New Analytical and CWENO Numerical Solutions for Axisymmetric Horizontal Flows with Heat Sources

A. Mohammadian

Department of Civil Engineering, University of Ottawa, 161 Louis Pasteur, Ottawa, Ontario, K1N6N5, Canada, E-mail: majid.mohammadian@uottawa.ca

Martin Charron

Section de Présision Numérique Atmosphérique, Environnement Canada, 2121, route Transcanadienne, Dorval, Quebec, H9P 1J3, Canada, E-mail: Martin.Charron@ec.gc.ca

Abstract

Some new nonlinear analytical solutions are found for axisymmetric horizontal flows dominated by strong heat sources. These flows are common in multiscale atmospheric and oceanic flows such as hurricane embryos and ocean gyres. The analytical solutions are illustrated with several examples. The proposed exact solutions provide analytical support for previous numerical observations and can be also used as benchmark problems for validating numerical models. A central weighted essentially non-oscillatory (CWENO) reconstruction is also employed for numerical simulation of the corresponding integro-differential equations. Due to the use of the same polynomial reconstruction for all derivatives and integral terms, the balance between those terms is well preserved, and the method can precisely reproduce the exact
solutions, which are hard to capture by traditional upwind schemes. The developed analytical solutions were employed to evaluate the performance of the numerical method, which showed an excellent performance of the numerical model in terms of numerical diffusion and oscillation.

Key words: Analytical solutions, horizontal flows, heat sources, mass sources, hot towers, axisymmetric flows, CWENO schemes

1 INTRODUCTION

Strong heat sources in nearly horizontal flows can generate vertical motion caused by changes in density. Such flows can be observed in various scales in the atmosphere, on scales from a few to several hundred kilometers; they are ubiquitous in several multiscale models in various spatio-temporal scales (Klein and Majda, 2006; Majda, 2007a and 2007b; Majda et al., 2010) for vertically sheared horizontal flows where two key assumptions, weak temperature gradients and low Froude numbers, are valid. Such flows with heat or mass sources are particularly common in the tropics.

Using numerical experiments, Hendricks et al. (2004) and Montgomery et al. (2006) showed that local heat sources can act as building blocks in generation of “hot towers” where the rise of warm air leads to a local converging flow in low levels and generates vorticity under the influence of the Coriolis effect, which leads to intense vortical zones. These intense vortical zones can then merge and generate large-scale cyclonic zones (anti-cyclonic in the southern hemisphere) which can eventually lead to hurricane embryos. On the other hand, random small-scale eddies and their interactions are also important in
the ocean, and can merge and lead to large-scale ocean gyres (Mohammadian & Marshall, 2010).

Majda et al. (2008) studied these flows in a canonical reduced balanced model. They performed a nonlinear stability analysis and examined the stability of these flows in some special cases, including linear heat sources in large-scale mean flows. They also studied the impact of a vertical shear on the stability of those flows. An elementary hot tower model was also proposed in Majda et al. (2008) to study the evolutions of hot towers in various large-scale background preconditionings, such as steady deep-convective, stratiform, and congestus mean flows, and showed that a deep-convective mean flow can lead to a more intense vortical zone. A first-order upwind scheme was used in that study for numerical solution of the corresponding equations. Later, Majda et al. (2010) developed a moist multiscale hurricane embryo model and employed their elementary hot tower model to seed small-scale vortical zones. They developed a new theory to explain how a hurricane embryo is created, based on the interaction of eddies in various spatio-temporal scales.

Mohammadian and Charron (2011) considered axisymmetric vertically-sheared horizontal flows with heat sources and developed some linear analytical solutions as well as nonlinear solutions in the absence of a background shear, and studied the stability of the flow in some special circumstances. Motivated by the above studies, the present paper aims at studying in more detail the evolution of axisymmetric vertically-sheared horizontal flows with heat sources. This paper presents new analytical solutions which lead to a better understanding of these flows and provide challenging benchmark problems to develop and evaluate highly accurate numerical methods. In future studies, the resulting model will be used in studying multiple-scale phenomena such as
interaction of small-scale and large-scale eddies and mean flows in the ocean and atmosphere.

As mentioned above, numerical simulation of these flows is also a challenging task especially in radial coordinates, due to the large gradient of vorticity close to the center. This is because the equations in the axisymmetric radial coordinates lead to integral terms that are in balance with other terms in analytical solutions. Solution of these equations involves both numerical differentiation and integration, and traditional total variation diminishing (TVD) schemes may lead to numerical oscillations. This is due to the upwind discretization of flux terms, which are hard to balance exactly at the discrete level with the high-order central methods used for the integral terms or due to truncation errors (Mohammadian & Le Roux, 2006 and 2008; Mohammadian, 2010). Therefore, a more rational approach is to use the same numerical methods for all terms. Although upwind discretization of source terms is also possible, it is usually expensive in terms of computational cost, and moreover, often some numerical oscillations still remain in the results. To overcome this problem, Mohammadian and Charron (2011) proposed using a Chebyshev pseudospectral method to solve the equations. Besides achieving a very high order of accuracy in spectral methods, the Chebyshev polynomials could be also used for numerical integration, and the imbalance problem is inherently resolved because the same method is used for all terms. However, a high level of numerical diffusion has to be added to eliminate the numerical noises induced by the spectral method due to the employed centered-type discretization method. As shown in the following, CWENO schemes (e.g. Xie et al., 2007, Cai et al., 2007) are better choices for these equations and lead to a lower level of numerical diffusion for the same computational cost. Moreover, they provide a
natural framework for evaluation of the integral terms, since the same types of polynomials that are constructed for the evaluation of derivatives could be used for numerical evaluation of the integral terms.

This paper is organized as follows. Section 2 deals with the reduced governing equations in the axisymmetric radial coordinates. In Section 3 some nonlinear exact solutions are presented, both with and without background shear. In Section 4 the numerical method is presented. Section 5 presents two numerical experiments. Concluding remarks complete the study.

2 GOVERNING EQUATIONS

In this paper we consider a canonical balanced model that arises systematically in several multiscale models developed for various spatio-temporal scales (Majda, 2007a, 2007b; Majda et al., 2008, 2010) and deals with a class of horizontal flows dominated by heat sources. The model in the canonical nondimensional form in the density coordinate is given by (see e.g. Majda et al., 2008).

\[
\frac{Du_h}{Dt} + fu_h^\perp = -\nabla_h p, \tag{1}
\]
\[
div u_h + w_z = 0, \tag{2}
\]
\[
w = S_\theta, \tag{3}
\]

where \((u_h, w)^t\) is the velocity vector, \(w\) is the vertical velocity in the \(z\) direction, \(u_h = (u, v)^t, u_h^\perp = (-v, u)^t\), \(u\) and \(v\) are the horizontal velocity components in the \(x\) and \(y\) directions respectively, \(f\) is the non-dimensional Coriolis parameter, \(S_\theta\) is the heat source, \(\text{div}_h\) and \(\nabla_h\) are the horizontal divergence and
gradient, and
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_h \cdot \nabla_h + w \frac{\partial}{\partial z}
\]  

(4)

The first two equations in (1)-(3) are momentum and continuity equations, respectively, and the last one is the leading-order part of the heat transport equation. As observed in (3), the heat source is a known parameter in this system, and therefore the vertical velocity \( w \) is also a known parameter which can be immediately obtained using (3). Note that a magnitude of \( \epsilon \) is considered for the heat source \( S_\theta \) in the potential temperature equation, which corresponds to a strong heating. Based on the horizontal and vertical reference magnitudes of \( L = H = 10 \text{km} \) and time scale of \( T = 15 \text{ minutes} \), this heat source amounts to 120\( \text{K/hour} \) and is consistent with the observed heating rate in hot towers (Hendricks et al., 2004; Montgomery et al., 2006). The reader is referred to Majda et al. (2008) for more details of (1)-(3) and the reference magnitudes of various scales.

Taking the curl of the momentum equation in (1), one obtains the vertical vorticity dynamics equation
\[
\frac{D\omega}{Dt} = (\omega + f) (S_\theta)_z + \left( \frac{\partial}{\partial z} u^\perp_h \right) \cdot \nabla_h w
\]

(5)

The above equation in the axisymmetric radial coordinate is reduced to (see Majda et al. 2008, Mohammadian and Charron, 2011)
\[
\frac{\partial \omega}{\partial t} - \left( \frac{1}{r} \int_0^r s \frac{\partial w}{\partial z} ds \right) \frac{\partial \omega}{\partial r} + w \frac{\partial \omega}{\partial z} - \frac{\partial w}{\partial z} (\omega + f) + \frac{\partial w}{\partial r} \frac{\partial}{\partial z} \left( \frac{1}{r} \int_0^r s \omega ds \right) = 0,
\]

(6)

where \( \omega = \text{curl} \ u_h \) and \( r \) represents the radial direction. The continuity equation in the radial axisymmetric case is written as
\[
\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru^r) + \frac{\partial w}{\partial z} = 0,
\]

(7)
Thus
\[-\frac{1}{r} \int_0^r s \frac{\partial w}{\partial z} ds = u^r\]  \hspace{1cm} (8)

Therefore, the term \( -\left( \frac{1}{r} \int_0^r s \frac{\partial w}{\partial z} ds \right) \frac{\partial \omega}{\partial r} \) in (6) represents radial advection. The term \( w \frac{\partial}{\partial z} \) represents the vertical advection and the term \( \frac{\partial w}{\partial r} \frac{1}{r} \int_0^r s \frac{\partial \omega}{\partial z} ds \) or equivalently \( \frac{\partial w}{\partial r} \left( \frac{1}{r} \int_0^r s \omega ds \right) \) is also a nonlinear advection term (see Mohammadian & Charron, 2011). Finally, the term \( -\frac{\partial w}{\partial z} (\omega + f) \) is the main source term which controls the stability of the flow.

3 NONLINEAR EXACT SOLUTIONS

Here, non-linear mean heat sources (i.e., with only vertical changes) are considered both with and without time dependence.

3.1 Nonlinear time-independent mean heat source \( w = w(z) \) in a mean initial vorticity field \( \omega_0 = \omega_0(z) \)

The two terms \( \frac{\partial w}{\partial r} \frac{\partial}{\partial z} \left( \frac{1}{r} \int_0^r s \omega ds \right) \) and \( \left( \frac{1}{r} \int_0^r s \frac{\partial w}{\partial z} ds \right) \frac{\partial \omega}{\partial r} \) are zero in this case, and (6) is reduced to
\[
\frac{\partial \omega}{\partial t} + w \frac{\partial \omega}{\partial z} = \frac{\partial w}{\partial z} (\omega + f) \]  \hspace{1cm} (9)

Exact solutions can thus be constructed in this case by using characteristic coordinates, as illustrated in the following examples. Although the flow in the \( r - z \) plane depends on both \( r \) and \( z \) components of velocity, the vorticity equation only includes \( z \) and \( t \) and therefore, one can define a characteristic coordinate in which only time dependence remains and, in the absence of source terms, vorticity will be constant along the characteristic line.
**Example 1.** For a quadratic heat source, \( w(z) = az^2 \), the radial velocity using the continuity equation is obtained as

\[
u_r = -raz
\] (10)

The characteristic coordinates are defined by

\[
\frac{dz}{dt} = az^2,
\] (11)

or

\[
z(t) = -\frac{1}{at - \frac{1}{z_0}}
\] (12)

Hence,

\[
z_0 = \frac{1}{at + \frac{1}{z}}
\] (13)

Let \( f = 0 \) (the case \( f \neq 0 \) is easily retrieved by a change of variable \( \hat{\omega} = \omega - f \))

Since \( \frac{\partial w}{\partial z} = 2az \), one obtains

\[
\frac{d\omega}{dt} = \frac{\partial w}{\partial z} \omega = 2az \omega
\] (14)

or

\[
\frac{d\omega}{dt} = -\frac{2az_0}{z_0 at - 1} \omega,
\] (15)

which is easily solved as

\[
\omega = \frac{\omega_0(z_0)}{(at z_0 - 1)^2}
\] (16)

or

\[
\omega = \frac{\omega_0(z)}{(z at + 1)^2}
\] (17)
It can be shown that for the case $f \neq 0$ the vorticity takes the following form

$$\omega(z, t) = \frac{\omega_0(zat + 1) + f}{(zat + 1)^2} - f$$

(18)

That is, the vorticity can grow unbounded even for $a < 0$, because the denominator goes to zero as $t$ increases.

For example, for a stratiform initial mean vorticity $\omega_0(z) := \sin(2z)$, the above exact solution leads to

$$\omega(z, t) = \frac{\sin\left(\frac{2z}{zat + 1}\right) + f}{\left(\frac{zat}{zat + 1} - 1\right)^2} - f,$$

(19)

which is unstable regardless of the sign of the vertical gradient of the vertical velocity field. Thus, a quadratic time-independent mean heat source will always lead to instability in axisymmetric vertically sheared horizontal flows.

**Example 2:** A deep convective cloud may be modeled by a polynomial, as

$$w = z(1 - z)$$

(20)

The characteristic transformation is written as

$$\frac{dz}{dt} = z(1 - z),$$

(21)

which has the following solution

$$z(t) = \frac{z_0e^t}{z_0(e^t - 1) + 1},$$

(22)

and one obtains

$$z_0 = \frac{z}{z + e^t(1 - z)}$$

(23)
The evolution of the vorticity in the characteristic coordinate is governed by

\[
\frac{d\omega}{dt} = w_z (\omega + f) = (1 - 2z) (\omega + f),
\]

which leads to the following exact solution

\[
\omega(z, t) = e^{-t} \left( z e^t - z - e^{-t} \right)^2 \left( \omega_0 (z_0) + f \right) - f
\]

In the absence of an initial vorticity, the vorticity is produced in the system as

\[
\omega(z, t) = e^{-t} \left( z e^t - z - e^{-t} \right)^2 f - f,
\]

which clearly shows that a deep convective heating can lead to a zone of intensified vorticity which can grow unbounded even in the absence of initial vorticity. Indeed, the small background vorticity induced by the Coriolis effect is enough to trigger the process. This is in agreement with the experiments of Majda et al. (2008) and provides analytical support for the numerical results reported there. Using the continuity equations, the radial velocity in this case is obtained as

\[
u_r = r \left( z - \frac{1}{2} \right)
\]

The radial and vertical velocity components in this case are shown in Figure 1 (left). As expected, a converging flow is observed at low levels which diverges at high levels. The maximum vertical velocity is observed at mid-levels. The term \( \frac{\partial \omega}{\partial z} \) is thus positive at low levels and positive at high levels. Vorticity profiles with \( f = 0.1 \) are also shown in Figure 1 (right) at times \( t = 0.5 \) (thick line) and \( t = 2 \) (thin line), where the horizontal axis represents vorticity and the vertical axis represents \( z \). Initially, a cyclonic zone is generated at low levels and an anticyclonic zone is generated at high levels. Because \( \frac{\partial \omega}{\partial z} \) is antisymmetric with respect to \( z = 1/2 \), initially the intensity of vorticity in
cyclonic and anticyclonic zones is comparable. However, the positive vorticity generated at low levels is advected to high levels and cancels the negative vorticity there. At low levels, on the other hand, there is no cancellation effect and vorticity continues to grow. Therefore, a zone of high vorticity is generated in low levels which eventually overcomes the anticyclonic zone in high levels, and a cyclonic flow is generated in the entire domain. The above analytical solution can thus explain how a deep convective cloud can lead to a zone of intensified vorticity. Note that the sign of the vorticity depends on the sign of $f$ and therefore, the same type of cloud that produces a cyclone in the northern hemisphere will lead to an anticyclone in the southern hemisphere, where $f$ is negative.
3.2 Nonlinear time-dependent mean heat source \( w = w(t, z) \) in a mean initial vorticity field \( \omega_0 = \omega_0(z) \)

Here, transient non-linear mean heat sources (i.e., with only vertical changes) are considered of the form \( w = w(t, z) \) in a mean initial vorticity field \( \omega_0 = \omega_0(z) \). Exact solutions can be constructed in this case by using characteristic coordinates in (9), as illustrated in the following. The characteristic coordinates in (9) are defined by

\[
\frac{dz}{dt} = w(t, z) \Rightarrow z = \mathcal{G}(t, z_0)
\]  

(28)

Setting

\[
\alpha(t, z_0) := \frac{\partial w(z(t, z_0), t)}{\partial z},
\]

(29)

and assuming \( f = 0 \), the equation (9) in the characteristic coordinates becomes

\[
\frac{d\omega}{dt} = \alpha(t, z_0)\omega
\]

(30)

or

\[
\frac{d}{dt} \left( e^{-\int_{t_0}^{t} \alpha(s, z_0) ds} \omega(t) \right) = 0
\]

(31)

and one obtains

\[
e^{-\int_{t_0}^{t} \alpha(s, z_0) ds} \omega(t) = \omega(z_0)
\]

(32)

The exact solution is thus given by

\[
\omega(z, t) = \left( e^{\int_{t_0}^{t} \alpha(s, z_0) ds} \right) \omega_0(z_0),
\]

(33)

and for the case \( f \neq 0 \), one can show that

- For axisymmetric horizontal flows with a with nonlinear time-dependent mean heat source \( w = w(t, z) \) in a mean initial vorticity field \( \omega_0 = \omega_0(z) \),
there are exact solutions of the form

\[ \omega(z, t) = \left( e^{\int_0^t \alpha(s, z_0) \, ds} \right) (\omega_0(z_0) + f) - f \]  

(34)

where \( z_0 = \mathcal{G}^{-1}(t, z) \) and \( \mathcal{G}(t, z_0) \) is the characteristic path defined by (28).

The stability of the flow depends on the sign of the vertical gradient of the vertical velocity component, i.e., \( w_z(z(t, z_0), t) \). If this sign of this quantity remains positive for a long enough period of time, the term \( e^{\int_0^t \alpha(s, z_0) \, ds} \) can lead to unbounded growth of the vorticity and instability of the flow. Indeed, even in the absence of an initial vorticity, the Coriolis effect (although small) can provide the required initial condition which will be intensified and will cause instability.

**Example 3:** Here, a deep convective heating with a sinusoidal time variations is modeled:

\[ w = z(1 - z) \sin(t) \]  

(35)

The characteristic transformation is written as

\[ \frac{dz}{dt} = z(1 - z) \sin(t), \]  

(36)

which has the following solution

\[ z(t) = \frac{ez_0}{ez_0 - \cos(t)(z_0 - 1)} \]  

(37)

and one obtains

\[ z_0 = \frac{-ez \cos(t)}{-e + ez - z \cos(t)} \]  

(38)
\begin{equation}
\alpha(t, z_0) := w_z(z(t, z_0), t) = (1 - \frac{e^{z_0}}{e^{z_0} - e^{\cos(t)}(z_0 - 1)}) \sin(t) \tag{39}
\end{equation}

and one can compute the term \( e^{\int_0^t \alpha(s, z_0) ds} \) in (34) The exact solution is thus given by

\begin{equation}
\omega(z, t) = e^{-1-\cos(t)} \left(-e + ez - z e^{\cos(t)}\right) (\omega_0(z_0) + f) - f \tag{40}
\end{equation}

Note that due to the sinusoidal variations of \( w_z(z(t, z_0), t) \), the value of \( e^{\int_0^t \alpha(s, z_0) ds} \) is bounded here. Therefore, based on the above exact solution, the flow is stable in this case for any form of initial vorticity.

3.3 The impact of a local heat source

Exact solutions may be obtained by using the method of separation of variables, as shown by the solution presented in the following example.

**Example 4.** A local heat source may be modeled with the following form

\[ w = \sin(t)e^{-\beta r^2}e^{\zeta z}, \tag{41} \]

with \( \beta \) and \( \zeta \) being constant coefficients. In the above solution, time variation of the heat source is assumed to be sinusoidal to represent diurnal temperature changes. The radial variation of the heat source is assumed to be of the form \( e^{-\beta r^2} \) to represent local heat sources such as clouds, and the vertical variation of the form \( e^{\zeta z} \), with \( \zeta < 0 \), allows us to model the rapid reduction of the heat source at higher elevations. It is verifiable that such a heat form leads to the following radial velocity and vorticity fields

\[ u_r = \frac{1}{2\beta r} \zeta \sin(t) \left(e^{-\beta r^2} - 1\right) e^{\zeta z}, \tag{42} \]
Fig. 2. Velocity field (left) and vorticity field (right) corresponding to example 4 with \( \beta = -\zeta = 1 \) and \( f = 0.1 \) at time \( t = \pi/4 \). Red lines in the left figure show stream-traces.

\[
\omega = -\zeta f \cos(t)e^{-\beta r^2}e^{\zeta z}.
\] (43)

The above exact solution shows that the vorticity generated by such a local heat source is bounded, and therefore the resulting flow is stable. The vertical and radial velocity components and vorticity field for this case, with \( \beta = -\zeta = 1 \) and \( f = 0.1 \) at time \( t = \pi/4 \), are shown in Figure 2. As is observed in (2), the flow corresponds to a local heat source with a maximum value at the center close to \( z = 0 \), and decreases in higher elevations and farther from the center. The heat source creates rising air (updraft) and leads to a cyclonic vorticity field with a maximum intensity at the center, close to the lower surface. Therefore, the above solution provides analytical evidence that a local heat source can trigger a cyclonic flow, which supports the numerical results reported by Montgomery et al. (2006) and Majda et al. (2008).
3.4 Effect of a background shear

Here, a general vertical velocity of the following form

\[ w = A(t)F(r)H(z) \]  \hspace{1cm} (44)

is considered in the following background shear

\[ \bar{U} = \bar{U}_\theta e_\theta \]  \hspace{1cm} (45)

with a \( \bar{U}_\theta \) of the form

\[ \bar{U}_\theta = C(t)P(r)Q(z) \]  \hspace{1cm} (46)

Special exact solutions for (6) of the form

\[ \omega = B(t)F(r)G(z) \]  \hspace{1cm} (47)

are sought. Plugging this form into (6), the following equation is obtained

\[
B'FG - ABF'(H'G - G'H) \frac{1}{2} \int rF - F^2 AB(H'G - G'H) + AF'HCPQ' = AFH'f
\]  \hspace{1cm} (48)

By imposing the conditions

\[ H'G - G'H = 0, \]  \hspace{1cm} (49)

\[ F'P = \gamma F, \]  \hspace{1cm} (50)

\[ C(t) = 1, \]  \hspace{1cm} (51)
one arrives at
\[ B'FG + \gamma AFHQ' = AFH'f, \] (52)
and by imposing another condition
\[ Q'H = \varsigma H', \] (53)
the equation (52) leads to
\[ \frac{B'}{A \varsigma(f - \gamma)} = \frac{H'}{G} = \lambda \] (54)
Therefore,
\[ G = \frac{H'}{\lambda}, \] (55)
\[ B' = \lambda A(f - \gamma \varsigma) \] (56)
In order to satisfy the above conditions, it is required that
\[ \frac{H''}{H'} = \frac{H'}{H}, \] (57)
and as before,
\[ H(z) = e^{\varsigma z} \] (58)
Hence,
\[ Q = \varsigma \varsigma z \] (59)
Omitting the details, the final result is stated in the following:

- For a heat source of the form
\[ w = B(t) \exp \left( \int_0^r \frac{\gamma}{P(s)} ds \right) e^{\varsigma z}, \] (60)
and the shear flow given by \( \overline{U} = \overline{U}_\theta e_\theta \) where
\[ \overline{U}_\theta = \zeta z P(r), \]  

for arbitrary functions \( B(t) \) and \( P(r) \) there are exact solutions for (6) of the form

\[ \omega = \zeta (f - \gamma) \left( \int_0^t B(\tau) d\tau \right) \left( \exp \left( \int_0^r \frac{\gamma}{P(s)} ds \right) \right) e^{\xi z} \]  

Based on the above exact solution, the vorticity is bounded if the integral \( \left( \int_0^t B(\tau) d\tau \right) \) is bounded, and the flow will be stable in this case. The shear does not have any impact on the stability of the flow, but it could intensify or suppress the vorticity based on the radial variations of the background shear \( (P(s)) \), which is reflected in the resulting vorticity field as \( \exp \left( \int_0^r \frac{\gamma}{P(s)} ds \right) \). The above solution also implies that a constant heat source leads to unbounded but linear growth of the vorticity, which can lead to instability of the flow.

**Example 5:** A simplified model of a local heat source of the form

\[ w(r, z, t) = \sin(t) e^{-\beta r^2} e^{\xi z}, \]  

with \( \beta \) and \( \zeta \) being constant coefficients, in the following mean shear flow

\[ \overline{U}_\theta = \frac{-\zeta z}{2\beta r} \]  

leads to the following vorticity field

\[ \omega(r, z, t) = -\zeta (f - 1) \cos(t) e^{-\beta r^2} e^{\xi z} \]  

The flow is thus stable in this case because \( \left( \int_0^t B(\tau) d\tau \right) \) is bounded and the background shear suppresses the vorticity in the regions far from the center.

This paper, only considers axisymmetric flows. Further analytical solutions can be found for the Cartesian case which is currently in progress by the authors.
4 NUMERICAL METHOD

Several methods are available for solving hyperbolic systems. Finite volume methods have been more popular due to their inherent conservation properties. Upwind finite volume methods are widely used for hyperbolic systems (e.g. Li et al., 2011, Pang et al., 2010, Mohammadian et al. 2007, Liu, 2006). However, in the presence of source terms, characteristic decomposition of flux terms may lead to an imbalance between the source and flux terms at the discrete level. CWENO schemes, on the other hand, are also very powerful in solving hyperbolic equations with source terms. Although they were originally developed for problems with shock waves, as we will show in this paper, they are also excellent choices for problems with source terms and smooth solutions. Because no decomposition is necessary in these methods, they approximate advection and source terms using the same approach. Therefore, the balance between flux and source terms is preserved much better than in upwind methods. Therefore, a fourth-order CWENO scheme is used in this paper for the numerical solution of (6). In order to solve (6), we first write it in the conservative form. This form is better for applying boundary conditions at the center, which will be a zero flux boundary, and therefore a grid point at the center will be avoided. A directional splitting scheme is used here to reduce computational costs associated with multidimensional CWENO schemes. As shown in the following, this approach leads to accurate yet efficient solutions.

In addition to the advection terms, there are two other terms in (6), which are $\frac{\partial w}{\partial r} \left( \frac{1}{r} \int_0^r s \omega ds \right)$ and $\frac{\partial w}{\partial z} (\omega + f)$. As explained earlier, the first term is also in the form of a transport term in the $z$ direction. Therefore we include it in the equations in the $z$-sweep. However, the source term $\frac{\partial w}{\partial z} (\omega + f)$ has no
directional preference, noting that $\frac{\partial w}{\partial z}$ is known from the heat source. Therefore, to keep the symmetry, we include half of it in the r-sweep and half in the z-sweep. Thus, the system is decomposed to

(i) The z-sweep

\[ \frac{\partial \omega}{\partial t} + \frac{\partial w \omega}{\partial z} = \frac{1}{2} \frac{\partial w}{\partial z} (\omega + f) - \omega \frac{\partial w}{\partial r} \frac{\partial}{\partial z} \left( \frac{1}{r} \int_0^r s \omega ds \right) \]

or

\[ \frac{\partial \omega}{\partial t} + \frac{\partial w \omega}{\partial z} = \frac{1}{2} \frac{\partial w}{\partial z} (-\omega + f) - \frac{\partial w}{\partial r} \frac{\partial}{\partial z} \left( \frac{1}{r} \int_0^r s \omega ds \right) \]

(ii) The r-sweep

\[ \frac{\partial \omega}{\partial t} + \frac{(u_r \omega)}{\partial r} = \frac{1}{2} \frac{\partial w}{\partial z} (\omega + f) - \frac{\partial u_r}{\partial r} \]

In order to simplify the calculations, a piecewise constant variation in time is assumed for the heat source. Therefore, the vertical and radial velocities, which are directly calculated from the heat source, remain constant over the entire step. Accordingly, in each time step, a part of the source term can be integrated analytically over the entire step by solving

\[ \frac{\partial \omega}{\partial t} = \frac{1}{2} \frac{\partial w}{\partial z} (-\omega + f) \]

which leads to

\[ \omega(t + \Delta t) = f + (\omega(t) - f) e^{-\frac{1}{2} \frac{\partial w}{\partial z} t} \]

and then the conservation law is solved

\[ \frac{\partial \omega}{\partial t} + \frac{\partial F}{\partial z} = R \]

where

\[ F = w \omega \]
The same approach is employed for the $r$ direction by integrating the source term analytically over the entire step using

$$\frac{\partial \omega}{\partial t} = \frac{1}{2} \frac{\partial w}{\partial z} (\omega + f) - \omega \frac{\partial u_r}{\partial r}$$

(74)

which leads to

$$\omega(t + \Delta t) = e^{a \Delta t} (2a \omega(t) + \frac{\partial w}{\partial z} f) - f \frac{\partial w}{\partial z}$$

(75)

where $a = \frac{1}{2} \frac{\partial w}{\partial z} - \frac{\partial u_r}{\partial r}$, and then the conservation equation is solved

$$\frac{\partial \omega}{\partial t} + \frac{\partial E}{\partial r} = 0,$$

(76)

where $E = u_r \omega$, using the CWENO scheme as described in the following.

Here we explain the solution method in the $z$ sweep. In order to simplify the notation, the discretization is given in a one dimensional form. The computational cells are denoted by $I_i = [z_i-1/2, z_i+1/2]$ and $I_{i+1/2} = [z_i, z_{i+1}]$, and cell averaged values are defined as

$$\overline{\omega}_i^n = \frac{1}{h} \int_{z_i-1/2}^{z_i+1/2} \omega(z, t^n) dz$$

(77)

and

$$\overline{\omega}_{i+1/2}^n = \frac{1}{h} \int_{z_i}^{z_{i+1}} \omega(z, t^n) dz$$

(78)

Numerical integration in the CWENO scheme is conducted on a time-staggered grid. That is, starting with $\overline{\omega}_i^n$, the solution is obtained for $\overline{\omega}_{i+1/2}^{n+1}$. Therefore, in each directional sweep, two time steps are needed to obtain the results on the same grid (i.e. $\overline{\omega}_{i+2}^{n+2}$). A numerical integration of (71) in time leads to
\[ \omega_{i+\frac{1}{2}}^{n+1} = \omega_{i+\frac{1}{2}}^n - \frac{1}{h} \left[ \int_{t^n}^{t^{n+1}} F(\omega(z_{i+1}, t)) \, dt - \int_{t^n}^{t^{n+1}} F(\omega(z_i, t)) \, dt \right] + \int_{t^n}^{t^{n+1}} R(\omega(z_{i+1}, t)) \, dt \] (79)

Hence, the solution involves calculation of \( \omega_{i+\frac{1}{2}}^n \) and the two temporal integrals \( \int_{t^n}^{t^{n+1}} F(\omega(z_i, t)) \, dt \) and \( \int_{t^n}^{t^{n+1}} R(\omega(z_{i+1}, t)) \, dt \), as explained in the following.

1. Calculation of \( \omega_{i+\frac{1}{2}}^n \) from \( \{\omega_i^n\} \) is performed by a WENO reconstruction via

\[ \omega_{i+\frac{1}{2}}^n = \frac{1}{h} \int_{z_i}^{z_{i+\frac{1}{2}}} \omega(z, t^n) \, dz = \frac{1}{h} \left[ \int_{z_i}^{z_{i+\frac{1}{2}}} \omega(z, t^n) \, dz + \int_{z_{i+\frac{1}{2}}}^{z_{i+1}} \omega(z, t^n) \, dz \right] \] (80)

The spatial integral \( \frac{1}{h} \int_{z_{i-\frac{1}{2}}}^{z_i} \omega(z, t^n) \, dz \) can be calculated using polynomials \( p_j(\omega) \), which satisfy

\[ \frac{1}{h} \int_{t_{i+j-1}}^{t_{i+j}} p_j(z) \, dz = \omega_{i+j-l}, \quad l = -3, \ldots, 3 \] (81)

as

\[ \frac{1}{h} \int_{z_{i-\frac{1}{2}}}^{z_{i}} \omega(z, t^n) \, dz = \sum_{j=0}^{2} s_j \int_{z_{i-\frac{1}{2}}}^{z_{i}} p_j(z) \, dz \] (82)

where \( s_j \) coefficients are nonlinear weights that are calculated based on the smoothness of \( \omega \), as explained in Appendix I. The same approach is employed for calculating \( \int_{z_{i-\frac{1}{2}}}^{z_{i+1}} \omega(z, t^n) \, dz \).

2. The approximation of temporal integrals \( \int_{t^n}^{t^{n+1}} F(\omega(z_i, t)) \, dt \) and \( \int_{t^n}^{t^{n+1}} R(\omega(z_{i+1}, t)) \, dt \) is performed using the three-point Gauss integration method. For example,
\[
\int_{t_n}^{t_{n+1}} F(\omega(z_i, t)) \, dt = \Delta t \sum_{l=0}^{3} \alpha_l F(\omega(z_i, t^n + \tau_l \Delta t)),
\]

(83)

where \(\alpha_1 = \frac{5}{18}, \alpha_2 = \frac{4}{9}, \) and \(\tau_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}, \tau_2 = \frac{1}{2}, \tau_3 = \frac{1}{2} + \frac{\sqrt{15}}{10}\) are the weights and knots of the Gauss quadrature formula.

The approximation of point values \(\omega(z_i, t^n + \tau_1 \Delta t)\) is performed by solving the ODE

\[
\frac{d \omega(z_i, t)}{dt} = R|_{z_i} - F|_{z_i},
\]

(84)

using the fourth-order natural continuous extension (NCE) Runge Kutta scheme (Zennaro, 1986). Therefore, point values \(\omega(z_i, t^n)\) and flux derivatives are required. WENO reconstruction of point values \(\omega(z_i, t^n)\) from \(\{\varpi^n_i\}\) is performed using

\[
\omega(z_i, t^n) = \sum_{j=0}^{2} s_j p_j(z_i),
\]

(85)

as explained in Appendix II. WENO reconstruction of derivatives is then performed as

\[
(F(u))|_{z_i} \approx \sum_{j=0}^{2} s_j p'_j(z_i),
\]

(86)

as explained in Appendix III. Note that the nonlinear weights \(s_j\) are different for derivatives and should be calculated again.

The integral \(\frac{1}{2} \int_0^r s \omega ds\) is calculated analytically using the WENO reconstruction of \(r \omega\), and then derivative \(\frac{p}{2} \left(\frac{1}{2} \int_0^r s \omega ds\right)\) is calculated using WENO reconstruction similar to the calculation of \((F(u))|_{z_i}\). The same approach is then conducted for the \(r\) sweep. A Strang splitting approach is used, where in the first step the equations are solved in the \(z\) direction first, and in the next step the equations are solved in the \(r\) direction first.
In this section, using two numerical experiments, we evaluate the accuracy of the proposed method for axisymmetric horizontal flows dominated by heat sources. In the first test case, we employ one of the exact solutions developed in this paper, and in the second test case we use an exact solution developed in Mohammadian and Charron (2011). We compare our results with a high-resolution total variation diminishing upwind scheme as well as with the Chebyshev pseudospectral method proposed in Mohammadian et al. (2011). Since the equations are used in non-dimensional form, the size of the computational domain is $1 \times 1$ in all cases. The boundary condition in the center is a zero flux. The inflow boundary conditions are specified from analytical solutions, and for outflow boundaries an extrapolation from the adjacent grid points inside the domain is performed.

5.1 Test 1

The first test case corresponds to example 4, in which the following heat source is supplied

$$w = \sin(t)e^{-\beta r^2}e^\zeta z, \quad (87)$$

and where $\beta$ and $\zeta$ are constant coefficients. The heat source leads to the following radial velocity

$$u_r = \frac{1}{2\beta r} \zeta \sin(t) \left( e^{-\beta r^2} - 1 \right) e^\zeta z \quad (88)$$

The analytical solution for the vorticity field is the following

$$\omega = -\zeta f \cos(t)e^{-\beta r^2}e^\zeta z \quad (89)$$
Numerical results using a grid with $40 \times 40$ points for for $\beta = \zeta = 1$ and $f = 0.1$ at time $t = 4$ are shown in Figure 3 and are compared with the exact solution where solid lines show the analytical solution and dashed lines show the numerical results. This conceptual model corresponds to a local heat source caused by clouds concentrated at higher levels. Since $\zeta > 0$ here, the vertical velocity gradient is positive and therefore, the vorticity is larger at higher levels. In particular, due to the exponential reduction of the heat source from the center, a zone of strong cyclonic flow is created at high levels close to the center. As observed in Figure 3, the model can successfully simulate the problem, and a very good match is obtained between the analytical and numerical solutions despite the coarse grid employed. The central core of intensified vorticity at higher levels is very well captured. Contours of numerical solutions for the vorticity closely follow the analytical solution which shows that the numerical method leads to a very low level of numerical diffusion as expected due to the fourth order accuracy of spatial discretization scheme. No numerical oscillations are observed which indicates that the balance between the source and flux terms is well preserved.

5.2 Test 2

Here, example 4 in Mohammadian and Charron (2011) is considered with the following heat source

$$w = \sin(t) \sin(2\pi r) e^{\zeta z},$$

where $\beta$ and $\zeta$ are constant coefficients. The resulting radial velocity is given by
Fig. 3. Vorticity field of test 2 corresponding to example 4 with $\beta=\zeta=1$ and $f=0.1$ at time $t=4$. Solid lines show the analytical solution and dashed lines show the numerical results.

$$u^* = \frac{\zeta \sin(t) e^{\zeta z}}{4r\pi^2} \left( -\sin(2\pi r) + 2\pi r \cos(2\pi r) \right),$$

which leads to the following analytical solution

$$\omega = -\zeta f \cos(t) \sin(2\pi r) e^{\zeta z}$$

Figure 6 shows numerical and analytical solutions for the vorticity field using $\zeta = \beta= 1$ and $f=0.1$ at time $t = \pi/4$ with a $100 \times 100$ grid. A mass-conservative cell-based fourth-order upwind scheme is also employed for comparison in Figure 5. The fourth-order upwind scheme leads to an unacceptable level of numerical noise due to the imbalance between the source and flux terms. The CWENO scheme, on the other hand, leads to very accurate results with no numerical noise. This is a clear demonstration of the superiority of the CWENO scheme for balanced axisymmetric horizontal flows with
heat sources in radial coordinates, where a homogeneous numerical scheme is needed with no directional preference in order to keep the balance between the flux and source terms. Further numerical experiments with a coarser grid (not shown) also lead to non-oscillatory results. Moreover, the method is also superior to the Chebyshev pseudo-spectral method proposed in Mohammadian and Charron (2011), shown in Figure 5, since the level of numerical diffusion in the model is even lower than the numerical diffusion required to dissipate the inherent numerical noise associated with the Chebyshev pseudo-spectral method, which is caused by the central differentiation scheme. Such a high level of required numerical noise leads to a destruction of the spectral accuracy expected from the Chebyshev method. Note that the CWENO scheme also uses a basically central approximation method. However, the use of a temporally staggered grid prevents the propagation and amplification of the noise of a central interpolation method. Another important feature of the CWENO scheme compared with the Chebyshev pseudo-spectral method is that unlike the latter, the CWENO method is not limited to a fixed arrangement of computational grid points, and one can use a variable grid size based on the required resolution at different regions of the problem. For example, as is seen in this test case, the middle of the simulation domain has a high gradient of vorticity but is associated with the coarsest part of the Chebyshev grid points and may not be resolved well with the Chebyshev method, while the grid used for the CWENO scheme can be easily refined here.
Fig. 4. Vorticity field of test 2 with $\zeta = \beta = 1$ and $f = 0.1$ at time $t = \pi$ using the Chebyshev pseudo spectral scheme (Mohammadian & Charron, 2011). Solid lines show the analytical solution and dashed lines show the numerical one.

Fig. 5. Vorticity field of test 2 corresponding to example 5 with $\zeta = \beta = 1$ and $f = 0.1$ at time $t = \pi$ using the fourth order upwind scheme. Solid lines show the numerical solution and dashed lines show the analytical one.
Fig. 6. Vorticity field of test 2 with $\zeta = \beta = 1$ and $f = 0.1$ at time $t = \pi$ using the proposed scheme. Solid lines show the analytical solution and dashed lines show the numerical one.

6 CONCLUSION

In this paper, new analytical nonlinear solutions were obtained for axisymmetric horizontal flows dominated by heat sources which are ubiquitous in several multiscale models recently developed for atmospheric and oceanic phenomena such as hurricane embryos. The exact solutions give further insight into the dynamics of these flows and their stability under various circumstances. Moreover, they provide analytical support for some results recently observed by numerical experiments. The proposed analytical solutions can also serve as challenging test cases for the evaluation of numerical methods used to solve those equations as building blocks of multiscale models. A fourth-order CWENO scheme was also developed for corresponding equations in the axisymmetric radial coordinates. The numerical method was shown to be superior to the Chebyshev pseudo-spectral method recently developed by the
authors, in terms of numerical diffusion. Moreover, the computational grid is more flexible in this method, unlike the Chebyshev scheme. On the other hand, the equations can be solved in the conservation Form, which enables a natural implementation of boundary conditions based on the numerical flux, which is not the case with the Chebyshev pseudo-spectral method. Due to the uniform central approximation of all source and flux terms, the balance between those terms in the analytical solutions is well preserved, and the computational results are free of numerical oscillations associated with upwind finite volume methods.

Acknowledgment

The research was supported by Environment Canada.

References


Appendix I: Calculation of nonlinear weights and smoothness indicators

A polynomial reconstruction $Q(z)$ is sought, such that

$$\frac{1}{h} \int_{l+1}^{l+1} Q(z)dz = \omega_{i+l}, \quad l = -3, ..., 3$$ (93)
It is obtained using polynomials \( p_j(z) \) which satisfy

\[
\frac{1}{h} \int_{I_{i+j-1}} p_j(z) \, dz = \vec{\omega}_{i+j-l}, \quad l = -3, \ldots, 3
\]  

(94)

using the following relationship

\[
\int_a^b Q(z) \, dz = \sum_{j=0}^{2} \gamma_j \int_a^b p_j(z) \, dz,
\]  

(95)

where \( \gamma_j \) values are given below. Therefore, in order to calculate, e.g., \( J_{i-\frac{1}{2}}^{z_i} \omega(z, t^n) \, dz \) in (80), the integrals of \( p_j(z) \) are calculated using

\[
\int_{z_i}^{z_i+1/2} p_j(z) \, dz = \sum_{l=0}^{2} a_{jl} \vec{\omega}_{i+j+l-2}, \quad j = 0, 1, 2,
\]  

(96)

where

\[
A = (a_{jl})_{3 \times 3} = \begin{pmatrix}
-\frac{1}{16} & \frac{1}{4} & \frac{5}{16} \\
\frac{1}{16} & \frac{1}{2} & -\frac{1}{16} \\
\frac{11}{16} & -\frac{1}{4} & \frac{1}{16}
\end{pmatrix},
\]

(97)

with \( \gamma_1 = \frac{3}{16} \), \( \gamma_2 = \frac{5}{8} \), \( \gamma_3 = \frac{3}{16} \). Similarly, \( J_{i-\frac{1}{2}}^{z_i+\frac{1}{2}} u(z, t^n) \, dz \) is calculated by integrating the corresponding polynomials as

\[
\int_{z_i}^{z_i+\frac{1}{2}} p_j(z) \, dz = \sum_{l=0}^{2} b_{jl} \vec{\omega}_{i+j+l-2}, \quad j = 0, 1, 2,
\]  

(98)

where

\[
B = (b_{jl})_{3 \times 3} = \begin{pmatrix}
\frac{1}{16} & -\frac{1}{4} & \frac{11}{16} \\
-\frac{1}{16} & \frac{1}{2} & \frac{5}{8} \\
\frac{1}{16} & \frac{1}{4} & -\frac{1}{16}
\end{pmatrix},
\]

(99)
with \( \gamma_1 = \frac{3}{16}, \gamma_2 = \frac{5}{8}, \gamma_3 = \frac{3}{16} \). Standard weights of WENO schemes have been used here (see e.g. Qiu and Shu, 2002), where the nonlinear weights are defined as

\[
s_j = \frac{s_j}{\sum_{l=0}^{2} s_l},
\]

(100)

\[
\pi_j = \frac{\gamma_j}{(\epsilon + \beta_j)^2},
\]

(101)

where \( \beta_j \) values are standard smoothness indicators (Qiu and Shu, 2002) and \( \epsilon = 10^{-8} \) is used here.

Appendix II: WENO reconstruction of point values

WENO reconstruction of point values \( \omega(z_i, t^n) \) from \( \{\overline{\omega}_i^n\} \) is performed using

\[
\omega(z_i, t^n) = \sum_{j=0}^{2} s_j p_j(z_i),
\]

(102)

where

\[
p_j(z_i) = \sum_{l=0}^{2} c_{jl} \overline{\omega}_{i+j+l-2}, \quad j = 0, 1, 2,
\]

(103)

with

\[
C = (c_{jl})_{3x3} =
\begin{pmatrix}
-\frac{1}{24} & \frac{1}{12} & \frac{23}{24} \\
\frac{1}{24} & \frac{13}{12} & -\frac{1}{24} \\
\frac{23}{24} & \frac{1}{12} & -\frac{1}{24}
\end{pmatrix},
\]

(104)
and $s_j$ values are calculated using $\gamma_1 = -\frac{9}{80}, \gamma_2 = \frac{49}{80}, \gamma_3 = -\frac{9}{80}$ as explained in Appendix I. The splitting technique of treating negative weights in WENO schemes proposed by Shi et al. (2002) is used here.

**Appendix III: WENO reconstruction of derivatives**

WENO reconstruction of derivatives is performed using a polynomial $Q$ which is obtained as

$$Q'(z_i) = \sum_{j=0}^{2} \gamma_j p_j'(z_i),$$

(105)

where

$$p_j(z_i) = \sum_{l=0}^{2} d_{jl} \phi_{i+j+l-2}, \quad j = 0, 1, 2,$$

(106)

with

$$D = (d_{jl})_{3 \times 3} = \begin{pmatrix} \frac{1}{2} & -2 & \frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3}{2} & 2 & -\frac{1}{2} \end{pmatrix},$$

(107)

and $\gamma_1 = \frac{1}{6}, \gamma_2 = \frac{2}{3}, \gamma_3 = \frac{1}{6}$. Smoothness indicators and nonlinear weights are calculated using the method explained in Appendix I.