_\phi)_{k-\frac{1}{2}} = -\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}} + \kappa\left(\varpi_{k-\frac{1}{2}}^+ P_k + \varpi_{k-\frac{1}{2}}^- P_{k-1}\right)\right] + \frac{g w_{k-\frac{1}{2}}}{\tau RT_*}$$

By further forming

$$-\frac{(L''_T)_{k+\frac{1}{2}} - (L''_T)_{k-\frac{1}{2}}}{\Delta\zeta_k} - \varpi_k^+ (L''_T)_{k+\frac{1}{2}} - \varpi_k^- (L''_T)_{k-\frac{1}{2}} =$$

$$\frac{\gamma}{\kappa\tau^2 RT_*}\left[\frac{1}{\Delta\zeta_k}\left(\frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} - \frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right) + \varpi_k^+ \frac{P_{k+1} - P_k}{\Delta\zeta_{k+\frac{1}{2}}} + \varpi_k^- \frac{P_k - P_{k-1}}{\Delta\zeta_{k-\frac{1}{2}}}\right]$$
$$-\frac{\gamma\varepsilon}{\kappa\tau^2 RT_*}\frac{1}{\Delta\zeta_k}\left[\left(\varpi_{k+\frac{1}{2}}^+ P_{k+1} + \varpi_{k+\frac{1}{2}}^- P_k\right) - \left(\varpi_{k-\frac{1}{2}}^+ P_k + \varpi_{k-\frac{1}{2}}^- P_{k-1}\right)\right]$$
$$-\frac{\gamma\varepsilon}{\kappa\tau^2 RT_*}\left[\varpi_k^+\left(\varpi_{k+\frac{1}{2}}^+ P_{k+1} + \varpi_{k+\frac{1}{2}}^- P_k\right) + \varpi_k^-\left(\varpi_{k-\frac{1}{2}}^+ P_k + \varpi_{k-\frac{1}{2}}^- P_{k-1}\right)\right]$$
$$+\frac{1}{\tau}\left(\frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\zeta_k} + \varpi_k^+ X_{k+\frac{1}{2}} + \varpi_k^- X_{k-\frac{1}{2}}\right)$$

and

$$-\boldsymbol{\varepsilon}\left(\boldsymbol{\varpi}_k^{+}\left(L''_{\phi}\right)_{k+\frac{1}{2}}+\boldsymbol{\varpi}_k^{-}\left(L''_{\phi}\right)_{k-\frac{1}{2}}\right)=\frac{\boldsymbol{\gamma\varepsilon}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\boldsymbol{\varpi}_k^{+}\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}+\boldsymbol{\varpi}_k^{-}\frac{P_k-P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right]$$
$$+\frac{\boldsymbol{\gamma\kappa\varepsilon}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\boldsymbol{\varpi}_k^{+}\left(\boldsymbol{\varpi}_{k+\frac{1}{2}}^{+}P_{k+1}+\boldsymbol{\varpi}_{k+\frac{1}{2}}^{-}P_k\right)+\boldsymbol{\varpi}_k^{-}\left(\boldsymbol{\varpi}_{k-\frac{1}{2}}^{+}P_k+\boldsymbol{\varpi}_{k-\frac{1}{2}}^{-}P_{k-1}\right)\right]$$
$$-\frac{g\boldsymbol{\varepsilon}}{\boldsymbol{\tau}RT_*}\left(\boldsymbol{\varpi}_k^{+}w_{k+\frac{1}{2}}+\boldsymbol{\varpi}_k^{-}w_{k-\frac{1}{2}}\right)$$

and finally

$$\left(L''_C\right)_k-\frac{\left(L''_T\right)_{k+\frac{1}{2}}-\left(L''_T\right)_{k-\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_k}-\boldsymbol{\varpi}_k^{+}\left[\left(L''_T\right)_{k+\frac{1}{2}}+\boldsymbol{\varepsilon}\left(L''_{\phi}\right)_{k+\frac{1}{2}}\right]-\boldsymbol{\varpi}_k^{-}\left[\left(L''_T\right)_{k-\frac{1}{2}}+\boldsymbol{\varepsilon}\left(L''_{\phi}\right)_{k-\frac{1}{2}}\right]=\left(L_P\right)_k$$

we succeed in eliminating *X* and *w*. In effect, we have

$$\left(L_P\right)_k=\nabla_{\zeta}^2 P_k+\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\frac{1}{\Delta\boldsymbol{\zeta}_k}\left(\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}-\frac{P_k-P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right)+\boldsymbol{\varpi}_k^{+}\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}+\boldsymbol{\varpi}_k^{-}\frac{P_k-P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right]$$
$$+\frac{\boldsymbol{\gamma\varepsilon}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\boldsymbol{\varpi}_k^{+}\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}+\boldsymbol{\varpi}_k^{-}\frac{P_k-P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}-\frac{\boldsymbol{\varpi}_{k+\frac{1}{2}}^{+}P_{k+1}+\boldsymbol{\varpi}_{k+\frac{1}{2}}^{-}P_k-\boldsymbol{\varpi}_{k-\frac{1}{2}}^{+}P_k-\boldsymbol{\varpi}_{k-\frac{1}{2}}^{-}P_{k-1}}{\Delta\boldsymbol{\zeta}_k}\right]$$
$$-\frac{\boldsymbol{\gamma\varepsilon}(1-\boldsymbol{\kappa})}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\boldsymbol{\varpi}_k^{+}\left(\boldsymbol{\varpi}_{k+\frac{1}{2}}^{+}P_{k+1}+\boldsymbol{\varpi}_{k+\frac{1}{2}}^{-}P_k\right)+\boldsymbol{\varpi}_k^{-}\left(\boldsymbol{\varpi}_{k-\frac{1}{2}}^{+}P_k+\boldsymbol{\varpi}_{k-\frac{1}{2}}^{-}P_{k-1}\right)\right]$$

The second bracket corresponds to the difference $\overline{\boldsymbol{\delta}_{\zeta}P}^{\zeta}-\boldsymbol{\delta}_{\zeta}\overline{P}^{\zeta}$ which vanishes by construction (commuting average, see **Appendix 5**). Therefore the final result is

$$\left(L_P\right)_k=\nabla_{\zeta}^2 P_k+\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\frac{1}{\Delta\boldsymbol{\zeta}_k}\left(\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}-\frac{P_k-P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right)+\boldsymbol{\varpi}_k^{+}\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}+\boldsymbol{\varpi}_k^{-}\frac{P_k-P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right]$$
$$-\frac{\boldsymbol{\gamma\varepsilon}(1-\boldsymbol{\kappa})}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\boldsymbol{\varpi}_k^{+}\left(\boldsymbol{\varpi}_{k+\frac{1}{2}}^{+}P_{k+1}+\boldsymbol{\varpi}_{k+\frac{1}{2}}^{-}P_k\right)+\boldsymbol{\varpi}_k^{-}\left(\boldsymbol{\varpi}_{k-\frac{1}{2}}^{+}P_k+\boldsymbol{\varpi}_{k-\frac{1}{2}}^{-}P_{k-1}\right)\right]$$

N equations and *N*+2 unknowns.

Appendix 4b. Detailed spatial discretization: *matrices of the elliptic problem*

The matrix of the elliptic problem is composed of the previous equations (k=1,N):

$$\Delta\boldsymbol{\zeta}_k(L_P)_k = \Delta\boldsymbol{\zeta}_k\nabla_\zeta^2 P_k + \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\left(\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} - \frac{P_k - P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right) + \Delta\boldsymbol{\zeta}_k\left(\boldsymbol{\varpi}_k^+\frac{P_{k+1}-P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} + \boldsymbol{\varpi}_k^-\frac{P_k - P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right)\right]$$
$$-\frac{\boldsymbol{\gamma\varepsilon}(1-\boldsymbol{\kappa})}{\boldsymbol{\kappa\tau}^2 RT_*}\Delta\boldsymbol{\zeta}_k\left[\boldsymbol{\varpi}_k^+\left(\boldsymbol{\varpi}_{k+\frac{1}{2}}^+ P_{k+1} + \boldsymbol{\varpi}_{k+\frac{1}{2}}^- P_k\right) + \boldsymbol{\varpi}_k^-\left(\boldsymbol{\varpi}_{k-\frac{1}{2}}^+ P_k + \boldsymbol{\varpi}_{k-\frac{1}{2}}^- P_{k-1}\right)\right]$$

(note that it has been multiplied through by $\Delta\zeta_k$) plus the boundary equations:

$$\left[\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}_\zeta P - \boldsymbol{\varepsilon}\overline{P}^\zeta\right)\right]_{\frac{1}{2}} = \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\frac{P_1 - P_0}{\Delta\boldsymbol{\zeta}_{\frac{1}{2}}} - \boldsymbol{\varepsilon}\left(\boldsymbol{\varpi}_{\frac{1}{2}}^+ P_1 + \boldsymbol{\varpi}_{\frac{1}{2}}^- P_0\right)\right) = -(L''_T)_{\frac{1}{2}}$$

$$\left[\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}_\zeta P + \boldsymbol{\kappa}\overline{P}^\zeta\right)\right]_{N+\frac{1}{2}} = \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\frac{P_{N+1} - P_N}{\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}}} + \boldsymbol{\kappa}\left(\boldsymbol{\varpi}_{N+\frac{1}{2}}^+ P_{N+1} + \boldsymbol{\varpi}_{N+\frac{1}{2}}^- P_N\right)\right) = -(L'''_T)_{N+\frac{1}{2}}$$

which are used to reduce the number of unknowns from N+2 to N. In effect, we find:

$$P_0 = \boldsymbol{\alpha}_T P_1 + C_T(L''_T)_{\frac{1}{2}}$$
$$P_{N+1} = \boldsymbol{\alpha}_S P_N - C_S(L'''_T)_{N+\frac{1}{2}}$$

with

$$\boldsymbol{\alpha}_T = \frac{1/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} - \boldsymbol{\varepsilon\varpi}_{\frac{1}{2}}^+}{1/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} + \boldsymbol{\varepsilon\varpi}_{\frac{1}{2}}^-}; \qquad C_T = \frac{\boldsymbol{\kappa\tau}^2 RT_*}{\boldsymbol{\gamma}}\frac{1}{1/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} + \boldsymbol{\varepsilon\varpi}_{\frac{1}{2}}^-}$$
$$\boldsymbol{\alpha}_S = \frac{1/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}} - \boldsymbol{\kappa\varpi}_{N+\frac{1}{2}}^-}{1/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}} + \boldsymbol{\kappa\varpi}_{N+\frac{1}{2}}^+}; \quad C_S = \frac{\boldsymbol{\kappa\tau}^2 RT_*}{\boldsymbol{\gamma}}\frac{1}{1/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}} + \boldsymbol{\kappa\varpi}_{N+\frac{1}{2}}^+}$$

Therefore we may rewrite the equations for $(L_P)_{ij1}$ and $(L_P)_{ijN}$ as follows

$$\Delta\boldsymbol{\zeta}_1(L'_P)_1 = \Delta\boldsymbol{\zeta}_1(L_P)_1 - C''_T(L''_T)_{\frac{1}{2}}$$
$$\Delta\boldsymbol{\zeta}_N(L'_P)_N = \Delta\boldsymbol{\zeta}_N(L_P)_N + C''_S(L'''_T)_{N+\frac{1}{2}}$$

to get respectively

$$\Delta\boldsymbol{\zeta}_1\left(L'_P\right)_1 = \Delta\boldsymbol{\zeta}_1\left(\nabla^2_\zeta P\right)_1 + \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\frac{P_2 - P_1}{\Delta\boldsymbol{\zeta}_{\frac{3}{2}}} - \frac{\left(1-\boldsymbol{\alpha}_T\right)P_1}{\Delta\boldsymbol{\zeta}_{\frac{1}{2}}} + \frac{\Delta\boldsymbol{\zeta}_1\boldsymbol{\varpi}_1^+}{\Delta\boldsymbol{\zeta}_{\frac{3}{2}}}\left(P_2 - P_1\right) + \frac{\Delta\boldsymbol{\zeta}_1\boldsymbol{\varpi}_1^-}{\Delta\boldsymbol{\zeta}_{\frac{1}{2}}}\left(1-\boldsymbol{\alpha}_T\right)P_1\right]$$

$$-\frac{\boldsymbol{\gamma\varepsilon}\left(1-\boldsymbol{\kappa}\right)}{\boldsymbol{\kappa\tau}^2 RT_*}\left\{\Delta\boldsymbol{\zeta}_1\boldsymbol{\varpi}_1^+\boldsymbol{\varpi}_{\frac{3}{2}}^+ P_2 + \Delta\boldsymbol{\zeta}_1\left[\boldsymbol{\varpi}_1^+\boldsymbol{\varpi}_{\frac{3}{2}}^- + \boldsymbol{\varpi}_1^-\left(\boldsymbol{\varpi}_{\frac{1}{2}}^+ + \boldsymbol{\varpi}_{\frac{1}{2}}^-\boldsymbol{\alpha}_T\right)\right]P_1\right\}$$

and

$$\Delta\boldsymbol{\zeta}_N\left(L'_P\right)_N = \Delta\boldsymbol{\zeta}_N\left(\nabla^2_\zeta P\right)_N + \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\frac{\left(\boldsymbol{\alpha}_S - 1\right)P_N}{\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}}} - \frac{P_N - P_{N-1}}{\Delta\boldsymbol{\zeta}_{N-\frac{1}{2}}} + \frac{\Delta\boldsymbol{\zeta}_N\boldsymbol{\varpi}_N^+}{\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}}}\left(\boldsymbol{\alpha}_S - 1\right)P_N + \frac{\Delta\boldsymbol{\zeta}_N\boldsymbol{\varpi}_N^-}{\Delta\boldsymbol{\zeta}_{N-\frac{1}{2}}}\left(P_N - P_{N-1}\right)\right)$$

$$-\frac{\boldsymbol{\gamma\varepsilon}\left(1-\boldsymbol{\kappa}\right)}{\boldsymbol{\kappa\tau}^2 RT_*}\left\{\Delta\boldsymbol{\zeta}_N\left[\boldsymbol{\varpi}_N^+\left(\boldsymbol{\varpi}_{N+\frac{1}{2}}^+\boldsymbol{\alpha}_S + \boldsymbol{\varpi}_{N+\frac{1}{2}}^-\right) + \boldsymbol{\varpi}_N^-\boldsymbol{\varpi}_{N-\frac{1}{2}}^+\right]P_N + \Delta\boldsymbol{\zeta}_N\boldsymbol{\varpi}_N^-\boldsymbol{\varpi}_{N-\frac{1}{2}}^- P_{N-1}\right\}$$

with

$$C''_T = \frac{1/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} - \Delta\boldsymbol{\zeta}_1\boldsymbol{\varpi}_1^-\left(1/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} + \boldsymbol{\varepsilon}\left(1-\boldsymbol{\kappa}\right)\boldsymbol{\varpi}_{\frac{1}{2}}^-\right)}{1/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} + \boldsymbol{\varepsilon\varpi}_{\frac{1}{2}}^-}$$

$$C''_S = \frac{1/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}} + \Delta\boldsymbol{\zeta}_N\boldsymbol{\varpi}_N^+\left(1/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}} - \boldsymbol{\varepsilon}\left(1-\boldsymbol{\kappa}\right)\boldsymbol{\varpi}_{N+\frac{1}{2}}^+\right)}{1/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}} + \boldsymbol{\kappa\varpi}_{N+\frac{1}{2}}^+}$$

The vertical matrix problem may be decomposed into a combination of a diagonal $\mathbf{P}$ and a set of tri-diagonal matrices, $\mathbf{P}_{\delta\delta}, \mathbf{P}_{\delta\mu} = \mathbf{P}_{\mu\delta}, \mathbf{P}_{\mu\mu}$, representing respectively a double difference, a mean followed by a difference or a difference followed by a mean and a double mean as follows:

$$\mathbf{P}\left(L'_P\right) = \mathbf{P}\nabla^2_\zeta P + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\mathbf{P}_{\delta\delta} + \mathbf{P}_{\delta\mu} - \varepsilon\left(1-\kappa\right)\mathbf{P}_{\mu\mu}\right)P$$

The tri-diagonal matrix elements for $1<k<N$ are given by:

$$\mathbf{P}_{k+1,k} = 0 \qquad \mathbf{P}_{k-1,k} = 0 \qquad \mathbf{P}_{k,k} = \Delta\boldsymbol{\zeta}_k$$

$$\left(\mathbf{P}_{\delta\delta}\right)_{k+1,k} = 1/\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}} \qquad \left(\mathbf{P}_{\delta\delta}\right)_{k-1,k} = 1/\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}} \qquad \left(\mathbf{P}_{\delta\delta}\right)_{k,k} = -\left(\mathbf{P}_{\delta\delta}\right)_{k-1,k} - \left(\mathbf{P}_{\delta\delta}\right)_{k+1,k}$$

$$\left(\mathbf{P}_{\delta\mu}\right)_{k+1,k} = \Delta\boldsymbol{\zeta}_k\boldsymbol{\varpi}_k^+/\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}} \qquad \left(\mathbf{P}_{\delta\mu}\right)_{k-1,k} = -\Delta\boldsymbol{\zeta}_k\boldsymbol{\varpi}_k^-/\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}} \qquad \left(\mathbf{P}_{\delta\mu}\right)_{k,k} = -\left(\mathbf{P}_{\delta\mu}\right)_{k-1,k} - \left(\mathbf{P}_{\delta\mu}\right)_{k+1,k}$$

$$\left(\mathbf{P}_{\mu\mu}\right)_{k+1,k} = \Delta\boldsymbol{\zeta}_k\boldsymbol{\varpi}_k^+\boldsymbol{\varpi}_{k+\frac{1}{2}}^+ \qquad \left(\mathbf{P}_{\mu\mu}\right)_{k-1,k} = \Delta\boldsymbol{\zeta}_k\boldsymbol{\varpi}_k^-\boldsymbol{\varpi}_{k-\frac{1}{2}}^- \qquad \left(\mathbf{P}_{\mu\mu}\right)_{k,k} = \Delta\boldsymbol{\zeta}_k\left(\boldsymbol{\varpi}_k^+\boldsymbol{\varpi}_{k+\frac{1}{2}}^- + \boldsymbol{\varpi}_k^-\boldsymbol{\varpi}_{k-\frac{1}{2}}^+\right)$$

and for k=1 and k=N by

$$\left(\mathbf{P}_{\delta\delta}\right)_{1,1} = -\left(\mathbf{P}_{\delta\delta}\right)_{2,1} - \left(1-\boldsymbol{\alpha}_T\right)/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} \qquad \left(\mathbf{P}_{\delta\delta}\right)_{N,N} = -\left(\mathbf{P}_{\delta\delta}\right)_{N-1,N} - \left(1-\boldsymbol{\alpha}_S\right)/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}}$$

$$\left(\mathbf{P}_{\delta\mu}\right)_{1,1} = -\left(\mathbf{P}_{\delta\mu}\right)_{2,1} + \left(1-\boldsymbol{\alpha}_T\right)\Delta\boldsymbol{\zeta}_1\boldsymbol{\varpi}_1^-/\Delta\boldsymbol{\zeta}_{\frac{1}{2}} \qquad \left(\mathbf{P}_{\delta\mu}\right)_{N,N} = -\left(\mathbf{P}_{\delta\mu}\right)_{N-1,N} - \left(1-\boldsymbol{\alpha}_S\right)\Delta\boldsymbol{\zeta}_N\boldsymbol{\varpi}_N^+/\Delta\boldsymbol{\zeta}_{N+\frac{1}{2}}$$

$$\left(\mathbf{P}_{\mu\mu}\right)_{1,1} = \Delta\boldsymbol{\zeta}_1\left[\boldsymbol{\varpi}_1^+\boldsymbol{\varpi}_{\frac{3}{2}}^- + \boldsymbol{\varpi}_1^-\left(\boldsymbol{\varpi}_{\frac{1}{2}}^+ + \boldsymbol{\varpi}_{\frac{1}{2}}^-\boldsymbol{\alpha}_T\right)\right] \qquad \left(\mathbf{P}_{\mu\mu}\right)_{N,N} = \Delta\boldsymbol{\zeta}_N\left[\boldsymbol{\varpi}_N^+\left(\boldsymbol{\varpi}_{N+\frac{1}{2}}^+\boldsymbol{\alpha}_S + \boldsymbol{\varpi}_{N+\frac{1}{2}}^-\right) + \boldsymbol{\varpi}_N^-\boldsymbol{\varpi}_{N-\frac{1}{2}}^+\right]$$

After solving the elliptic problem and therefore knowing P_1 to P_N, we calculate P_0 and P_{N+1} using the relations:

$$P_0 = C_T\left(L''_T\right)_{\frac{1}{2}} + \boldsymbol{\alpha}_T P_1$$

$$P_{N+1} = \boldsymbol{\alpha}_S P_N - C_S\left(L'''_T\right)_{N+\frac{1}{2}}$$

Appendix 5. How were chosen the averaging operators and note about commutation

Let us consider two variables, *G* and *H*, defined on separate staggered grids as follows:

$$G_{k-\frac{1}{2}} = G(\zeta_{k-\frac{1}{2}}) \quad ; \quad H_k = H(\zeta_k)$$

indicating that *G* is defined on half-levels while *H* is defined on full ones. Only the independent variable $\boldsymbol{\zeta}$ could and was defined on both types of levels and thus take the two types of indices. The metric parameter could also sometimes be defined on both types of level, hence two different symbols (*B* on full and B on half levels). To obtain the variables *G* and *H* on their alternative grids, averaging operators $\boldsymbol{\alpha}$ and *a* such that:

$$(\boldsymbol{\alpha}G)_k = \boldsymbol{\alpha}_k G_{k+\frac{1}{2}} + (1-\boldsymbol{\alpha}_k)G_{k-\frac{1}{2}} \quad ; \quad (aH)_{k-\frac{1}{2}} = a_{k-\frac{1}{2}} H_k + \left(1 - a_{k-\frac{1}{2}}\right)H_{k-1}$$

are introduced. In the following discussion, difference operators will be needed and we define them:

$$(\boldsymbol{\delta}G)_k = \frac{G_{k+\frac{1}{2}} - G_{k-\frac{1}{2}}}{\boldsymbol{\zeta}_{k+\frac{1}{2}} - \boldsymbol{\zeta}_{k-\frac{1}{2}}} = \frac{G_{k+\frac{1}{2}} - G_{k-\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_k} \quad ; \quad (\boldsymbol{\delta}H)_{k-\frac{1}{2}} = \frac{H_k - H_{k-1}}{\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_{k-1}} = \frac{H_k - H_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}$$

Now, let us consider the discretized elliptic equation derived in **section 15** and which we write formally as follows:

$$\nabla_\zeta^2 P_k + \left[\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left(\boldsymbol{\delta}^2 + \boldsymbol{\alpha\delta} + \boldsymbol{\varepsilon}(\boldsymbol{\alpha\delta} - \boldsymbol{\delta}a) - \boldsymbol{\varepsilon}(1-\boldsymbol{\kappa})\boldsymbol{\alpha}a\right)P\right]_k = (L_P)_k$$

There is a term, $\boldsymbol{\varepsilon}(\boldsymbol{\alpha\delta} - \boldsymbol{\delta}a)$, which was assumed to vanish, which has no analytic equivalent but which vanishes only if the mean and difference operators commute. Let us impose this condition and examine the consequences. We get

$$(\boldsymbol{\alpha\delta}P)_k = \boldsymbol{\alpha}_k(\boldsymbol{\delta}P)_{k+\frac{1}{2}} + (1-\boldsymbol{\alpha}_k)(\boldsymbol{\delta}P)_{k-\frac{1}{2}} = \boldsymbol{\alpha}_k \frac{P_{k+1} - P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} + (1-\boldsymbol{\alpha}_k)\frac{P_k - P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}$$

$$(\boldsymbol{\delta}aP)_k = \frac{(aP)_{k+\frac{1}{2}} - (aP)_{k-\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_k} = \frac{a_{k+\frac{1}{2}}P_{k+1} + \left(1 - a_{k+\frac{1}{2}}\right)P_k - a_{k-\frac{1}{2}}P_k - \left(1 - a_{k-\frac{1}{2}}\right)P_{k-1}}{\Delta\boldsymbol{\zeta}_k}$$

implying

$$\frac{\Delta\boldsymbol{\zeta}_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} = \frac{a_{k+\frac{1}{2}}}{\boldsymbol{\alpha}_k}; \qquad \frac{\Delta\boldsymbol{\zeta}_k}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} = \frac{1 - a_{k-\frac{1}{2}}}{1-\boldsymbol{\alpha}_k}$$

and either

$$\text{(a)} \qquad \frac{\Delta\boldsymbol{\zeta}_{k+1}}{\Delta\boldsymbol{\zeta}_k} = \frac{\boldsymbol{\alpha}_k}{1-\boldsymbol{\alpha}_{k+1}} \frac{1-a_{k+\frac{1}{2}}}{a_{k+\frac{1}{2}}} \quad \text{or} \quad \text{(b)} \; \frac{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} = \frac{\boldsymbol{\alpha}_k}{1-\boldsymbol{\alpha}_k} \frac{1-a_{k-\frac{1}{2}}}{a_{k+\frac{1}{2}}}$$

If the relation between the half and full levels is given, for example if, as we have chosen:

$$\boldsymbol{\zeta}_{k-\frac{1}{2}} = \frac{\boldsymbol{\zeta}_k + \boldsymbol{\zeta}_{k-1}}{2}$$

then we most likely want

$$a_{k-\frac{1}{2}} = \frac{1}{2}$$

From (b) we get

$$\boldsymbol{\alpha}_k = \frac{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}} + \Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} = \frac{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}}{2\Delta\boldsymbol{\zeta}_k}$$

and thus

$$(\boldsymbol{\alpha}G)_k = \frac{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}} G_{k+\frac{1}{2}} + \Delta\boldsymbol{\zeta}_{k-\frac{1}{2}} G_{k-\frac{1}{2}}}{2\Delta\boldsymbol{\zeta}_k} \quad ; \quad (aH)_{k-\frac{1}{2}} = \frac{H_k + H_{k-1}}{2}$$

Instead of choosing $a_{k-\frac{1}{2}}$ off-hand as we have done, we might have imposed another condition such as the symmetry of matrix M formed by the product of the matrix obtained from the double averaging operator αa and the diagonal matrix with elements $\Delta\zeta_k$, i.e. if we had imposed that the tri-diagonal matrix M whose elements are

$$\begin{aligned}(\Delta\boldsymbol{\zeta}\boldsymbol{\alpha}aP)_k &= \Delta\boldsymbol{\zeta}_k\boldsymbol{\alpha}_k\left(a_{k+\frac{1}{2}}P_{k+1} + \left(1-a_{k+\frac{1}{2}}\right)P_k\right) + (1-\boldsymbol{\alpha}_k)\Delta\boldsymbol{\zeta}_k\left(a_{k-\frac{1}{2}}P_k + \left(1-a_{k-\frac{1}{2}}\right)P_{k-1}\right) \\ &= M_{k+1,k}P_{k+1} + M_{k,k}P_k + M_{k-1,k}P_{k-1}\end{aligned}$$

be symmetric, i.e. setting $M_{k+1,k} = M_{k,k+1}$, i.e. $\Delta\boldsymbol{\zeta}_k\boldsymbol{\alpha}_k a_{k+\frac{1}{2}} = (1-\boldsymbol{\alpha}_{k+1})\Delta\boldsymbol{\zeta}_{k+1}\left(1-a_{k+\frac{1}{2}}\right)$, i.e.

$$\text{(c)} \qquad \frac{\Delta\boldsymbol{\zeta}_{k+1}}{\Delta\boldsymbol{\zeta}_k} = \frac{\boldsymbol{\alpha}_k}{1-\boldsymbol{\alpha}_{k+1}} \frac{a_{k+\frac{1}{2}}}{1-a_{k+\frac{1}{2}}}$$

Then, combining (c) with (a), we would have again found

$$a_{k-\frac{1}{2}} = \frac{1}{2}$$

In the original formulation of the staggered-grid version of the model, we indeed wanted to obtain symmetric matrices in the vertical (maintaining a property of the regular-grid version of the model) and commutation occurred naturally (only one mean being explicit in the code, the second one occurring only in the elimination process). With the new coordinate we lost the symmetric property due to the presence of a first derivative in the analytic problem. But the requirement that half-levels be exactly in the middle of full levels is good for the accuracy of the hydrostatic relation and the commutation requirement, besides simplifying the code, may serve in improving the conservative properties of the scheme.

So far we have dealt with the difference and average operators away from the boundaries. Let us now look at them near the boundaries. The equations defined on half levels apply to the top and bottom where difference and average operators operate on some variables, namely ϕ' and q. But their values are required at one of the boundaries [$\phi'_S = \phi_S$ at the surface and q_T=0 at the top] while their values at the other can be obtained by numerical integration provided the difference operator leading to them is defined which it has been (it is by construction an off-centered difference though). This is why we consider the top and bottom to be full levels as far as ϕ' and q are concerned, respectively labeled 0 and N+1. The averaging operator then simply selects the corresponding value.

One last item remains to be explained referring to **Figure 1** describing Charney-Phillips grid. In the preceding paragraph, we said that some equations applied to the boundaries. With the presence of ϕ' and q, all required variables were also apparently defined there but, if so, the difference operators were off-centered and therefore only first order. Centered differences are recovered if we displace the thermodynamic and vertical momentum equations as well as the variables T and w to the middle of the half-layers nearing the boundaries [to levels ¾ and N+ ¼ as shown in the figure]. This is what we have done. We believe this is beneficial for temperature in particular which is shifted from the surface to a better place from the physical as well as numerical point of view. To better assess what we have done, here is a formal representation of the three linear equations affected by the change for the top (a similar change occurs at the bottom):

$$\left(L'_w\right)_{\frac{3}{4}} = \frac{w_{\frac{3}{4}}}{\boldsymbol{\tau}} - g\frac{q_1 - q_{\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_{\frac{3}{4}}}$$

$$\left(L'_T\right)_{\frac{3}{4}} = \frac{1}{\boldsymbol{\tau}}\frac{q_1 - q_{\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_{\frac{3}{4}}} - \frac{1}{\boldsymbol{\tau}RT_*}\frac{P_1 - P_{\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_{\frac{3}{4}}} - \boldsymbol{\kappa}\left[\dot{\boldsymbol{\zeta}} + \frac{\mathrm{B}s + q}{\boldsymbol{\tau}}\right]_{\frac{1}{2}}$$

$$\left(L_{\boldsymbol{\phi}}\right)_{\frac{1}{2}} = \frac{P_{\frac{1}{2}}}{\boldsymbol{\tau}} - RT_*\dot{\boldsymbol{\zeta}}_{\frac{1}{2}} - gw_{\frac{3}{4}}$$

In the thermodynamic equation defined at level ¾ the term in brackets remains evaluated at the boundary, level ½. In the geopotential equation defined at level ½, w is taken at level ¾. This can be interpreted, in the first case as an interpolation, in the second case as an extrapolation, constant in both cases [$f_{\frac{3}{4}} = \boldsymbol{\alpha} f_{\frac{3}{2}} + (1-\boldsymbol{\alpha}) f_{\frac{1}{2}}$ with $\boldsymbol{\alpha}$=0].

Appendix 6. The Dynamic Core Code and vertical discretization: *A brief description*

The dynamic core code is essentially organized as follows:

set_zeta, set_dync, set_oprz, preverln: *compute* constants *and* parameters *of the vertical discretization*

***Timestep** Loop*

tstpdyn: *performs a* dynamical time step *calling* **rhs, adw, pre, nli, sol, bac**

- **rhs:** compute the 6 basic Right-Hand-Side terms: $\mathbf{R}_h$, R_w, R_T, R_C, R_ϕ (**section 14**)

$$\mathbf{R}_h = \frac{\mathbf{V}_h}{\boldsymbol{\tau}} \qquad - \boldsymbol{\beta}\left(f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_\zeta (Bs+q) + \left(1+\overline{\boldsymbol{\mu}}^{\zeta}\right)\nabla_\zeta \boldsymbol{\phi}' \right)$$

$$R_w = \frac{w}{\boldsymbol{\tau}} \qquad - \boldsymbol{\beta}(-g\boldsymbol{\mu})$$

$$R_T = \frac{1}{\boldsymbol{\tau}}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left(\mathrm{B}s + \overline{q}^{\zeta}\right) \right] \qquad - \boldsymbol{\beta}\left(-\boldsymbol{\kappa}\dot{\boldsymbol{\zeta}}\right)$$

$$R_C = \frac{1}{\boldsymbol{\tau}}\left[\overline{\mathrm{B}}^{\zeta} s + \ln\left(1+\boldsymbol{\delta}_\zeta \mathrm{B}s\right) \right] \qquad - \boldsymbol{\beta}\left(\nabla_\zeta \cdot \mathbf{V}_h + \boldsymbol{\delta}_\zeta \dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta} \right)$$

$$R_\phi = \frac{\overline{\boldsymbol{\phi}'}^{\zeta}}{\boldsymbol{\tau}} \qquad - \boldsymbol{\beta}\left(-RT_*\dot{\boldsymbol{\zeta}} - gw\right)$$

***Departure** Outer-Loop*

- **adw:** **adw_pos**: Compute the next estimate of the departure points.
  adw_int: Evaluate Right-Hand-Side terms at departure points.

- **pre:** combine R_T, R_w, R_ϕ into R''_T, R''_ϕ, combine $\mathbf{R}_h, R_w, R_C$ into R''_C and finally R''_C, R''_T, R''_ϕ into R_P: (**section 16**)

$$\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(R_T + \frac{R_w}{g\boldsymbol{\tau}} + \frac{\boldsymbol{\varepsilon}}{RT_*}R_\phi \right) \equiv R''_T$$

$$\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(R_T + \frac{R_w}{g\boldsymbol{\tau}} - \frac{\boldsymbol{\kappa}}{RT_*}R_\phi \right) \equiv R''_\phi$$

$$\nabla_\zeta \cdot \mathbf{R}_h - \frac{1}{\boldsymbol{\tau}}\left(R_C - \frac{\overline{R_w}^{\zeta}}{g\boldsymbol{\tau}} \right) \equiv R''_C$$

$$R''_C - \left(\boldsymbol{\delta}_\zeta R''_T + \overline{R''_T}^{\zeta} \right) - \boldsymbol{\varepsilon} \overline{R''_{\boldsymbol{\phi}}}^{\zeta} \equiv R_P$$

The final version of the Right-Hand sides are: $\mathbf{R}_h$, R_w, R''_T, $R''_{\boldsymbol{\phi}}$, R_P

Non-linear *Inner_Loop*

- **nli:** compute non-linear Left-Hand sides: $\mathbf{N}_h, N'_w, N'_T, N_C, N_{\boldsymbol{\phi}}$

(section 12)

$$\mathbf{N}_h = f\mathbf{k} \times \mathbf{V}_h + R\overline{T'}^{\zeta} \nabla_\zeta (Bs + q) + \overline{\boldsymbol{\mu}}^{\zeta} \nabla_\zeta \boldsymbol{\phi}'$$

$$N'_w = -g(\boldsymbol{\mu} - \boldsymbol{\delta}_\zeta q)$$

$$N'_T = \frac{1}{\boldsymbol{\tau}}\left[\ln\left(\frac{T}{T_*} \right) + \frac{\boldsymbol{\delta}_\zeta (\boldsymbol{\phi}' + RT_* Bs)}{RT_*} \right]$$

$$N_C = \frac{1}{\boldsymbol{\tau}}\left[\ln(1 + \boldsymbol{\delta}_\zeta \mathrm{B}s) - \boldsymbol{\delta}_\zeta \mathrm{B}s \right]$$

$$N_{\boldsymbol{\phi}} = 0$$

and combine them into $N''_T, N''_{\boldsymbol{\phi}}, N''_C, N_P$

(section 16)

$$\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(N'_T + \frac{N'_w}{g\boldsymbol{\tau}} + \frac{\boldsymbol{\varepsilon}}{RT_*} N_{\boldsymbol{\phi}} \right) \equiv N''_T$$

$$\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}}\left(N'_T + \frac{N'_w}{g\boldsymbol{\tau}} - \frac{\boldsymbol{\kappa}}{RT_*} N_{\boldsymbol{\phi}} \right) \equiv N''_{\boldsymbol{\phi}}$$

$$\nabla_\zeta \cdot \mathbf{N}_h - \frac{1}{\boldsymbol{\tau}}\left(N_C - \frac{\overline{N'_w}^{\zeta}}{g\boldsymbol{\tau}} \right) \equiv N''_C$$

$$N''_C - \left(\boldsymbol{\delta}_\zeta N''_T + \overline{N''_T}^{\zeta} \right) - \boldsymbol{\varepsilon} \overline{N''_{\boldsymbol{\phi}}}^{\zeta} \equiv N_P$$

and obtain final Right-Hand Side of the Elliptic Problem $L_P = R_P - N_P$, including modifications imposed by boundary conditions $(L'_P)_1$ and $(L'_P)_N$.

(appendix 4b)

- **sol:** solve the Elliptic Problem

(section 15 & appendices 4a and 4b)

$$\mathbf{P}\left(L'_P\right) = \mathbf{P}\nabla_\zeta^2 P + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\mathbf{P}_{\delta\delta} + \mathbf{P}_{\delta\mu} - \varepsilon(1-\kappa)\mathbf{P}_{\mu\mu}\right)P$$

- **bac:** back substitution: compute variables for next iteration/time step

(section 17)

$$\frac{\mathbf{V}_h}{\tau} = \left[\mathbf{R}_h - \mathbf{N}_h - \nabla_\zeta P\right]$$

$$\frac{w}{\tau} = -\frac{RT_*}{g}\left[R''_\phi - N''_\phi + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P + \kappa \overline{P}^\zeta\right)\right]$$

$$\delta_\zeta q + \overline{q}^\zeta = -\frac{1}{g}\left[R_w - N'_w - \frac{w}{\tau}\right]$$

$$s = \frac{P_S - \phi_S}{RT_*} - q_S$$

$$\frac{\dot{\zeta}}{\tau} = -\left[R''_T - N''_T + \frac{\gamma}{\kappa\tau^2 RT_*}\left(\delta_\zeta P - \varepsilon \overline{P}^\zeta\right)\right] - \frac{\mathrm{B}s + \overline{q}^\zeta}{\tau^2}$$

$$\phi = P - RT_*(Bs + q)$$

$$1 + \mu = e^{\overline{q}^\zeta}\left[\frac{\delta_\zeta q}{\delta_\zeta(\zeta + Bs)} + 1\right]$$

$$\frac{T}{T_*} = e^{\overline{q}^\zeta}\,\frac{\delta_\zeta(\zeta - \phi / RT_*)}{\delta_\zeta(\zeta + Bs)}$$

end inner loop

end outer loop

end timestep loop

Appendix 7. The hydrostatic option

We start with the final form of the equations given in **section 5**:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\boldsymbol{\mu})\nabla_\zeta\boldsymbol{\phi}' = 0$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}(Bs+q)\right] - \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}} = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial \boldsymbol{\zeta}}s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \left(\frac{\partial}{\partial \boldsymbol{\zeta}} + 1\right)\dot{\boldsymbol{\zeta}} = 0$$

$$\frac{d\boldsymbol{\phi}'}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$

$$1 + \boldsymbol{\mu} - e^q\left[\frac{\partial q}{\partial(\boldsymbol{\zeta} + Bs)} + 1\right] = 0$$

$$\frac{T}{T_*} - e^q\frac{\partial(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*)}{\partial(\boldsymbol{\zeta} + Bs)} = 0$$

The hydrostatic approximation may be considered to consist in neglecting non-hydrostatic pressure effects, therefore assuming q=0. Then μ=0 also and the vertical acceleration *dw/dt* is neglected. In fact, the vertical motion w becomes irrelevant. Neither the vertical momentum nor the geopotential tendency equations are required in the solution system although we may still solve the geopotential tendency equation to diagnose w. Therefore, we only need to solve:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta\boldsymbol{\phi}' = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}Bs\right] - \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}} = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial \boldsymbol{\zeta}}s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \left(\frac{\partial}{\partial \boldsymbol{\zeta}} + 1\right)\dot{\boldsymbol{\zeta}} = 0$$

$$\frac{T}{T_*} - \frac{\partial(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*)}{\partial(\boldsymbol{\zeta} + Bs)} = 0$$

All the terms involving the prognostic vertical momentum and diagnostic μ equations which were not already equal to zero are set to vanish: F_w, G_w, L_w, N_w, R_w, L_μ, N_μ, L'_w, N'_w . The parameter ε =0, hence γ=1.

Appendix 8. The autobarotropic model

We build an auto-barotropic model (Dutton, *The Ceaseless Wind*, pp 186-7) from the three-dimensional code of GEM in order to simulate a barotropic model. We do that in

i) eliminating the physical effects,
ii) making the hydrostatic hypothesis,
iii) introducing a key $\delta_{autobarot}=0$ to eliminate the pressure tendency $d(Bs)/dt$ in both the thermodynamic and continuity equations,
iv) in initializing with barotropic conditions :

$$\mathbf{V}_h \neq \mathbf{V}_h(\zeta);\ T = T_* = const;\ \dot{\zeta} = 0;\ \phi' + RT_*Bs = \phi'_T = \phi_S + RT_*s\ ,$$

conditions which will be maintained afterwards, hence the name autobarotropic model.

From the complete equations:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\boldsymbol{\mu})\nabla_\zeta\boldsymbol{\phi}' = \mathbf{F}_h$$

$$\boldsymbol{\delta}_H\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left[\frac{d}{dt}(Bs+q) + \dot{\boldsymbol{\zeta}}\right] = \frac{Q}{c_pT}$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial\boldsymbol{\zeta}}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\boldsymbol{\zeta}}}{\partial\boldsymbol{\zeta}} + \dot{\boldsymbol{\zeta}} = 0$$

$$\frac{d\boldsymbol{\phi}'}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$

$$1 + \boldsymbol{\mu} - e^q\left[\frac{\partial q}{\partial(\boldsymbol{\zeta}+Bs)} + 1\right] = 0$$

$$\frac{T}{T_*} - e^q\frac{\partial(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*)}{\partial(\boldsymbol{\zeta}+Bs)} = 0$$

with B defined simply as $B = \dfrac{\zeta - \zeta_T}{\zeta_S - \zeta_T}$, we eliminate sources and sinks of momentum and heat and we make the hydrostatic approximation, reducing the number of equations and variables to (see **Appendix 7**):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta \phi' = 0$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}(Bs) + \dot{\zeta}\right] = 0$$

$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial \zeta}s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

$$\frac{T}{T_*} - \frac{\partial(\zeta - \phi'/RT_*)}{\partial(\zeta + Bs)} = 0$$

Considering barotropic initial conditions ($\mathbf{V}_h \neq \mathbf{V}_h(\zeta);\ T = T_* = const;\ \dot{\zeta} = 0$), we derive from the hydrostatic equation that P is uniform in the vertical:

$$P = \phi' + RT_*Bs = \phi'_T = \phi_S + RT_*s \neq P(\zeta)$$

and we note that

$$s = \frac{\phi'_T - \phi_S}{RT_*}$$

Indeed,

$$1 - \frac{\partial(\zeta - \phi'/RT_*)}{\partial(\zeta + Bs)} = \frac{\partial(\phi'/RT_* + Bs)}{\partial(\zeta + Bs)} = \frac{\partial(\phi'/RT_* + Bs)}{\partial \zeta}\frac{\partial \zeta}{\partial(\zeta + Bs)} = 0$$

$$\frac{\partial(\phi' + RT_*Bs)}{\partial \zeta} = \frac{\partial P}{\partial \zeta} = 0$$

We therefore have in the momentum equation:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + \nabla_\zeta \phi_T = 0$$

And since $P = \phi_T \neq P(\zeta)$, then $\mathbf{V}_h$ stays $\mathbf{V}_h \neq \mathbf{V}_h(\zeta)$.

Now, even though $\dot{\zeta} = 0$ and $T = T_* = const$ initially, temperature will change since the thermodynamic equation still says:

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) = \kappa\frac{d}{dt}(Bs) \neq 0$$

However if we write

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\delta_{autobarot}\frac{d}{dt}(Bs) = 0$$

making $\delta_{autobarot} = 0$, then $\frac{d}{dt}\ln\left(\frac{T}{T_*}\right)$=0 et T will remain constant and equal to T_*. Similarly, in the continuity equation, $\dot{\zeta} = 0$ initially and introducing $\delta_{autobarot} = 0$:

$$\frac{d}{dt}\left[\boldsymbol{\delta}_{autobarot} Bs + \ln\left(1 + \frac{\partial B}{\partial \boldsymbol{\zeta}} s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\boldsymbol{\zeta}}}{\partial \boldsymbol{\zeta}} + \dot{\boldsymbol{\zeta}} = 0$$

we get

$$\frac{d}{dt}\ln\left(1 + \frac{\partial B}{\partial \zeta} s\right) + \nabla_\zeta \cdot \mathbf{V}_h = 0$$

$$\frac{d}{dt}\ln\left(1 + \frac{\phi'_T - \phi_S}{RT_*(\zeta_S - \zeta_T)}\right) + \nabla_\zeta \cdot \mathbf{V}_h = 0$$

$$\frac{d}{dt}\ln\left(\frac{\phi_{*T} + \phi'_T - \phi_S}{\phi_{*T}}\right) + \nabla_\zeta \cdot \mathbf{V}_h = 0$$

$$\frac{d}{dt}\ln(\phi_T - \phi_S) + \nabla_\zeta \cdot \mathbf{V}_h = 0$$

And this relation is invariant in the vertical, hence $\dot{\zeta} \neq \dot{\zeta}(\zeta)$ and $\dot{\zeta}$ =0 is maintained Hence, the model equations:

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\text{x}\mathbf{V}_h + RT\nabla_\zeta Bs + \nabla_\zeta \phi' = 0$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\delta_{autobarot}\frac{d}{dt}(Bs) + \dot{\zeta}\right] = 0$$

$$\frac{d}{dt}\left[\delta_{autobarot} Bs + \ln\left(1 + \frac{\partial B}{\partial \zeta} s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

$$\frac{T}{T_*} - \frac{\partial(\zeta - \phi'/RT_*)}{\partial(\zeta + Bs)} = 0$$

with a vertical structure (many levels, at least 3: e.g. hyb = 0.583333, 0.75, 0.9166666 with p_{top}=50000., to satisfy the operations), but starting with barotropic conditions, simulates the barotropic equations:

$$\boxed{\begin{aligned} \frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\text{x}\mathbf{V}_h + \nabla_\zeta \phi_T &= 0 \\ \frac{d}{dt}\ln(\phi_T - \phi_S) + \nabla_\zeta \cdot \mathbf{V}_h &= 0 \end{aligned}}$$

It is autobarotropic.

Appendix 9. *Open top* boundary conditions

The goal is to develop an open boundary condition at the top, i.e. a condition with $X_T = \left(\dot{\zeta} + \frac{Bs+q}{\tau}\right)_T \neq 0$, not only $\dot{\zeta}_T \neq 0$ but also $B_T \neq 0$ (the top no more being necessarily a hydrostatic pressure level) and $q_T \neq 0$ (in the non-hydrostatic case).

First, let us deal with the linear system (**Appendix 4a**):

$$\left(\mathbf{L}_h\right)_k = \frac{\mathbf{V}_{h\,k}}{\boldsymbol{\tau}} + \nabla_\zeta P_k$$

$$\left(L'_w\right)_{k-\frac{1}{2}} = \frac{w_{k-\frac{1}{2}}}{\boldsymbol{\tau}} - g\left(\frac{q_k - q_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} + \boldsymbol{\varpi}^+_{k-\frac{1}{2}} q_k + \boldsymbol{\varpi}^-_{k-\frac{1}{2}} q_{k-1}\right)$$

$$\left(L_T\right)_{k-\frac{1}{2}} = \frac{1}{\boldsymbol{\tau}}\frac{T'_{k-\frac{1}{2}}}{T_*} - \boldsymbol{\kappa} X_{k-\frac{1}{2}}$$

$$\left(L_C\right)_k = -\frac{1}{\boldsymbol{\tau}}\left[\boldsymbol{\varpi}^+_k\left(\frac{q_{k+1} - q_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} + \boldsymbol{\varpi}^+_{k+\frac{1}{2}} q_{k+1} + \boldsymbol{\varpi}^-_{k+\frac{1}{2}} q_k\right) + \boldsymbol{\varpi}^-_k\left(\frac{q_k - q_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}} + \boldsymbol{\varpi}^+_{k-\frac{1}{2}} q_k + \boldsymbol{\varpi}^-_{k-\frac{1}{2}} q_{k-1}\right)\right]$$
$$+ \nabla_\zeta \cdot \mathbf{V}_{h\,k} + \frac{X_{k+\frac{1}{2}} - X_{k-\frac{1}{2}}}{\Delta\boldsymbol{\zeta}_k} + \boldsymbol{\varpi}^+_k X_{k+\frac{1}{2}} + \boldsymbol{\varpi}^-_k X_{k-\frac{1}{2}}$$

$$\left(L_\phi\right)_{k-\frac{1}{2}} = \frac{1}{\boldsymbol{\tau}}\left(\boldsymbol{\varpi}^+_{k-\frac{1}{2}} P_k + \boldsymbol{\varpi}^-_{k-\frac{1}{2}} P_{k-1}\right) - RT_* X_{k-\frac{1}{2}} - g w_{k-\frac{1}{2}}$$

We know we can combine these equations into a set of only N equations in the vertical for N+2 unknowns P_k (k=k_0,N):

$$\left(L_P\right)_k = \nabla^2_\zeta P_k + \frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\frac{1}{\Delta\boldsymbol{\zeta}_k}\left(\frac{P_{k+1} - P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} - \frac{P_k - P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right) + \boldsymbol{\varpi}^+_k \frac{P_{k+1} - P_k}{\Delta\boldsymbol{\zeta}_{k+\frac{1}{2}}} + \boldsymbol{\varpi}^-_k \frac{P_k - P_{k-1}}{\Delta\boldsymbol{\zeta}_{k-\frac{1}{2}}}\right]$$
$$- \frac{\boldsymbol{\gamma\varepsilon}(1-\boldsymbol{\kappa})}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\boldsymbol{\varpi}^+_k\left(\boldsymbol{\varpi}^+_{k+\frac{1}{2}} P_{k+1} + \boldsymbol{\varpi}^-_{k+\frac{1}{2}} P_k\right) + \boldsymbol{\varpi}^-_k\left(\boldsymbol{\varpi}^+_{k-\frac{1}{2}} P_k + \boldsymbol{\varpi}^-_{k-\frac{1}{2}} P_{k-1}\right)\right]$$

and therefore requiring two additional equations (top and bottom boundary conditions) for its solution. A closed top boundary condition occurring at k_0=1 ($\dot{\boldsymbol{\zeta}}_T = 0; \mathrm{B}_T = 0; q_T = 0$) is satisfied by using

$$\left(L''_T\right)_{k_0-\frac{1}{2}} + \frac{X_T}{\boldsymbol{\tau}} = -\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\frac{P_{k_0} - P_{k_0-1}}{\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}} - \boldsymbol{\varepsilon}\left(\boldsymbol{\varpi}^+_{k_0-\frac{1}{2}} P_k + \boldsymbol{\varpi}^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right]$$

to obtain a boundary condition in terms of X (*generalized vertical motion* $\dot{\zeta}$) since $X_T = \left[\dot{\boldsymbol{\zeta}} + (\mathrm{B}s + q)/\boldsymbol{\tau}\right]_T = 0$. For an open top occurring at $k_0 \neq 1$, we have none of the above conditions ($\dot{\boldsymbol{\zeta}}_T \neq 0; \mathrm{B}_T \neq 0; q_T \neq 0$). Another relation must be found. There are two possibilities:

- (i) using L''_{ϕ} to obtain a boundary condition in terms of *vertical motion w,* specifying w_{top}:

$$\left(L''_{\phi}\right)_{k_0-\frac{1}{2}} - \frac{g w_{\mathrm{top}}}{\boldsymbol{\tau} R T_*} = -\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa}\boldsymbol{\tau}^2 R T_*}\left[\frac{P_{k_0} - P_{k_0-1}}{\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}} + \boldsymbol{\kappa}\left(\boldsymbol{\varpi}^+_{k_0-\frac{1}{2}} P_{k_0} + \boldsymbol{\varpi}^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right]$$

- (ii) combining L''_T with L_T as follows

$$\left(L''_T\right)_{k_0-\frac{1}{2}} - \frac{1}{\boldsymbol{\kappa}\boldsymbol{\tau}}\left(L_T\right)_{k_0-\frac{1}{2}} + \frac{1}{\boldsymbol{\kappa}\boldsymbol{\tau}^2}\frac{T'_{top}}{T_*} = -\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa}\boldsymbol{\tau}^2 R T_*}\left[\frac{P_{k_0} - P_{k_0-1}}{\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}} - \boldsymbol{\varepsilon}\left(\boldsymbol{\varpi}^+_{k_0-\frac{1}{2}} P_k + \boldsymbol{\varpi}^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right]$$

to obtain a boundary condition in terms of *temperature T*, specifying T'_{top}:

$$-\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa}\boldsymbol{\tau}^2 R T_*}\left(\frac{P_{k_0} - P_{k_0-1}}{\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}} - \boldsymbol{\varepsilon}\left(\boldsymbol{\varpi}^+_{k_0-\frac{1}{2}} P_{k_0} + \boldsymbol{\varpi}^-_{k_0-\frac{1}{2}} P_{k_0-1}\right)\right) = L_B = R_B - N_B + \frac{T'_{top}}{\boldsymbol{\kappa}\boldsymbol{\tau}^2 T_*}$$

Although vertical motion *w* seems the logical choice, there are two big objections: first, it is well known that vertical motion can be quite noisy and it could be difficult to get a suitably balanced field; second, in the hydrostatic case, *w* is not even a prognostic variable of the model.

The open top case (ii) in fact leads to equations for P_{k_0-1}, ($k_0 \neq 1$) formally identical to the closed top case with k_0=1 (see Appendices 4a and 4b). In effect, we write

$$P_{k_0-1} = \boldsymbol{\alpha}_T P_{k_0} + C_T L_B$$

with

$$\boldsymbol{\alpha}_T = \frac{1/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}} - \boldsymbol{\varepsilon}\boldsymbol{\varpi}^+_{k_0-\frac{1}{2}}}{1/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}} + \boldsymbol{\varepsilon}\boldsymbol{\varpi}^-_{k_0-\frac{1}{2}}}; \qquad C_T = \frac{\boldsymbol{\kappa}\boldsymbol{\tau}^2 R T_*}{\boldsymbol{\gamma}}\frac{1}{1/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}} + \boldsymbol{\varepsilon}\boldsymbol{\varpi}^-_{k_0-\frac{1}{2}}}$$

Therefore

$$\Delta\boldsymbol{\zeta}_{k_0}\left(L_P\right)_{k_0} - C''_T L_B = \Delta\boldsymbol{\zeta}_{k_0}\left(\nabla_\zeta^2 P\right)_{k_0}$$

$$+\frac{\boldsymbol{\gamma}}{\boldsymbol{\kappa\tau}^2 RT_*}\left[\frac{P_{k_0+1}-P_{k_0}}{\Delta\boldsymbol{\zeta}_{k_0+\frac{1}{2}}} - \frac{(1-\boldsymbol{\alpha}_T)P_{k_0}}{\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}} + \frac{\Delta\boldsymbol{\zeta}_{k_0}\boldsymbol{\varpi}^+_{k_0}}{\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}}\left(P_{k_0+1}-P_{k_0}\right) + \frac{\Delta\boldsymbol{\zeta}_{k_0}\boldsymbol{\varpi}^-_k}{\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}}(1-\boldsymbol{\alpha}_T)P_{k_0}\right]$$

$$-\frac{\boldsymbol{\gamma\varepsilon}(1-\boldsymbol{\kappa})}{\boldsymbol{\kappa\tau}^2 RT_*}\Delta\boldsymbol{\zeta}_{k_0}\left\{\boldsymbol{\varpi}^+_{k_0}\boldsymbol{\varpi}^+_{k_0+\frac{1}{2}}P_{k_0+1} + \left[\boldsymbol{\varpi}^+_{k_0}\boldsymbol{\varpi}^-_{k_0+\frac{1}{2}} + \boldsymbol{\varpi}^-_{k_0}\left(\boldsymbol{\varpi}^+_{k_0-\frac{1}{2}} + \boldsymbol{\varpi}^-_{k_0-\frac{1}{2}}\boldsymbol{\alpha}_T\right)\right]P_{k_0}\right\}$$

with

$$C''_T = \frac{1/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}} - \Delta\boldsymbol{\zeta}_{k_0}\boldsymbol{\varpi}^-_{k_0}\left(1/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}} + \boldsymbol{\varepsilon}(1-\boldsymbol{\kappa})\boldsymbol{\varpi}^-_{k_0-\frac{1}{2}}\right)}{1/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}} + \boldsymbol{\varepsilon}\boldsymbol{\varpi}^-_{k_0-\frac{1}{2}}}$$

and the matrix elements for $k=k_0$ are

$$\left(\mathbf{P}_{\delta\delta}\right)_{k_0,k_0} = -\left(\mathbf{P}_{\delta\delta}\right)_{k_0+1,k_0} - (1-\boldsymbol{\alpha}_T)/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}$$

$$\left(\mathbf{P}_{\delta\mu}\right)_{k_0,k_0} = -\left(\mathbf{P}_{\delta\mu}\right)_{k_0+1,k_0} + (1-\boldsymbol{\alpha}_T)\Delta\boldsymbol{\zeta}_{k_0}\boldsymbol{\varpi}^-_{k_0}/\Delta\boldsymbol{\zeta}_{k_0-\frac{1}{2}}$$

$$\left(\mathbf{P}_{\mu\mu}\right)_{k_0,k_0} = \Delta\boldsymbol{\zeta}_{k_0}\left[\boldsymbol{\varpi}^+_{k_0}\boldsymbol{\varpi}^-_{k_0+\frac{1}{2}} + \boldsymbol{\varpi}^-_{k_0}\left(\boldsymbol{\varpi}^+_{k_0-\frac{1}{2}} + \boldsymbol{\varpi}^-_{k_0-\frac{1}{2}}\boldsymbol{\alpha}_T\right)\right]$$

All of this is trivial then (in **Appendices 4a** and **4b**, replace indices 1, 3/2, 1/2, respectively by k_0, k_0+1/2, k_0-1/2), except for the calculation of the right-hand sides corresponding to L_B, i.e. R_B and N_B:

$$R_B = \left(R''_T\right)_{k_0-\frac{1}{2}} - \frac{1}{\boldsymbol{\kappa\tau}}\left(R_T\right)_{k_0-\frac{1}{2}}$$

$$N_B = \left(N''_T\right)_{k_0-\frac{1}{2}} - \frac{1}{\boldsymbol{\kappa\tau}}\left(N_T\right)_{k_0-\frac{1}{2}}$$

More explicitly for N_B,

$$N_B = \left(N''_T\right)_{k_0-\frac{1}{2}} - \frac{1}{\boldsymbol{\kappa\tau}^2}\left[\ln\left(\frac{T_{k_0-\frac{1}{2}}}{T_*}\right) - \frac{T'_{k_0-\frac{1}{2}}}{T_*}\right]$$

In the non-hydrostatic case, another condition is needed, namely the true pressure at the top, p_{top}, from which we may calculate

$$q_T = \ln(p_{top}/\boldsymbol{\pi}_T) = \ln p_{top} - \left(\boldsymbol{\zeta}_T + \mathrm{B}_T s\right) = \ln p_{top} - \left(\boldsymbol{\zeta}_{k_0-\frac{1}{2}} + \mathrm{B}_{k_0-\frac{1}{2}}s\right)$$

Appendix 10. Aspects of horizontal discretization

First of all, note that by 'horizontal' is meant a model 'quasi-spherical' constant ζ surface. In the horizontal then, the equations in spherical coordinates are discretized on an Arakawa C grid, with the wind image components $U_{i+\frac{1}{2},j}\ (i=0,N_i,\ j=1,N_j)$ and $V_{i,j+\frac{1}{2}}\ (i=1,N_i,\ j=0,N_j)$ staggered with respect to all the other variables $w_{i,j}, T_{i,j}, (\dot{\boldsymbol{\zeta}}_{i,j}, s_{i,j}), \boldsymbol{\phi}'_{i,j}, \boldsymbol{\mu}_{i,j}\ (i=1,N_i,\ j=1,N_j)$, an Arakawa C grid with the *U* points with indices *i*=0 and *i*=N_i coinciding by symmetry and with the *V* points with indices *j*=0 and *j*=N_j respectively landing on the south and north pole and therefore vanishing. Looking at the equations (**section 5**), we find that only three equations require attention: the two *horizontal momentum* equations

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta\left(Bs+q\right) + \left(1+\boldsymbol{\mu}\right)\nabla_\zeta\boldsymbol{\phi}' = 0$$

with

$$\mathbf{V}_h = u\hat{\boldsymbol{\lambda}} + v\hat{\boldsymbol{\theta}} = \frac{a}{\cos\boldsymbol{\theta}}\left[U\hat{\boldsymbol{\lambda}} + V\hat{\boldsymbol{\theta}}\right]$$

defined in terms of its longitudinal component *u* in the direction $\hat{\boldsymbol{\lambda}}$ and its latitudinal component *v* in the direction $\hat{\boldsymbol{\theta}}$, or the so-called corresponding wind images *U* and *V,* and with the gradient operator given by:

$$\nabla_\zeta = \frac{\hat{\boldsymbol{\lambda}}}{a\cos\boldsymbol{\theta}}\frac{\partial}{\partial\boldsymbol{\lambda}} + \frac{\hat{\boldsymbol{\theta}}}{a}\frac{\partial}{\partial\boldsymbol{\theta}} = \frac{1}{a\cos\boldsymbol{\theta}}\left[\hat{\boldsymbol{\lambda}}\frac{\partial}{\partial X} + \hat{\boldsymbol{\theta}}\frac{\partial}{\partial Y}\right]$$

with $dX=d\boldsymbol{\lambda}$ and $dY=d\boldsymbol{\theta}/\cos\boldsymbol{\theta}=d\sin\boldsymbol{\theta}/\cos^2\boldsymbol{\theta}$, and the *continuity* equation written as follows:

$$\frac{d}{dt}\ln\left(\boldsymbol{\pi}\frac{\partial\ln\boldsymbol{\pi}}{\partial\boldsymbol{\zeta}}\right) + D + \frac{\partial\dot{\boldsymbol{\zeta}}}{\partial\boldsymbol{\zeta}} = 0$$

where

$$D = \nabla_\zeta\cdot\mathbf{V}_h = \frac{1}{a\cos\boldsymbol{\theta}}\left[\frac{\partial u}{\partial\boldsymbol{\lambda}} + \frac{\partial\left(v\cos\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right] = \frac{1}{\cos^2\boldsymbol{\theta}}\left[\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right]$$

is the horizontal divergence discretized very simply as follows:

$$D_{ij} = \frac{1}{\cos^2\boldsymbol{\theta}_j}\left[\left(\boldsymbol{\delta}_X U\right)_{ij} + \left(\boldsymbol{\delta}_Y V\right)_{ij}\right]$$

$$\left(\boldsymbol{\delta}_X U\right)_{ij} = \frac{U_{i+\frac{1}{2},j} - U_{i-\frac{1}{2},j}}{\Delta X_i}; \qquad \left(\boldsymbol{\delta}_Y V\right)_{ij} = \frac{V_{i,j+\frac{1}{2}} - V_{i,j-\frac{1}{2}}}{\Delta Y_j}$$

Note that in a global integration we have periodicity in the λ-(X-)direction, so $U_{\frac{1}{2},j} = U_{N_i+\frac{1}{2},j}$, and, since V vanishes at both poles, $V_{i,\frac{1}{2}} = V_{i,N+\frac{1}{2}} = 0$, the problem is closed in the horizontal.

The 'horizontal' vector momentum equation is modified to be solved as a three-dimensional vector equation in Cartesian coordinates subject however to the constraint that the wind keeps parallel to the earth's surface (Côté, MWR 1988):

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta(Bs+q) + (1+\boldsymbol{\mu})\nabla_\zeta\boldsymbol{\phi}' + \mathrm{m}\mathbf{r} = 0$$

The constraint, $\mathrm{m}\mathbf{r}$, where $\mathbf{r}$ is the earth's radius and m a Lagrange multiplier, acts as a supplementary force normal to the surface. We then introduce the semi-Lagrangian implicit discretization (**section 8**) directly on the vector equation:

$$\frac{\mathbf{V}_h^A - \mathbf{V}_h^D}{\Delta t} + b^A\left(\mathbf{G}_h^A + \mathrm{m}\mathbf{r}^A\right) + \left(1-b^A\right)\left(\mathbf{G}_h^D + \mathrm{m}\mathbf{r}^D\right) = 0$$

$$\frac{\mathbf{V}_h^A}{\boldsymbol{\tau}} + \mathbf{G}_h^A + \mathrm{m}\mathbf{c} = \frac{\mathbf{V}_h^D}{\boldsymbol{\tau}} - \boldsymbol{\beta}\mathbf{G}_h^D \equiv \mathbf{R}_h^D$$

$$\frac{\mathbf{V}_h^A}{\boldsymbol{\tau}} + \mathbf{G}_h^A = \mathbf{R}_h^D - \mathrm{m}\mathbf{c} \equiv \mathbf{R}_h^C$$

with $\mathbf{c} = \mathbf{r}^A + \boldsymbol{\beta}\,\mathbf{r}^D$. Multiplying through scalarly by $\mathbf{r}^A$

$$\mathbf{r}^A \cdot \left[\frac{\mathbf{V}_h^A}{\boldsymbol{\tau}} + \mathbf{G}_h^A + \mathrm{m}\mathbf{c}\right] = \mathbf{r}^A \cdot \mathbf{R}_h^D$$

gives $\mathrm{m} = \dfrac{\mathbf{r}^A \cdot \mathbf{R}_h^D}{\mathbf{r}^A \cdot \mathbf{c}}$, since by construction $\mathbf{r}^A$ is $\perp$ to both $\mathbf{V}_h^A$ and $\mathbf{G}_h^A$. In Cartesian coordinates

$$\mathrm{m} = \frac{x^A R_x^D + y^A R_y^D + z^A R_z^D}{x^A c_x + y^A c_y + z^A c_z}$$

Therefore the metric correction to be applied to $\mathbf{R}_h^D$, in order for the result to remain on the sphere, is:

$$\begin{Bmatrix} R_x^C \\ R_y^C \\ R_z^C \end{Bmatrix} = \begin{Bmatrix} R_x^D - \mathrm{m}c_\mathrm{x} \\ R_y^D - \mathrm{m}c_\mathrm{y} \\ R_z^D - \mathrm{m}c_\mathrm{z} \end{Bmatrix} = \begin{Bmatrix} R_x^D - \mathrm{m}\left(x^A + \boldsymbol{\beta}\,x^D\right) \\ R_y^D - \mathrm{m}\left(y^A + \boldsymbol{\beta}\,y^D\right) \\ R_z^D - \mathrm{m}\left(z^A + \boldsymbol{\beta}\,z^D\right) \end{Bmatrix}$$

However, $\mathbf{R}_h$ is given in spherical coordinates in terms of wind images:

$$\mathbf{R}_h = \frac{a}{\cos\boldsymbol{\theta}}\left[R_U\hat{\boldsymbol{\lambda}} + R_V\hat{\boldsymbol{\theta}}\right]$$

with

$$R_U = \frac{U}{\boldsymbol{\tau}} - \boldsymbol{\beta}\left[- fV + \frac{1}{a^2}\left(RT\frac{\partial(Bs+q)}{\partial X} + (1+\boldsymbol{\mu})\frac{\partial \boldsymbol{\phi}'}{\partial X} \right)\right]$$
$$R_V = \frac{V}{\boldsymbol{\tau}} - \boldsymbol{\beta}\left[+ fU + \frac{1}{a^2}\left(RT\frac{\partial(Bs+q)}{\partial Y} + (1+\boldsymbol{\mu})\frac{\partial \boldsymbol{\phi}'}{\partial Y} \right)\right]$$

To obtain the Cartesian coordinates (R_x^D, R_y^D, R_z^D) of $\mathbf{R}_h^D$ from its spherical coordinates (R_U^D, R_V^D), we apply the coordinate transformation law at the departure point:

$$\begin{Bmatrix} R_x^D / a \\ R_y^D / a \\ R_z^D / a \end{Bmatrix} = \begin{bmatrix} -\sin\lambda^D & -\sin\theta^D \cos\lambda^D & \cos\theta^D \cos\lambda^D \\ \cos\lambda^D & -\sin\theta^D \sin\lambda^D & \cos\theta^D \sin\lambda^D \\ 0 & \cos\theta^D & \sin\theta^D \end{bmatrix} \begin{Bmatrix} R_U^D / \cos\theta^D \\ R_V^D / \cos\theta^D \\ 0 \end{Bmatrix} = \begin{bmatrix} -\sin\lambda^D / \cos\theta^D & -\tan\theta^D \cos\lambda^D \\ \cos\lambda^D / \cos\theta^D & -\tan\theta^D \sin\lambda^D \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} R_U^D \\ R_V^D \end{Bmatrix}$$

Hence

$$\begin{Bmatrix} R_x^D / a \\ R_y^D / a \\ R_z^D / a \end{Bmatrix} = \begin{bmatrix} -y^D / \cos^2\theta^D & -x^D z^D / \cos^2\theta^D \\ x^D / \cos^2\theta^D & -y^D z^D / \cos^2\theta^D \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} R_U^D \\ R_V^D \end{Bmatrix}$$

using

$$\begin{Bmatrix} x^D \\ y^D \\ z^D \end{Bmatrix} = \begin{bmatrix} -\sin\lambda^D & -\sin\theta^D \cos\lambda^D & \cos\theta^D \cos\lambda^D \\ \cos\lambda^D & -\sin\theta^D \sin\lambda^D & \cos\theta^D \sin\lambda^D \\ 0 & \cos\theta^D & \sin\theta^D \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \cos\theta^D \cos\lambda^D \\ \cos\theta^D \sin\lambda^D \\ \sin\theta^D \end{Bmatrix}$$

Finally, to obtain the spherical coordinates (R_U^C, R_V^C) of $\mathbf{R}_h^C$, from its Cartesian coordinates (R_x^C, R_y^C, R_z^C), we apply the inverse transformation at the arrival point:

$$\begin{Bmatrix} R_U^C / \cos\theta^A \\ R_V^C / \cos\theta^A \\ 0 \end{Bmatrix} = \begin{bmatrix} -\sin\lambda^A & \cos\lambda^A & 0 \\ -\sin\theta^A \cos\lambda^A & -\sin\theta^A \sin\lambda^A & \cos\theta^A \\ \cos\theta^A \cos\lambda^A & \cos\theta^A \sin\lambda^A & \sin\theta^A \end{bmatrix} \begin{Bmatrix} R_x^C / a \\ R_y^C / a \\ R_z^C / a \end{Bmatrix}$$

hence

$$\begin{Bmatrix} R_U^C \\ R_V^C \\ 0 \end{Bmatrix} = \begin{bmatrix} -y^A & x^A & 0 \\ -z^A x^A & -z^A y^A & \cos^2\theta^A \\ x^A & y^A & z^A \end{bmatrix} \begin{Bmatrix} R_x^C / a \\ R_y^C / a \\ R_z^C / a \end{Bmatrix}$$

using

$$\begin{Bmatrix} x^A \\ y^A \\ z^A \end{Bmatrix} = \begin{Bmatrix} \cos\theta^A \cos\lambda^A \\ \cos\theta^A \sin\lambda^A \\ \sin\theta^A \end{Bmatrix}$$

The vanishing of the last row of $\mathbf{R}_h^C$ is true by construction, $\mathbf{r}^A \cdot \mathbf{R}_h^C = 0$. We use the information to simplify the middle row getting finally:

$$\begin{Bmatrix} R_U^D \\ R_V^D \end{Bmatrix} = \begin{Bmatrix} -y^A & x^A & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} R_x^C / a \\ R_y^C / a \\ R_z^C / a \end{Bmatrix}$$

In summary then, having (R_U^D, R_V^D), i.e. $\mathbf{R}_h^D$ in spherical coordinates, we

1) transform $\mathbf{R}_h^D$ to Cartesian coordinates, computing R_x^D, R_y^D, R_z^D,
2) compute **c**, m and $\mathbf{R}_h^C$ in Cartesian coordinates,
3) transform $\mathbf{R}_h^C$ back to spherical coordinates, i.e. compute R_U^C, R_V^C

In order to solve this semi-Lagrangian equation, in fact all of the other equations as well, we must first solve the equation for the displacements themselves. Consider

$$\frac{d\mathbf{r}}{dt} = \mathbf{V} = |\mathbf{V}|\mathbf{t}$$

where $\mathbf{t}$ is a unit vector tangent to the spherical earth in the direction of **V** and $|\mathbf{V}|$ is the module of **V** *assumed constant* during the displacement. For simplicity, we have taken the radius of the earth **r** as a unit vector all along. Then, in the plane of the displacement, the trajectory is an arc of circle, a great circle displacement. If $\mathbf{r}^D$, $\mathbf{t}^D$ and $\mathbf{r}^A$, $\mathbf{t}^A$ are unit vectors respectively at the *departure* and *arrival* points, we have

$$\begin{Bmatrix} \mathbf{r}^A \\ \mathbf{t}^A \end{Bmatrix} = \begin{bmatrix} \cos\Delta & \sin\Delta \\ -\sin\Delta & \cos\Delta \end{bmatrix} \begin{Bmatrix} \mathbf{r}^D \\ \mathbf{t}^D \end{Bmatrix} \text{ or } \begin{Bmatrix} \mathbf{r}^D \\ \mathbf{t}^D \end{Bmatrix} = \begin{bmatrix} \cos\Delta & -\sin\Delta \\ \sin\Delta & \cos\Delta \end{bmatrix} \begin{Bmatrix} \mathbf{r}^A \\ \mathbf{t}^A \end{Bmatrix}$$

where $\Delta = |\mathbf{V}|\Delta t$. We therefore can write

$$\mathbf{r}^D = \mathbf{r}^A \cos\Delta - \sin\Delta\, \mathbf{t}^A \qquad \text{or} \qquad \mathbf{r}^D = \frac{\mathbf{r}^A - \sin\Delta\, \mathbf{t}^D}{\cos\Delta}$$

where $\mathbf{t}^D = \mathbf{V}^D / |\mathbf{V}|$.

Assuming that **V** is known in spherical coordinates (U,V) and having a first estimate of the location (λ^M,θ^M) of the *mid-point* $\mathbf{r}^M$ between the departure and arrival points, we first obtain $\mathbf{V}^M$ by interpolating **V** at that position. Then we proceed to **improve** the estimate of $\mathbf{r}^M$ by performing a great circle displacement solving the above equation. We may proceed as follows:

1) compute $|\mathbf{V}| = \sqrt{\dfrac{\left(U^M\right)^2 + \left(V^M\right)^2}{\cos^2\theta^M}}$ and $\Delta = \dfrac{\Delta t}{2}|\mathbf{V}|$

2) compute the arrival position $\mathbf{r}^A$ in Cartesian coordinates $\left(x^A, y^A, z^A\right)$

3) compute $\mathbf{V}^M$ in Cartesian coordinates using old $\mathbf{r}^M$ position

$$\mathbf{V}^M = \dot{\mathbf{r}}^M = \begin{Bmatrix} \dot{x}^M \\ \dot{y}^M \\ \dot{z}^M \end{Bmatrix} = \begin{Bmatrix} -\sin\lambda^M \cos\theta^M \dot{\lambda}^M - \cos\lambda^M \sin\theta^M \dot{\theta}^M \\ \cos\lambda^M \cos\theta^M \dot{\lambda}^M - \sin\lambda^M \sin\theta^M \dot{\theta}^M \\ \cos\theta^M \dot{\theta}^M \end{Bmatrix}$$

$$= \begin{Bmatrix} -\left(y^M U^M + x^M z^M V^M\right)/\cos^2\theta^M \\ \left(x^M U^M - y^M z^M V^M\right)/\cos^2\theta^M \\ U^M \end{Bmatrix}$$

4) compute new $\mathbf{r}^M$ in Cartesian coordinates: $\mathbf{r}^M \equiv \begin{Bmatrix} x^M \\ y^M \\ z^M \end{Bmatrix} = \dfrac{1}{\cos\Delta} \begin{Bmatrix} x^A \\ y^A \\ z^A \end{Bmatrix} - \dfrac{\tan\Delta}{|\mathbf{V}|} \begin{Bmatrix} \dot{x}^M \\ \dot{y}^M \\ \dot{z}^M \end{Bmatrix}$

5) obtain $\mathbf{r}^M$ in spherical coordinates: $\begin{Bmatrix} \lambda^M \\ \theta^M \end{Bmatrix} = \begin{Bmatrix} \tan^{-1}\left(y^M / x^M\right) \\ \sin^{-1}\left(z^M\right) \end{Bmatrix}$

In the model, the process is an iterative one (**section 8**). So we repeat the procedure until convergence. Once the new mid-point position $\mathbf{r}^M$ valid at t-$\Delta t/2$ is found, the true departure position $\mathbf{r}^D$ valid at t-Δt is obtained by doubling the great circle displacement:

6) obtain $\mathbf{r}^D = 2\cos\Delta \mathbf{r}^M - \mathbf{r}^A = \begin{Bmatrix} x^D \\ y^D \\ z^D \end{Bmatrix} = 2\cos\Delta \begin{Bmatrix} x^M \\ y^M \\ z^M \end{Bmatrix} - \begin{Bmatrix} x^A \\ y^A \\ z^A \end{Bmatrix}$, first in Cartesian and

7) finally in spherical coordinates: $\begin{Bmatrix} \lambda^D \\ \theta^D \end{Bmatrix} = \begin{Bmatrix} \tan^{-1}\left(y^D / x^D\right) \\ \sin^{-1}\left(z^D\right) \end{Bmatrix}$

We are now ready for the discretization in the horizontal. The equation

$$\frac{\mathbf{V}_h^A}{\boldsymbol{\tau}} + \mathbf{G}_h^A = \mathbf{L}_h + \mathbf{N}_h = \mathbf{R}_h^C$$

is decomposed into its components (**section 10**):

$$L_U + N_U = \left[\frac{U}{\boldsymbol{\tau}} - fV + \frac{1}{a^2}\left(RT\frac{\partial(Bs+q)}{\partial X} + (1+\boldsymbol{\mu})\frac{\partial \boldsymbol{\phi}'}{\partial X}\right)\right]^A = R_U^C$$

$$L_V + N_V = \left[\frac{V}{\boldsymbol{\tau}} + fU + \frac{1}{a^2}\left(RT\frac{\partial(Bs+q)}{\partial Y} + (1+\boldsymbol{\mu})\frac{\partial \boldsymbol{\phi}'}{\partial Y}\right)\right]^A = R_V^C$$

and horizontally discretized as follows

$$\left(L_U+N_U\right)_{i+\frac{1}{2},j}=\left[\frac{U}{\boldsymbol{\tau}}-f\langle V\rangle^{YX}+\frac{R\overline{T}^{X}}{a^2}\boldsymbol{\delta}_X\left(Bs+q\right)+\frac{1+\overline{\boldsymbol{\mu}}^{X}}{a^2}\boldsymbol{\delta}_X\boldsymbol{\phi}'\right]^{A}_{i+\frac{1}{2},j}=\left(R_U^C\right)_{i+\frac{1}{2},j}$$

$$\left(L_V+N_V\right)_{i,j+\frac{1}{2}}=\left[\frac{V}{\boldsymbol{\tau}}+f\langle U\rangle^{XY}+\frac{R\overline{T}^{Y}}{a^2}\boldsymbol{\delta}_Y\left(Bs+q\right)+\frac{1+\overline{\boldsymbol{\mu}}^{Y}}{a^2}\boldsymbol{\delta}_Y\boldsymbol{\phi}'\right]^{A}_{i,j+\frac{1}{2}}=\left(R_V^C\right)_{i,j+\frac{1}{2}}$$

using the following simple two-point *difference* and *mean* operators:

$$\left(\boldsymbol{\delta}_X A\right)_{i+\frac{1}{2},j}=\frac{A_{i+1,j}-A_{i,j}}{\Delta X_{i+\frac{1}{2}}};\qquad \left(\overline{A}^{X}\right)_{i+\frac{1}{2},j}=\boldsymbol{\varpi}^{X}{}_{i+\frac{1}{2}}A_{i+1,j}+\left(1-\boldsymbol{\varpi}^{X}{}_{i+\frac{1}{2}}\right)A_{i,j}$$

$$\left(\boldsymbol{\delta}_Y A\right)_{i,j+\frac{1}{2}}=\frac{A_{i,j+1}-A_{i,j}}{\Delta Y_{j+\frac{1}{2}}};\qquad \left(\overline{A}^{Y}\right)_{i,j+\frac{1}{2}}=\boldsymbol{\varpi}^{Y}{}_{j+\frac{1}{2}}A_{i,j+1}+\left(1-\boldsymbol{\varpi}^{Y}{}_{j+\frac{1}{2}}\right)A_{i,j}$$

as well as the four-point (cubic interpolation) mean operators:

$$\langle V\rangle^{Y}_{i,j}=\boldsymbol{\alpha}_j\, V_{i,j-\frac{3}{2}}+\boldsymbol{\beta}_j\, V_{i,j-\frac{1}{2}}+\boldsymbol{\gamma}_j\, V_{i,j+\frac{1}{2}}+\boldsymbol{\delta}_j\, V_{i,j+\frac{3}{2}}$$

$$\langle V\rangle^{YX}_{i+\frac{1}{2},j}=\left\langle\langle V\rangle^{Y}\right\rangle^{X}_{i+\frac{1}{2},j}=\boldsymbol{\alpha}_{i+\frac{1}{2}}\langle V\rangle^{Y}_{i-1,j}+\boldsymbol{\beta}_{i+\frac{1}{2}}\langle V\rangle^{Y}_{i,j}+\boldsymbol{\gamma}_{i+\frac{1}{2}}\langle V\rangle^{Y}_{i+1,j}+\boldsymbol{\delta}_{i+\frac{1}{2}}\langle V\rangle^{Y}_{i+2,j}$$

$$\langle U\rangle^{X}_{i,j}=\boldsymbol{\alpha}_i\, U_{i-\frac{3}{2},j}+\boldsymbol{\beta}_i\, U_{i-\frac{1}{2},j}+\boldsymbol{\gamma}_i\, U_{i+\frac{1}{2},j}+\boldsymbol{\delta}_i\, U_{i+\frac{3}{2},j}$$

$$\langle U\rangle^{XY}_{i,j+\frac{1}{2}}=\left\langle\langle U\rangle^{X}\right\rangle^{Y}_{i,j+\frac{1}{2}}=\boldsymbol{\alpha}_{j+\frac{1}{2}}\langle U\rangle^{X}_{i,j-1}+\boldsymbol{\beta}_{j+\frac{1}{2}}\langle U\rangle^{X}_{i,j}+\boldsymbol{\gamma}_{j+\frac{1}{2}}\langle U\rangle^{X}_{i,j+1}+\boldsymbol{\delta}_{j+\frac{1}{2}}\langle U\rangle^{X}_{i,j+2}$$

The left-hand-sides (dropping the superscript ^{A}) are linearized separately (**section 11**):

$$\left(L_U\right)_{i+\frac{1}{2},j}=\frac{U_{i+\frac{1}{2},j}}{\boldsymbol{\tau}}+\frac{1}{a^2}\left(\boldsymbol{\delta}_X P\right)_{i+\frac{1}{2},j}$$

$$P=\boldsymbol{\phi}'+RT_*\left(Bs+q\right)$$

$$\left(L_V\right)_{i,j+\frac{1}{2}}=\frac{V_{i,j+\frac{1}{2}}}{\boldsymbol{\tau}}+\frac{1}{a^2}\left(\boldsymbol{\delta}_Y P\right)_{i,j+\frac{1}{2}}$$

leaving as non-linear terms (**section 12**):

$$\left(N_U\right)_{i+\frac{1}{2},j}=-f\langle V\rangle^{YX}_{i+\frac{1}{2},j}+\frac{R\overline{T'}^{X}_{i+\frac{1}{2},j}}{a^2}\left[\boldsymbol{\delta}_X\left(Bs+q\right)\right]_{i+\frac{1}{2},j}+\frac{\overline{\boldsymbol{\mu}}^{X}_{i+\frac{1}{2},j}}{a^2}\left(\boldsymbol{\delta}_X\boldsymbol{\phi}'\right)_{i+\frac{1}{2},j}$$

$$\left(N_V\right)_{i,j+\frac{1}{2}}=+f\langle U\rangle^{XY}_{i,j+\frac{1}{2}}+\frac{R\overline{T'}^{Y}_{i,j+\frac{1}{2}}}{a^2}\left[\boldsymbol{\delta}_Y\left(Bs+q\right)\right]_{i,j+\frac{1}{2}}+\frac{\overline{\boldsymbol{\mu}}^{Y}_{i,j+\frac{1}{2}}}{a^2}\left(\boldsymbol{\delta}_Y\boldsymbol{\phi}'\right)_{i,j+\frac{1}{2}}$$

Table 1. The equations of GEM in 4 transformations

$$
\begin{aligned}
&\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F} \\
&\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T} \\
&\frac{d\ln \rho}{dt} + \nabla\cdot\mathbf{V} = 0 \\
&\rho = \frac{p}{RT}
\end{aligned}
$$

Vertical coordinate transformation: z to ζ (unspecified)

$$
\begin{aligned}
&\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta} \\
&\frac{\partial}{\partial z} \equiv \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}
\end{aligned}
$$

1 →

$$
\begin{aligned}
&\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p - \nabla_\zeta z\frac{\partial\zeta}{\partial z}\frac{\partial\ln p}{\partial\zeta}\right) = \mathbf{F}_h \\
&\frac{dw}{dt} + RT\frac{\partial\zeta}{\partial z}\frac{\partial\ln p}{\partial\zeta} + g = F_w \\
&\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T} \\
&\frac{d}{dt}\ln\left(\rho\frac{\partial z}{\partial\zeta}\right) + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} = 0 \\
&\frac{dz}{dt} - w = 0 \\
&\rho = \frac{p}{RT}
\end{aligned}
$$

Vertical coordinate transformation: z to ζ (specified)

$$
\begin{aligned}
&RT = -\frac{p}{\pi}\frac{\partial\phi}{\partial\ln\pi} \\
&\phi = gz \\
&\mu = \frac{p}{\pi}\frac{\partial\ln p}{\partial\ln\pi} - 1
\end{aligned}
$$

2 →

$$
\begin{aligned}
&\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\mu)\nabla_\zeta\phi = \mathbf{F}_h \\
&\frac{dw}{dt} - g\mu = F_w \\
&\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T} \\
&\frac{d}{dt}\ln\left(\pi\frac{\partial\ln\pi}{\partial\zeta}\right) + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} = 0 \\
&\frac{d\phi}{dt} - gw = 0 \\
&1 + \mu - \frac{p}{\pi}\frac{\partial\ln p}{\partial\ln\pi} = 0 \\
&RT + \frac{p}{\pi}\frac{\partial\phi}{\partial\ln\pi} = 0
\end{aligned}
$$

Going to model thermodynamic variables $\phi', q, s, \dot{\zeta}$

$$\boldsymbol{\phi}' = \boldsymbol{\phi} - \boldsymbol{\phi}_*$$
$$\ln p = \ln \boldsymbol{\pi} + q$$
$$\ln \boldsymbol{\pi} = \boldsymbol{\zeta} + Bs$$

(3) →

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta\left(Bs+q\right) + \left(1+\boldsymbol{\mu}\right)\nabla_\zeta\boldsymbol{\phi}' = \mathbf{F}_h$$
$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$
$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left[\frac{d}{dt}\left(Bs+q\right) + \dot{\boldsymbol{\zeta}}\right] = \frac{Q}{c_pT}$$
$$\frac{d}{dt}\left[Bs + \ln\left(1 + \frac{\partial B}{\partial\boldsymbol{\zeta}}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\boldsymbol{\zeta}}}{\partial\boldsymbol{\zeta}} + \dot{\boldsymbol{\zeta}} = 0$$
$$\frac{d\boldsymbol{\phi}'}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$
$$1 + \boldsymbol{\mu} - e^q\left[\frac{\partial q}{\partial\left(\boldsymbol{\zeta} + Bs\right)} + 1\right] = 0$$
$$\frac{T}{T_*} - e^q\frac{\partial\left(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*\right)}{\partial\left(\boldsymbol{\zeta} + Bs\right)} = 0$$

Discretizing in the vertical

$$\overline{\left(\ \right)}^{\zeta}$$
$$\boldsymbol{\delta}_\zeta\left(\ \right)$$

(4) →

———	$w, T, \dot{\boldsymbol{\zeta}}, \boldsymbol{\mu}$
– – – – –	$\mathbf{V}_h, \boldsymbol{\phi}'$
———	$w, T, \dot{\boldsymbol{\zeta}}, \boldsymbol{\mu}$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_\zeta\left(Bs+q\right) + \left(1+\overline{\boldsymbol{\mu}}^{\zeta}\right)\nabla_\zeta\boldsymbol{\phi}' = \mathbf{F}_h$$
$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$
$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left(\mathrm{B}s + \overline{q}^{\zeta}\right)\right] - \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}} = \frac{Q}{c_pT}$$
$$\frac{d}{dt}\left[\overline{\mathrm{B}}^{\zeta}s + \ln\left(1 + \boldsymbol{\delta}_\zeta\mathrm{B}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \boldsymbol{\delta}_\zeta\dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta} = 0$$
$$\frac{d\overline{\boldsymbol{\phi}'}^{\zeta}}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$
$$1 + \boldsymbol{\mu} - e^{\overline{q}^{\zeta}}\left[\frac{\boldsymbol{\delta}_\zeta q}{\boldsymbol{\delta}_\zeta\left(\boldsymbol{\zeta} + Bs\right)} + 1\right] = 0$$
$$\frac{T}{T_*} - e^{\overline{q}^{\zeta}}\frac{\boldsymbol{\delta}_\zeta\left(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*\right)}{\boldsymbol{\delta}_\zeta\left(\boldsymbol{\zeta} + Bs\right)} = 0$$

Table 2. The Equations of GEM *vertically discretized* on **Charney-Phillips grid**

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\boldsymbol{\mu}}^{\zeta}\right)\nabla_{\zeta}\boldsymbol{\phi}' = \mathbf{F}_h \quad \text{----}$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w \quad \text{——}$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left(\mathrm{B}s + \overline{q}^{\zeta}\right)\right] - \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}} = \frac{Q}{c_p T} \quad \text{——}$$

$$\frac{d}{dt}\left[\overline{\mathrm{B}}^{\zeta} s + \ln\left(1 + \boldsymbol{\delta}_{\zeta}\mathrm{B}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \boldsymbol{\delta}_{\zeta}\dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta} = 0 \quad \text{----}$$

$$\frac{d\overline{\boldsymbol{\phi}'}^{\zeta}}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0 \quad \text{——}$$

$$1 + \boldsymbol{\mu} - e^{\overline{q}^{\zeta}} - e^{\overline{q}^{\zeta}}\,\frac{\boldsymbol{\delta}_{\zeta} q}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta} + Bs)} = 0 \quad \text{——}$$

$$\frac{T}{T_*} - e^{\overline{q}^{\zeta}}\,\frac{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*)}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta} + Bs)} = 0 \quad \text{——}$$

——	$w, T, \dot{\zeta}, \boldsymbol{\mu}$
- - -	$\mathbf{V}_h, \phi, q$
——	$w, T, \dot{\zeta}, \boldsymbol{\mu}$

$\mathbf{V}_h$: *horizontal wind*; f : *Coriolis parameter*

w : *vertical velicity*; $\boldsymbol{\kappa} = R/c_p$

T : *temperature*; $T' = T\text{-}T_*$; $T_* = const$

$\boldsymbol{\phi}$: *geopotential*; $\boldsymbol{\phi}' = \boldsymbol{\phi} - \boldsymbol{\phi}_*$; $\boldsymbol{\phi}_* = -RT_*(\boldsymbol{\zeta} - \boldsymbol{\zeta}_S)$

$q = \ln(p/\boldsymbol{\pi})$: *non-hydrostatic log-pressure deviation*

p : *pressure*; $\boldsymbol{\pi}$: *hydrostatic pressure*; $\partial\boldsymbol{\phi}/\partial\boldsymbol{\pi} = -RT/p$

$\boldsymbol{\mu} = \partial p/\partial\boldsymbol{\pi} - 1$: *ratio of vertical acceleration to gravitational acceleration*

$s = \ln(\boldsymbol{\pi}_S / p_{ref})$: *log-surface-pressure*; B, $\mathrm{B} = \overline{B}^{\zeta}$: *metric parameter*

$\dot{\boldsymbol{\zeta}} = d\boldsymbol{\zeta}/dt$; $\boldsymbol{\zeta}$: *model vertical coordinate*

$\overline{(\ )}^{\zeta}$: *averaging operator*; $\boldsymbol{\delta}_{\zeta}$: *differencing operator*

$p_{top}/p_{ref} = \eta_T < \eta < 1$: *specified* $\boldsymbol{\pi}$*-like model levels*; $p_{ref} = 10^5\,Pa$

$$\zeta = \zeta_S + \ln(\eta)$$

$\ln p_{top} = \zeta_T \le \zeta \le \zeta_S = \ln p_{ref}$: *calculated* $\ln\boldsymbol{\pi}$ – like *model coordinate* levels

$$\ln\pi = A + Bs$$

$$A = \zeta; \qquad B = \left(\frac{\zeta - \zeta_T}{\zeta_S - \zeta_T}\right)^r; \qquad 0 < r = r_{\max} - (r_{\max} - r_{\min})\left(\frac{\zeta - \zeta_T}{\zeta_S - \zeta_T}\right) < 30$$

Boundary Conditions: $\dot{\zeta}_S = \dot{\zeta}_T = 0$ $[\, q_T = \ln(p_{top}/\pi_T) = 0; \phi_S = gz_{topo}; p_{top} = \text{const}\,]$