Une question d'équations
❁
A simple matter of equations

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by Claude Girard
❁
The most beautiful equations of GEM

meteorology

The scientist does not study nature because it is useful;
he studies it because he delights in it,
and he delights in it because it is beautiful.
If nature were not beautiful,
it would not be worth knowing,
and if nature were not worth knowing,
life would not be worth living.

Henri Poincaré

N.B.: Certain document features can be better viewed in the ppt document available at:
http://iweb/~armncrg/PRESENTATIONS/

- <u>Derivation</u> of the Euler equations
in vertical ζ(dzeta)-coordinate
of the log-hydrostatic-pressure type
with a modified definition of hydrostatic pressure
vertically discretized on a Charney-Phillips grid

-Implementation in GEM

-A comparison: ALADIN-NH/IFS-ECMWF

~armncrg/GEM_DOC/**GEM4.0.pdf**

35 years in RPN 1973-2008

learning dynamics & numerics … besides physical parameterization

-Semi-Implicit Scheme (Robert, **1972**)
-Spectral Method (Robert, 1966, Daley et al. 1976, ECMWF)
-The primitive equations in σ-coordinate
-Vertical discretization in σ-coordinate (SDF, SEC,CCRN)

Spectral Models

-Semi-Lagrangian Scheme (Robert, **1982**)
-The Euler equations in generalized height-base *Z*-coordinate
-Vertical discretization in *Z*-coordinate (MC2)

MC2

-The Euler equations in mass-base η-coordinate (Laprise, **1992**)
-hydrostatic-pressure type

-The Euler equations in *hydrostatic-pressure* η-coordinate (Yeh et al., **2002**)
-Staggering in the vertical of the Euler equations in η-coordinate

GEM

-The Euler equations in ***log****-hydrostatic-pressure type* ζ-coordinate
-Staggering in the vertical of the Euler equations in ζ-coordinate

… *learning* mostly from Canadians

The Meteorological Equations (1)

Momentum:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \mathbf{F}$$

Thermodynamic:

$$\frac{dT}{dt} - \frac{1}{\rho c_p}\frac{dp}{dt} = \frac{Q}{c_p}$$

Continuity:

$$\frac{d\ln\rho}{dt} + \nabla\cdot\mathbf{V} = 0$$

Gas State:

$$p = \rho RT$$

6 Equations ⟷ 6 dependent Variables: $\mathbf{V}=(\mathbf{V}_h, w), p, \rho, T$

6 Equations: 5 prognostic + 1 diagnostic

The Meteorological Equations (2)

First Law		
	Momentum:	$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \mathbf{F}$$
	Thermodynamic:	$$\frac{dT}{dt} - \frac{1}{\rho c_p}\frac{dp}{dt} = \frac{Q}{c_p}$$
	Continuity:	$$\frac{d\ln\rho}{dt} + \nabla\cdot\mathbf{V} = 0$$

2nd Law → **Gas State:**

$$p = \rho R T$$

6 Equations ⟷ 6 dependent Variables: $\mathbf{V}=(\mathbf{V}_h, w), p, \rho, T$

6 Equations: 5 prognostic + 1 diagnostic

The Reduced Meteorological Equations (1)

(elimination of ρ)

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + \frac{RT}{p}\nabla p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \mathbf{F}$$

$$p = \rho RT$$

The Reduced Meteorological Equations (2)

(elimination of ρ)

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d \ln T}{dt} - \kappa \frac{d \ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{d \ln T}{dt} - \frac{R}{c_p} \frac{d \ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{dT}{dt} - \frac{R}{c_p} \frac{T}{p} \frac{dp}{dt} = \frac{Q}{c_p}$$

$$\frac{dT}{dt} - \frac{1}{\rho c_p} \frac{dp}{dt} = \frac{Q}{c_p}$$

$$p = \rho R T$$

The Reduced Meteorological Equations (3)

(elimination of ρ)

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d \ln T}{dt} - \kappa \frac{d \ln p}{dt} = \frac{Q}{c_p T}$$

$$(1 - \kappa)\frac{d \ln p}{dt} + \nabla \cdot \mathbf{V} = \frac{Q}{c_p T}$$

$$\frac{d \ln p / T}{dt} + \nabla \cdot \mathbf{V} = 0$$

$$\frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{V} = 0$$

$$p = \rho RT$$

The Reduced Meteorological Equations (4)

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d \ln T}{dt} - \kappa \frac{d \ln p}{dt} = \frac{Q}{c_p T}$$

$$(1 - \kappa)\frac{d \ln p}{dt} + \nabla \cdot \mathbf{V} = \frac{Q}{c_p T}$$

5 prognostic Equations

Vertical coordinate transformation: *z* to ζ (unspecified) (1)

These <u>transformation rules</u> which can all be recovered from the invariance of the <u>total derivative</u> are sufficient

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta}$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta}$$

$$\frac{d}{dt} \equiv \frac{d}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx_i}{dt} \frac{\partial f}{\partial x_i}$$

$$\frac{\partial}{\partial t} + \mathbf{V}_h \cdot \nabla_z + w \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + \mathbf{V}_h \cdot \nabla_\zeta + \dot{\zeta} \frac{\partial}{\partial \zeta}$$

No transformation of vector components!!!

The 3 winds components are treated as 3 independent scalars !!!

Incomplete transformation

ζ is an oblique (non-orthogonal) coordinate

~armncrg/GEM_DOC/**GEM4.0.pdf**

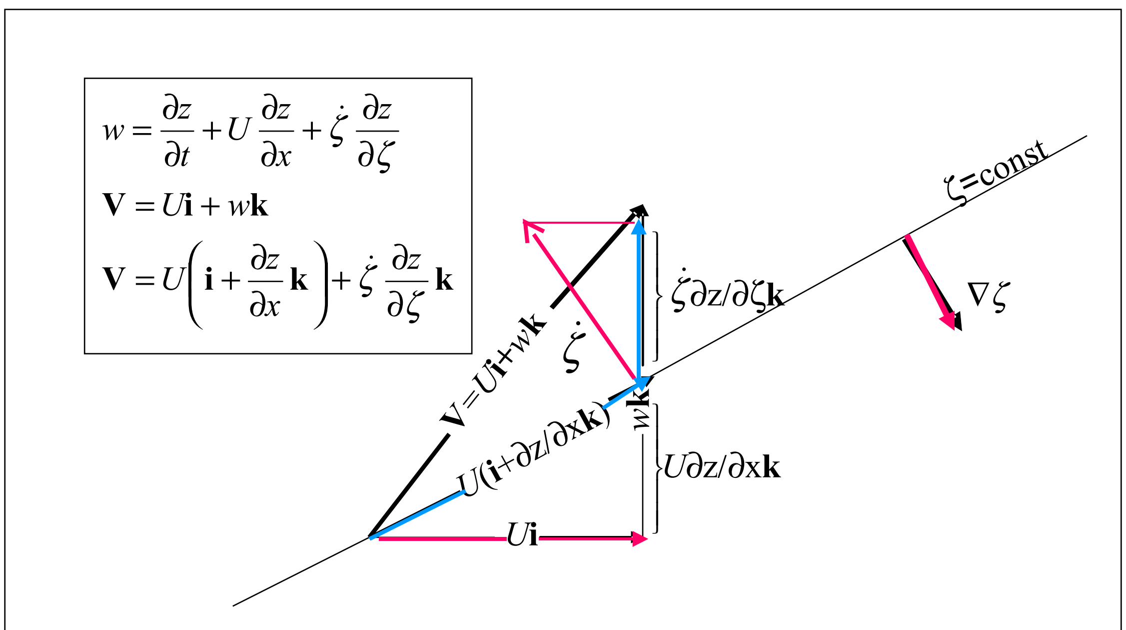
$$\mathbf{V}\cdot\nabla f = V^i \boldsymbol{\tau}_i \cdot \boldsymbol{\eta}^j \frac{\partial f}{\partial \hat{x}^j} = V^i \frac{\partial f}{\partial \hat{x}^i}$$

$$= \left[U\left(\mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}\right) + V\left(\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}\right) + \dot{\zeta}\frac{\partial z}{\partial \zeta}\mathbf{k} \right] \cdot \left[\left(\frac{\partial f}{\partial x}\right)_{\zeta}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{\zeta}\mathbf{j} + \frac{\partial f}{\partial \zeta}\nabla\zeta \right]$$

$$= U\left(\frac{\partial f}{\partial x}\right)_{y,\zeta} + V\left(\frac{\partial f}{\partial y}\right)_{x,\zeta} + \dot{\zeta}\left(\frac{\partial f}{\partial \zeta}\right)_{x,y}$$

$\dfrac{\partial f}{\partial \hat{x}^j}$: covariant components

$V^i = U, V, \dot{\zeta}$: contravariant components



$$\dot{\zeta} = \mathbf{V}\cdot\nabla\zeta$$

Vertical coordinate transformation: *z* to ζ (unspecified) (2)

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla \ln p + g\mathbf{k} = \mathbf{F}$$

$$\rightarrow \frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_z \ln p = \mathbf{F}_h$$

$$\rightarrow \frac{dw}{dt} + RT\frac{\partial \ln p}{\partial z} + g = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T} \longrightarrow \frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$(1 - \kappa)\frac{d\ln p}{dt} + \nabla\cdot\mathbf{V} = \frac{Q}{c_p T} \longrightarrow (1 - \kappa)\frac{d\ln p}{dt} + \nabla_z\cdot\mathbf{V}_h + \frac{\partial w}{\partial z} = \frac{Q}{c_p T}$$

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

Vertical coordinate transformation: *z* to ζ (unspecified) (3)

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta} + g = F_w$$

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}$$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_z \ln p = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial \ln p}{\partial z} + g = F_w$$

Vertical coordinate transformation: *z* to ζ (unspecified) (4)

$$(1-\kappa)\frac{d\ln p}{dt}+\frac{d}{dt}\ln\left(\frac{\partial z}{\partial \zeta}\right)+\nabla_\zeta\cdot\mathbf{V}_h+\frac{\partial\dot{\zeta}}{\partial\zeta}=\frac{Q}{c_pT}$$

$$\nabla_z\equiv\nabla_\zeta-\nabla_\zeta z\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

$$\frac{\partial}{\partial z}\equiv\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

$$(1-\kappa)\frac{d\ln p}{dt}+\nabla_z\cdot\mathbf{V}_h+\frac{\partial w}{\partial z}=\frac{Q}{c_pT}$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}$$

$$\frac{\partial w}{\partial z} = \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}(w)$$

$$= \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\left(\frac{dz}{dt}\right)$$

$$= \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial t} + \mathbf{V}_h \cdot \nabla_\zeta z + \dot{\zeta}\frac{\partial z}{\partial \zeta}\right)$$

$$= \frac{\partial \zeta}{\partial z}\left(\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \zeta}\right) + \mathbf{V}_h \cdot \nabla_\zeta\left(\frac{\partial z}{\partial \zeta}\right) + \dot{\zeta}\frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial \zeta}\right) + \frac{\partial \mathbf{V}_h}{\partial \zeta} \cdot \nabla_\zeta z + \frac{\partial \dot{\zeta}}{\partial \zeta}\frac{\partial z}{\partial \zeta}\right)$$

$$= \frac{\partial \zeta}{\partial z}\left(\frac{d}{dt}\left(\frac{\partial z}{\partial \zeta}\right) + \frac{\partial \mathbf{V}_h}{\partial \zeta} \cdot \nabla_\zeta z + \frac{\partial \dot{\zeta}}{\partial \zeta}\frac{\partial z}{\partial \zeta}\right)$$

$$\frac{\partial w}{\partial z} = \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \zeta}\right) + \frac{\partial \zeta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \zeta} \cdot \nabla_\zeta z + \frac{\partial \dot{\zeta}}{\partial \zeta}$$

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta}$$

$$\nabla_z \cdot \mathbf{V}_h = \nabla_\zeta \cdot \mathbf{V}_h - \frac{\partial \zeta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \zeta} \cdot \nabla_\zeta z$$

+

$$\frac{\partial w}{\partial z} = \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \zeta}\right) + \frac{\partial \zeta}{\partial z}\frac{\partial \mathbf{V}_h}{\partial \zeta} \cdot \nabla_\zeta z + \frac{\partial \dot{\zeta}}{\partial \zeta}$$

$$\nabla_z \cdot \mathbf{V}_h + \frac{\partial w}{\partial z} = \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta}$$

Vertical coordinate transformation: *z* to ζ (unspecified) (5)

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta} + g = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$(1-\kappa)\frac{d\ln p}{dt} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} = \frac{Q}{c_p T}$$

5 equations

2 vertical velocities

$$\frac{dz}{dt} - w = 0$$

7 dependent variables: $\mathbf{V}_h, w, p, T, z, \dot{\zeta}$?

$$z = z(\mathbf{r}_h, \zeta, t)$$

The equations of GEM4

1

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \mathbf{F}$$

$$\frac{dT}{dt} - \frac{1}{\rho c_p}\frac{dp}{dt} = \frac{Q}{c_p}$$

$$\frac{d\ln\rho}{dt} + \nabla\cdot\mathbf{V} = 0$$

$$p = \rho RT$$

$\Rightarrow$

2

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla\ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$(1-\kappa)\frac{d\ln p}{dt} + \nabla\cdot\mathbf{V} = \frac{Q}{c_p T}$$

Vertical coordinate transformation: z to ζ (unspecified)

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

$\Rightarrow$

3

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta\ln p - \nabla_\zeta z\frac{\partial\zeta}{\partial z}\frac{\partial\ln p}{\partial\zeta}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial\zeta}{\partial z}\frac{\partial\ln p}{\partial\zeta} + g = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$(1-\kappa)\frac{d\ln p}{dt} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial\zeta}\right) + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} = \frac{Q}{c_p T}$$

$$\frac{dz}{dt} - w = 0$$

Vertical coordinate transformation: *z* to ζ (MC2)

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta} + g = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$(1-\kappa)\frac{d\ln p}{dt} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} = \frac{Q}{c_p T}$$

diagnostic equation

$$\mathbf{V}_h \cdot \nabla_\zeta z + \dot{\zeta}\frac{\partial z}{\partial \zeta} = w$$

←

$$\frac{\partial z}{\partial t_\zeta} = 0 \qquad \frac{dz}{dt} - w = 0$$

$$\zeta = \frac{z - z_S}{z_T - z_S} \quad or \quad z = \zeta z_T + (1-\zeta)z_S$$

Vertical coordinate transformation (GEM4): *z* to lnπ to ζ (1)

$$\boldsymbol{\phi} = gz$$

$$RT = -\frac{\partial \boldsymbol{\phi}}{\partial \ln \boldsymbol{\pi}}$$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p - \nabla_\zeta z \frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta} + g = F_w$$

$$\boldsymbol{\mu} = \frac{\partial \ln(p/\boldsymbol{\pi})}{\partial \ln \boldsymbol{\pi}}$$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta \ln p + \nabla_\zeta z \frac{g}{RT}(1+\mu)\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\left(-\frac{g}{RT}(1+\mu)\right) + g = F_w$$

$$\begin{aligned}
\frac{\partial \zeta}{\partial z}\frac{\partial \ln p}{\partial \zeta} &= \frac{\partial \ln p}{\partial z} \\
&= \frac{\partial \ln p}{\partial \ln \pi}\frac{\partial \ln \pi}{\partial z} \\
&= \left(1 + \frac{\partial \ln p/\pi}{\partial \ln \pi}\right)\frac{\partial \ln \pi}{\partial z} \\
&= g(1+\mu)\frac{\partial \ln \pi}{\partial \phi} \\
&= -\frac{g}{RT}(1+\mu)
\end{aligned}$$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\boldsymbol{\mu})\nabla_\zeta \boldsymbol{\phi} = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$

$$\boldsymbol{\mu} = \frac{dw}{dt} / g \sim \frac{\text{vertical acceleration}}{\text{gravitational acceleration}}$$

Vertical coordinate transformation (GEM4): z to $\ln\pi$ to ζ (2)

$$(1-\kappa)\frac{d\ln p}{dt}+\frac{d}{dt}\ln\left(\frac{\partial z}{\partial\zeta}\right)+\nabla_\zeta\cdot\mathbf{V}_h+\frac{\partial\dot{\zeta}}{\partial\zeta}=\frac{Q}{c_pT}$$

$$(1-\kappa)\frac{d\ln p}{dt}+\frac{d}{dt}\ln\left(T\frac{\partial\ln\pi}{\partial\zeta}\right)+\nabla_\zeta\cdot\mathbf{V}_h+\frac{\partial\dot{\zeta}}{\partial\zeta}=\frac{Q}{c_pT}$$

$$(1-\kappa)\frac{d\ln p}{dt}+\frac{d}{dt}\ln T+\frac{d}{dt}\ln\left(\frac{\partial\ln\pi}{\partial\zeta}\right)+\nabla_\zeta\cdot\mathbf{V}_h+\frac{\partial\dot{\zeta}}{\partial\zeta}=\frac{Q}{c_pT}$$

$$\frac{d\ln p}{dt}+\frac{d}{dt}\ln\left(\frac{\partial\ln\pi}{\partial\zeta}\right)+\nabla_\zeta\cdot\mathbf{V}_h+\frac{\partial\dot{\zeta}}{\partial\zeta}=0$$

$$RT=-\frac{\partial\phi}{\partial\ln\pi}$$

$$\phi=gz$$

$$\frac{\partial z}{\partial\zeta}=\frac{1}{g}\frac{\partial\phi}{\partial\zeta}=\frac{1}{g}\frac{\partial\phi}{\partial\ln\pi}\frac{\partial\ln\pi}{\partial\zeta}=-\frac{RT}{g}\frac{\partial\ln\pi}{\partial\zeta}$$

Vertical coordinate transformation (GEM4): z to $\ln\pi$ to ζ (3)

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\mu)\nabla_\zeta \phi = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\mu = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{d\ln p}{dt} + \frac{d}{dt}\ln\left(\frac{\partial \ln \pi}{\partial \zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0$$

$$RT + \frac{\partial \phi}{\partial \ln \pi} = 0$$

9 equations

$$\mu - \frac{\partial \ln p/\pi}{\partial \ln \pi} = 0$$

9 dependent variables:

$$\mathbf{V}_h, w, p, T, \boldsymbol{\phi}, \dot{\boldsymbol{\zeta}}, \boldsymbol{\mu}, \ln\boldsymbol{\pi}$$

$$\frac{d\phi}{dt} - gw = 0$$

$$\ln\boldsymbol{\pi} = \boldsymbol{\zeta} + Bs; \qquad s = \ln(\boldsymbol{\pi}_S / p_{ref})$$

$$p_{top} / p_{ref} = \boldsymbol{\eta}_T < \boldsymbol{\eta} < 1 : \textit{specified}\ \ \boldsymbol{\pi}\text{-}\textit{like model levels}; \qquad p_{ref} = 10^5\, Pa$$

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_S + \ln(\boldsymbol{\eta})$$

$$\ln p_{top} = \boldsymbol{\zeta}_T \le \boldsymbol{\zeta} \le \boldsymbol{\zeta}_S = \ln p_{ref} : \textit{calculated}\ \ln \boldsymbol{\pi} - \text{like}\ \textit{model coordinate}\ \text{levels}$$

$$\ln \boldsymbol{\pi} = A(\boldsymbol{\zeta}) + B(\boldsymbol{\zeta}) s$$

$$A = \boldsymbol{\zeta}; \qquad B = \left(\frac{\boldsymbol{\zeta} - \boldsymbol{\zeta}_T}{\boldsymbol{\zeta}_S - \boldsymbol{\zeta}_T} \right)^r$$

Going to model variables ϕ', q, s from ϕ, p, $\ln\pi$ (1)

$$\phi' = \phi - \phi_*$$

$$q = \ln p / \pi$$

$$s = \ln(\pi_S / p_{ref})$$

$$\phi_*(\zeta) = -RT_*(\zeta - \zeta_S)$$

$$\frac{\partial \phi_*}{\partial \zeta} = -RT_*; \quad T_* = 240K$$

$$\frac{d\phi_*}{dt} = -RT_*\dot{\zeta}$$

$$\ln\pi = \zeta + Bs$$

$$\ln p = \ln p / \pi + \ln\pi$$

$$\ln p = \quad q \quad + \zeta + Bs$$

$q = \ln p - \ln \pi$: non-hydrostatic log-pressure deviation

Going to model variables ϕ', q, s (2)

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta\left(Bs+q\right) + \left(1+\mu\right)\nabla_\zeta\phi' = \mathbf{F}_h$$

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + \left(1+\mu\right)\nabla_\zeta\phi = \mathbf{F}_h$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left[\frac{d}{dt}\left(Bs+q\right) + \dot{\boldsymbol{\zeta}}\right] = \frac{Q}{c_p T}$$

$$\phi = \phi_* + \phi'$$

$$\ln p = q + \zeta + Bs$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$\nabla_\zeta\phi = \nabla_\zeta\phi'$$

$$\nabla_\zeta \ln p = \nabla_\zeta\left(Bs+q\right)$$

$$\frac{d\ln p}{dt} = \frac{d}{dt}\left(Bs+q\right) + \dot{\zeta}$$

Going to model variables ϕ', q, s (3)

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta\left(Bs+q\right) + \left(1+\mu\right)\nabla_\zeta\phi' = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\mu = F_w$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}\left(Bs+q\right) + \dot{\zeta}\right] = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\left[\left(Bs+q\right) + \ln\left(1 + \frac{\partial B}{\partial\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} + \dot{\zeta} = 0$$

$$\frac{d\ln p}{dt} + \frac{d}{dt}\ln\left(\frac{\partial\ln\pi}{\partial\zeta}\right) + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} = 0$$

$$\frac{d\ln p}{dt} = \frac{d}{dt}\left(Bs+q\right) + \dot{\zeta}$$

$$\ln\pi = \zeta + Bs$$

Going to model variables ϕ', q, s (4)

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta\left(Bs+q\right) + \left(1+\mu\right)\nabla_\zeta\phi' = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\mu = F_w$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left[\frac{d}{dt}\left(Bs+q\right) + \dot{\boldsymbol{\zeta}}\right] = \frac{Q}{c_pT}$$

$$\frac{d}{dt}\left[\left(Bs+q\right) + \ln\left(1+\frac{\partial B}{\partial\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} + \dot{\zeta} = 0$$

$$\frac{d\phi}{dt} - gw = 0 \qquad \frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$\mu - \frac{\partial\ln\left(p/\pi\right)}{\partial\ln\pi} = 0 \quad \longrightarrow \quad \mu - \frac{\partial q}{\partial\left(\zeta+Bs\right)} = 0$$

$$RT + \frac{\partial\phi}{\partial\ln\pi} = 0 \qquad \frac{T}{T_*} + \frac{\partial\left(\phi' - RT_*\zeta\right)}{RT_*\partial\left(\zeta+Bs\right)} = 0$$

The 8 non-hydrostatic equations of GEM

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta\left(Bs+q\right) + \left(1+\boldsymbol{\mu}\right)\nabla_\zeta\boldsymbol{\phi}' = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\mu = F_w$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}\left(Bs+q\right) + \dot{\zeta}\right] = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\left[\left(Bs+q\right) + \ln\left(1+\frac{\partial B}{\partial\boldsymbol{\zeta}}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\boldsymbol{\zeta}}}{\partial\boldsymbol{\zeta}} + \dot{\boldsymbol{\zeta}} = 0$$

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

The 8 variables:

$$\mathbf{V}_h, T, \boldsymbol{\phi}', \dot{\boldsymbol{\zeta}}, s$$

$$w, q, \boldsymbol{\mu}$$

$$\mu - \frac{\partial q}{\partial\left(\zeta + Bs\right)} = 0$$

$$\frac{T}{T_*} + \frac{\partial\left(\phi' - RT_*\zeta\right)}{RT_*\partial\left(\zeta + Bs\right)} = 0$$

The 5 hydrostatic equations of GEM

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta\left(Bs\right) + \nabla_\zeta\phi' = \mathbf{F}_h$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}\left(Bs\right) + \dot{\zeta}\right] = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\left[\left(Bs\right) + \ln\left(1 + \frac{\partial B}{\partial \zeta}s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

The 5 variables:

$\mathbf{V}_h, T, \phi', \dot{\zeta}, s$

~~w, q, μ~~

$$\frac{T}{T_*} + \frac{\partial\left(\phi' - RT_*\zeta\right)}{RT_*\partial\left(\zeta + Bs\right)} = 0$$

Discretizing in the vertical with staggering

- - - $$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}\nabla_\zeta(Bs+q) + (1+\overline{\mu})\nabla_\zeta\phi' = \mathbf{F}_h$$

——— $$\frac{dw}{dt} - g\mu = F_w$$

——— $$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \kappa\left[\frac{d}{dt}(Bs+\overline{q}) + \dot{\zeta}\right] = \frac{Q}{c_p T}$$

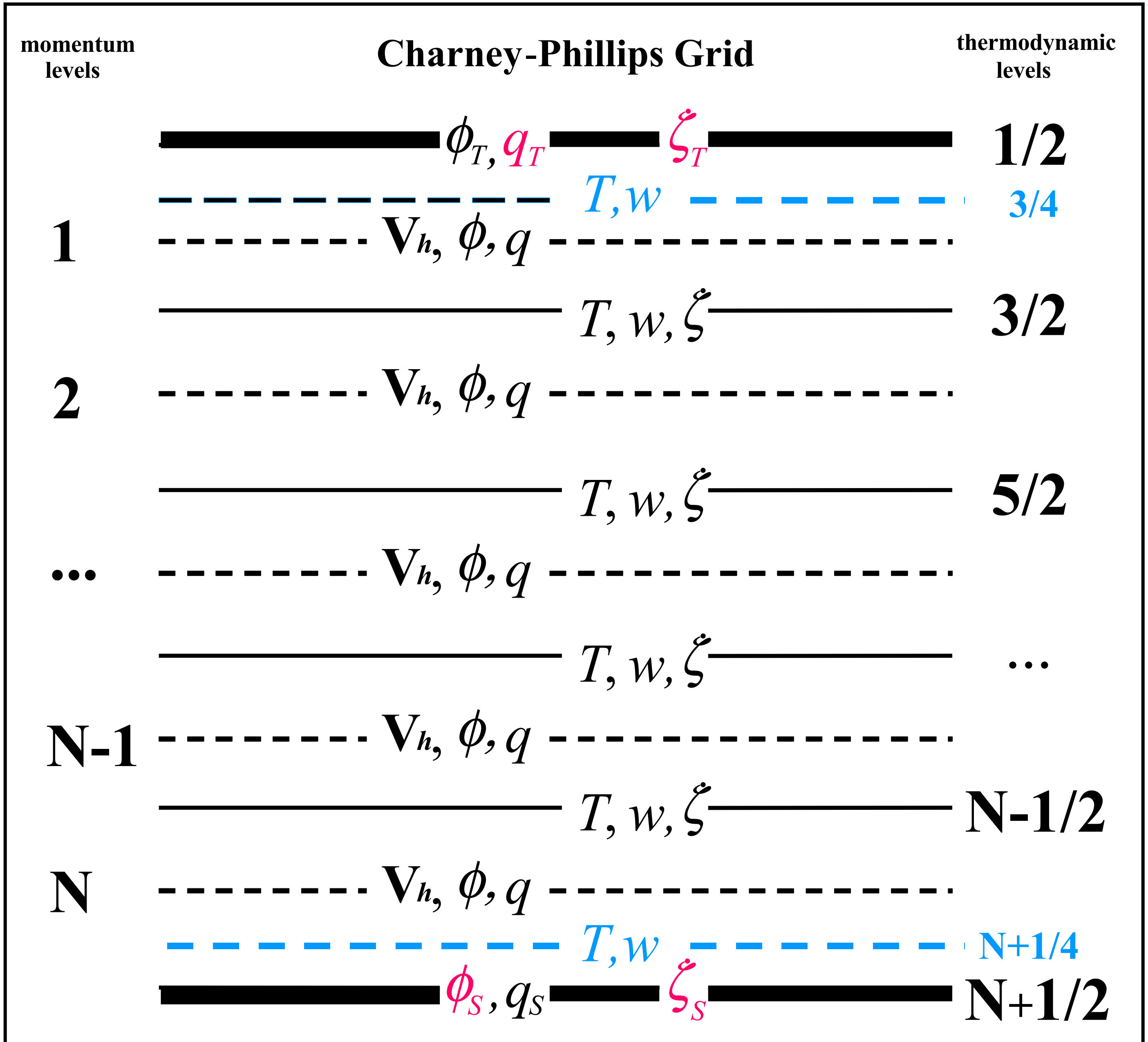
- - - $$\frac{d}{dt}\left[(Bs+q) + \ln\left(1+\frac{\partial B}{\partial\zeta}s\right)\right] + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} + \overline{\dot{\zeta}} = 0$$

——— $$\frac{d\overline{\phi}'}{dt} - RT_*\dot{\zeta} - gw = 0$$

——— $$\mu - \frac{\partial q}{\partial(\zeta+Bs)} = 0$$

——— $$\frac{T}{T_*} + \frac{\partial(\phi' - RT_*\zeta)}{RT_*\partial(\zeta+Bs)} = 0$$

———	$w, T', \dot{\zeta}, \mu$
- - - - -	$\mathbf{V}_h, \phi, q$
———	$w, T', \dot{\zeta}, \mu$



N mom. layers

N mom. levels

N+1 th. levels

N-1 th. layers

2 th. ½ layers

2 boundaries

The 8 non-linear equations of GEM4

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}x\mathbf{V}_h + R(T_* + \cancel{T'})\nabla_\zeta(Bs+q) + (1+\cancel{\mu})\nabla_\zeta\phi' = \cancel{\mathbf{F}_h}$$

$$\frac{dw}{dt} - g\mu = \cancel{F_w}$$

$$\frac{d}{dt}\cancel{\ln\left(1+\frac{T'}{T_*}\right)} - \kappa\left[\frac{d}{dt}(Bs+q)+\dot{\zeta}\right] = \cancel{\frac{Q}{c_p T}}$$

$$\frac{d}{dt}\left[(Bs+q)+\cancel{\ln\left(1+\frac{\partial B}{\partial\zeta}s\right)}\right] + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} + \dot{\zeta} = 0$$

$$\frac{d\phi'}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$\mu - \frac{\partial q}{\partial(\zeta+\cancel{Bs})} = 0$$

$$\frac{T'}{T_*} + \frac{\partial(\phi'+RT_*Bs)}{RT_*\partial(\zeta+\cancel{Bs})} = 0$$

The 8 linear equations of GEM4

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R(T_* \cdot \quad)\nabla_\zeta (Bs+q) + \qquad \nabla_\zeta \phi' = 0$$

$$\frac{dw}{dt} - g\mu = 0$$

$$\frac{d}{dt}\left(\qquad \frac{T'}{T_*} \right) - \kappa \left[\frac{d}{dt}(Bs+q) + \dot{\zeta} \right] = 0$$

$$\frac{d}{dt}\left[(Bs+q) + \qquad \frac{\partial B}{\partial \zeta} s \quad \right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} + \dot{\zeta} = 0$$

$$\frac{d\phi'}{dt} - RT_* \dot{\zeta} - gw = 0$$

$$\mu - \frac{\partial q}{\partial (\zeta \qquad)} = 0$$

$$\frac{T'}{T_*} + \frac{\partial (\phi' + RT_* Bs)}{RT_* \partial (\zeta \qquad)} = 0$$

The equations derived in 6 steps

1

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \mathbf{F}$$

$$\frac{dT}{dt} - \frac{1}{\rho c_p}\frac{dp}{dt} = \frac{Q}{c_p}$$

$$\frac{d\ln\rho}{dt} + \nabla\cdot\mathbf{V} = 0$$

$$p = \rho RT$$

$\Rightarrow$

2

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V} + RT\nabla\ln p + g\mathbf{k} = \mathbf{F}$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$(1-\kappa)\frac{d\ln p}{dt} + \nabla\cdot\mathbf{V} = \frac{Q}{c_p T}$$

Vertical coordinate transformation: *z* to ζ (unspecified)

$$\nabla_z \equiv \nabla_\zeta - \nabla_\zeta z\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

$$\frac{\partial}{\partial z} \equiv \frac{\partial\zeta}{\partial z}\frac{\partial}{\partial\zeta}$$

$\Rightarrow$

3

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\left(\nabla_\zeta\ln p - \nabla_\zeta z\frac{\partial\zeta}{\partial z}\frac{\partial\ln p}{\partial\zeta}\right) = \mathbf{F}_h$$

$$\frac{dw}{dt} + RT\frac{\partial\zeta}{\partial z}\frac{\partial\ln p}{\partial\zeta} + g = F_w$$

$$\frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$(1-\kappa)\frac{d\ln p}{dt} + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial\zeta}\right) + \nabla_\zeta\cdot\mathbf{V}_h + \frac{\partial\dot{\zeta}}{\partial\zeta} = \frac{Q}{c_p T}$$

$$\frac{dz}{dt} - w = 0$$

Vertical coordinate transformation: z to $\ln\pi$ to ζ

$$RT = -\frac{\partial \phi}{\partial \ln \pi}$$

$$\phi = gz$$

$$\mu = \frac{\partial \ln p}{\partial \ln \pi} - 1$$

$$\Rightarrow$$

$$\ln \pi = \zeta + Bs$$

$$s = \ln \pi_S - \zeta_S$$

4

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta \ln p + (1+\mu)\nabla_\zeta \phi = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\mu = F_w$$

$$\frac{d\ln T}{dt} - \kappa \frac{d\ln p}{dt} = \frac{Q}{c_p T}$$

$$\frac{d\ln p}{dt} + \frac{d}{dt}\ln\left(\frac{\partial \ln \pi}{\partial \zeta}\right) + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0$$

$$\frac{d\phi}{dt} - gw = 0$$

$$\mu - \frac{\partial \ln(p/\pi)}{\partial \ln \pi} = 0$$

$$RT + \frac{\partial \phi}{\partial \ln \pi} = 0$$

$$\ln \pi = \zeta + Bs$$

Going to model variables ϕ', q, s

$$\boldsymbol{\phi}' = \boldsymbol{\phi} - \boldsymbol{\phi}_*$$
$$\ln p = \ln \boldsymbol{\pi} + q$$
$$\ln \boldsymbol{\pi} = \boldsymbol{\zeta} + Bs$$

$\Rightarrow$

5

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + RT\nabla_\zeta (Bs+q) + (1+\boldsymbol{\mu})\nabla_\zeta \boldsymbol{\phi}' = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left[\frac{d}{dt}(Bs+q) + \dot{\boldsymbol{\zeta}}\right] = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\left[(Bs+q) + \ln\left(1 + \frac{\partial B}{\partial \boldsymbol{\zeta}} s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \frac{\partial \dot{\boldsymbol{\zeta}}}{\partial \boldsymbol{\zeta}} + \dot{\boldsymbol{\zeta}} = 0$$

$$\frac{d\boldsymbol{\phi}'}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$

$$\boldsymbol{\mu} - \frac{\partial q}{\partial(\boldsymbol{\zeta} + Bs)} = 0$$

$$\frac{T}{T_*} + \frac{\partial(\boldsymbol{\phi}' - RT_*\boldsymbol{\zeta})}{RT_*\partial(\boldsymbol{\zeta} + Bs)} = 0$$

Discretizing in the vertical with staggering

$$\overline{(\quad)}^{\zeta}$$

$$\delta_{\zeta}(\quad)$$

$$\Rightarrow$$

——— (solid)	$w, T', \dot{\zeta}, \boldsymbol{\mu}$
- - - (dashed)	$\mathbf{V}_h, \phi', q$
——— (solid)	$w, T', \dot{\zeta}, \boldsymbol{\mu}$

6

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^{\zeta}\nabla_{\zeta}(Bs+q) + \left(1+\overline{\boldsymbol{\mu}}^{\zeta}\right)\nabla_{\zeta}\boldsymbol{\phi}' = \mathbf{F}_h$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$

$$\frac{d}{dt}\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\left[\frac{d}{dt}\overline{(Bs+q)}^{\zeta} + \dot{\boldsymbol{\zeta}}\right] = \frac{Q}{c_p T}$$

$$\frac{d}{dt}\left[(Bs+q) + \ln\left(1+\frac{\partial B}{\partial \boldsymbol{\zeta}}s\right)\right] + \nabla_{\zeta}\cdot\mathbf{V}_h + \boldsymbol{\delta}_{\zeta}\dot{\boldsymbol{\zeta}} + \overline{\dot{\boldsymbol{\zeta}}}^{\zeta} = 0$$

$$\frac{d\overline{\boldsymbol{\phi}'}^{\zeta}}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$

$$\boldsymbol{\mu} - \frac{\boldsymbol{\delta}_{\zeta}q}{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} = 0$$

$$\frac{T}{T_*} + \frac{\boldsymbol{\delta}_{\zeta}(\boldsymbol{\phi}' - RT_*\boldsymbol{\zeta})}{RT_*\boldsymbol{\delta}_{\zeta}(\boldsymbol{\zeta}+Bs)} = 0$$

The equations implemented in 5 steps

1. *Introduction of Staggering in Z (hydrostatic pressure).*
2. *Logarithmic differencing in the hydrostatic equation. From* ΔZ *to* Δ
3. *Incomplete coordinate transformation. From Z to* lnZ.
4. *Complete coordinate transformation. From* lnZ *to* ζ. *Vertical motion* $\dot{\zeta}$
5. *A modified definition of hydrostatic pressure* $\partial\phi / \partial \ln \pi = -RT$

$$Z = \pi_* \rightarrow \zeta = \ln \pi_*$$

$$\partial\phi / \partial\pi = -1/\rho \rightarrow \partial\phi / \partial \ln \pi = -RT$$

Changing the Vertical Coordinate

Step1

$$Z = A(\eta) + b(\eta); \qquad b = Bp_0$$

$$\pi = A(\eta) + B(\eta)\pi_S = A(\eta) + b(\eta)e^s$$

$$s = \ln(\pi_S / p_0)$$

$$A = (n - B)p_{ref}; \qquad B = \left(\frac{\eta - \eta_T}{1 - \eta_T}\right)^r$$

$$\pi = Z + b(e^s - 1)$$

$$q = \ln p / \pi$$

$$\mathbf{q} = \ln p / Z = q + \ln\left(1 + \frac{b}{Z}(e^s - 1)\right)$$

$$\ln p = \mathbf{q} + \ln Z$$

$$-RT = -\frac{p}{\rho} = p\frac{\partial \phi}{\partial \pi} = e^q \frac{1 + b/Z(e^s - 1)}{1 + \partial b/\partial Z(e^s - 1)} Z\frac{\partial \phi}{\partial Z}$$

Step5

$$\zeta = \zeta_S + \ln(\eta); \qquad \zeta_S = \ln p_{ref}$$

$$\ln \pi = A(\zeta) + B(\zeta)s$$

$$s = \ln \pi_S / p_{ref}$$

$$A = \zeta; \qquad B = \left(\frac{\zeta - \zeta_T}{\zeta_S - \zeta_T}\right)^r$$

$$\ln \pi = \zeta + Bs$$

$$q = \ln p / \pi$$

$$\ln p = q + Bs + \zeta$$

$$-RT = \frac{\partial \phi}{\partial \ln \pi} = \frac{\partial \phi}{\partial(\zeta + Bs)}$$

Changing the Non-linear Dynamics System

Step 1

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^Z \nabla_Z \mathbf{q} + \left(1 + \overline{\boldsymbol{\mu}}^Z\right)\nabla_Z \boldsymbol{\phi}' = 0$$

$$\frac{dw}{dt} - g\boldsymbol{\mu} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}\overline{\mathbf{q}}^Z\right] - \boldsymbol{\kappa}\frac{\dot{Z}}{Z} = 0$$

$$\frac{d}{dt}\ln\left[1 + \frac{\partial b}{\partial Z}\left(e^s - 1\right)\right] + \nabla_Z \cdot \mathbf{V}_h + \boldsymbol{\delta}_Z \dot{Z} = 0$$

$$\frac{d\overline{\boldsymbol{\phi}'}^Z}{dt} - RT_* \frac{\dot{Z}}{Z} - gw = 0$$

$$\mathbf{q} - \left\{q + \ln\left[1 + \left(b/Z\right)\left(e^s - 1\right)\right]\right\} = 0$$

$$1 + \boldsymbol{\mu} - e^{\overline{q}^Z}\left[1 + Z\boldsymbol{\delta}_Z q\,\frac{1 + \overline{\left(b/Z\right)}^Z\left(e^s - 1\right)}{1 + \boldsymbol{\delta}_Z b\left(e^s - 1\right)}\right] = 0$$

$$\frac{T}{T_*} - e^{\overline{q}^Z}\,\frac{1 + \overline{\left(b/Z\right)}^Z\left(e^s - 1\right)}{1 + \boldsymbol{\delta}_Z b\left(e^s - 1\right)}\left(1 - \frac{Z\boldsymbol{\delta}_Z \boldsymbol{\phi}'}{RT_*}\right) = 0$$

Step 5

$$\frac{d\mathbf{V}_h}{dt} + f\mathbf{k}\mathrm{x}\mathbf{V}_h + R\overline{T}^\zeta \nabla_\zeta \left(Bs + q\right) + \left(1 + \overline{\mu}^\zeta\right)\nabla_\zeta \phi' = 0$$

$$\frac{dw}{dt} - g\mu = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \kappa\left(\overline{Bs + q}^\zeta\right)\right] - \kappa\dot{\zeta} = 0$$

$$\frac{d}{dt}\left[Bs + q + \ln\left(\frac{\partial\left(\zeta + Bs\right)}{\partial\zeta}\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \delta_\zeta \dot{\zeta} + \overline{\dot{\zeta}}^\zeta = 0$$

$$\frac{d\overline{\phi'}^\zeta}{dt} - RT_*\dot{\zeta} - gw = 0$$

$$\mu - \frac{\delta_\zeta q}{\delta_\zeta\left(\zeta + Bs\right)} = 0$$

$$\frac{T}{T_*} - \frac{\delta_\zeta\left(\zeta - \phi'/RT_*\right)}{RT_*\delta_\zeta\left(\zeta + Bs\right)} = 0$$

Changing the Linear Dynamics Systems

Step 1

$$\frac{\mathbf{V}_h}{\boldsymbol{\tau}} + \nabla_Z P = \mathbf{L}'_h$$

$$\frac{w}{\boldsymbol{\tau}} - g\boldsymbol{\delta}_Z Q = L'_w$$

$$\frac{\boldsymbol{\delta}_Z Q}{\boldsymbol{\tau}} - \frac{Z\boldsymbol{\delta}_Z P}{\boldsymbol{\tau} RT_*} - \boldsymbol{\kappa}\frac{X + \overline{Q}^Z}{Z} = L'_T$$

$$\nabla_Z \cdot \mathbf{V}_h + \boldsymbol{\delta}_Z X = L_C$$

$$\frac{\overline{P}^Z}{\boldsymbol{\tau}} - RT_* \frac{X + \overline{Q}^Z}{Z} - gw = L_P$$

$$P - \boldsymbol{\phi}' - RT_*\left(\frac{b}{Z}s + q\right) = 0$$

$$\frac{X + \overline{Q}^Z}{Z} - \frac{\dot{Z}}{Z} - \frac{1}{\boldsymbol{\tau}}\left(\overline{\frac{b}{Z}s + q}^Z\right) = 0$$

$$Q - Zq = 0$$

Step 5

$$\frac{\mathbf{V}_h}{\boldsymbol{\tau}} + \nabla_\zeta P = \mathbf{L}'_h$$

$$\frac{w}{\boldsymbol{\tau}} - g\boldsymbol{\delta}_\zeta q = L'_w$$

$$\frac{\boldsymbol{\delta}_\zeta q}{\boldsymbol{\tau}} - \frac{\boldsymbol{\delta}_\zeta P}{\boldsymbol{\tau} RT_*} - \boldsymbol{\kappa} X = L'_T$$

$$-\frac{\overline{\boldsymbol{\delta}_\zeta q}^\zeta}{\boldsymbol{\tau}} + \nabla_\zeta \cdot \mathbf{V}_h + \boldsymbol{\delta}_\zeta X + \overline{X}^\zeta = L_C$$

$$\frac{\overline{P}^\zeta}{\boldsymbol{\tau}} - RT_* X - gw = L_{\boldsymbol{\phi}}$$

$$P - \boldsymbol{\phi}' - RT_*(Bs + q) = 0$$

$$X - \dot{\boldsymbol{\zeta}} - \frac{1}{\boldsymbol{\tau}}\left(\overline{Bs + q}^Z\right) = 0$$

A comparison: ALADIN-NH, IFS-ECMWF

$\mathbf{V}_h :$	$\frac{d\mathbf{V}_h}{dt} + \frac{RT}{p}\nabla_{\boldsymbol{\eta}} p + \frac{1}{m}\frac{\partial p}{\partial \boldsymbol{\eta}}\nabla_{\boldsymbol{\eta}}\boldsymbol{\phi} = \mathbf{F}_h$
$w :$	$\frac{dw}{dt} + g\left(1 - \frac{1}{m}\frac{\partial p}{\partial \boldsymbol{\eta}}\right) = F_w$
$T :$	$\frac{dT}{dt} - \frac{RT}{c_v} D_3 = \frac{Q}{c_v}$
$p :$	$\frac{dp}{dt} - \frac{c_p}{c_v} p D_3 = \frac{Qp}{c_v T}$
$m, \dot{\boldsymbol{\eta}} :$	$\frac{\partial m}{\partial t} + \nabla_{\boldsymbol{\eta}} \cdot m\mathbf{V}_h + \frac{\partial}{\partial \boldsymbol{\eta}} m\dot{\boldsymbol{\eta}} = 0$
$\boldsymbol{\phi} :$	$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\eta}} + m\frac{RT}{p} = 0$
$D_3 :$	$D_3 - \left[\nabla_{\boldsymbol{\eta}} \cdot \mathbf{V}_h - \frac{gp}{mRT}\frac{\partial w}{\partial \boldsymbol{\eta}} + \frac{p}{mRT}\nabla_{\boldsymbol{\eta}}\boldsymbol{\phi} \cdot \frac{\partial \mathbf{V}_h}{\partial \boldsymbol{\eta}}\right] = 0$

$$\frac{d\boldsymbol{\phi}}{dt} - gw = 0$$

Definitions:

$$\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\eta}} = m$$

$$\boldsymbol{\pi} = A + B\boldsymbol{\pi}_S$$

8 variables, 8 equations (6 prognostic)

A comparison: ALADIN-NH, IFS-ECMWF

Change of Variables

$$q = \ln\left(p / \boldsymbol{\pi}\right)$$

$$\mathrm{d} = -\frac{gp}{mRT}\frac{\partial w}{\partial \boldsymbol{\eta}}$$

$$\mathrm{X} = \frac{p}{mRT}\nabla_{\boldsymbol{\eta}}\boldsymbol{\phi}\cdot\frac{\partial \mathbf{V}_h}{\partial \boldsymbol{\eta}}$$

$$D_3 = \nabla_{\boldsymbol{\eta}}\cdot\mathbf{V}_h - \frac{gp}{mRT}\frac{\partial w}{\partial \boldsymbol{\eta}} + \frac{p}{mRT}\nabla_{\boldsymbol{\eta}}\boldsymbol{\phi}\cdot\frac{\partial \mathbf{V}_h}{\partial \boldsymbol{\eta}} = \nabla_{\boldsymbol{\eta}}\cdot\mathbf{V}_h + \mathrm{d} + \mathrm{X}$$

A comparison: ALADIN-NH, IFS-ECMWF

$\mathbf{V}_h$:
$$\frac{d\mathbf{V}_h}{dt} + \frac{RT}{p}\nabla_{\boldsymbol{\eta}} p + \frac{1}{m}\frac{\partial p}{\partial \boldsymbol{\eta}}\nabla_{\boldsymbol{\eta}}\boldsymbol{\phi} = \mathbf{F}_h$$

d :
$$\frac{d\mathrm{d}}{dt} + \mathrm{d}\left(\mathrm{d} + \mathrm{X}\right) + \frac{g^2 p}{mRT}\frac{\partial}{\partial \boldsymbol{\eta}}\left(\frac{1}{m}\frac{\partial(p - \boldsymbol{\pi})}{\partial \boldsymbol{\eta}}\right) - \frac{gp}{mRT}\frac{\partial \mathbf{V}_h}{\partial \boldsymbol{\eta}} \cdot \nabla_{\boldsymbol{\eta}} w = -\frac{gp}{mRT}\frac{\partial F_w}{\partial \boldsymbol{\eta}}$$

T :
$$\frac{dT}{dt} - \frac{RT}{c_v}\left(\nabla_{\boldsymbol{\eta}} \cdot \mathbf{V}_h + \mathrm{d} + \mathrm{X}\right) = \frac{Q}{c_v}$$

q :
$$\frac{dq}{dt} + \frac{\dot{\boldsymbol{\pi}}}{\boldsymbol{\pi}} - \frac{c_p}{c_v}\left(\nabla_{\boldsymbol{\eta}} \cdot \mathbf{V}_h + \mathrm{d} + \mathrm{X}\right) = \frac{Q}{c_v T}$$

$m, \dot{\boldsymbol{\eta}}$:
$$\frac{\partial m}{\partial t} + \nabla_{\boldsymbol{\eta}} \cdot m\mathbf{V}_h + \frac{\partial}{\partial \boldsymbol{\eta}} m\dot{\boldsymbol{\eta}} = 0$$

$\boldsymbol{\phi}$:
$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\eta}} + m\frac{RT}{p} = 0$$

$$q - \ln(p/\boldsymbol{\pi}) = 0$$

p :

w :
$$\mathrm{d} + \frac{gp}{mRT}\frac{\partial w}{\partial \boldsymbol{\eta}} = 0$$

$$\mathrm{X} - \frac{p}{mRT}\nabla_{\eta}\boldsymbol{\phi} \cdot \frac{\partial \mathbf{V}_h}{\partial \boldsymbol{\eta}} = 0$$

X :

$\dot{\boldsymbol{\pi}}$:
$$\dot{\boldsymbol{\pi}} = \ldots$$

11 variables, 11 equations (6 prognostic)

A comparison: GEM

$\mathbf{V}_h$:
$$\frac{d\mathbf{V}_h}{dt} + RT\nabla_\zeta (Bs+q) + (1+\boldsymbol{\mu})\nabla_\zeta \boldsymbol{\phi}' = \mathbf{F}_h$$

w :
$$\frac{dw}{dt} - g\boldsymbol{\mu} = F_w$$

T :
$$\frac{d}{dt}\left[\ln\left(\frac{T}{T_*}\right) - \boldsymbol{\kappa}(Bs+q)\right] - \boldsymbol{\kappa}\dot{\boldsymbol{\zeta}} = \frac{Q}{c_p T}$$

$s, \dot{\boldsymbol{\zeta}}$:
$$\frac{d}{dt}\left[(Bs+q) + \ln\left(1 + \frac{\partial B}{\partial \boldsymbol{\zeta}} s\right)\right] + \nabla_\zeta \cdot \mathbf{V}_h + \left(\frac{\partial}{\partial \boldsymbol{\zeta}} + 1\right)\dot{\boldsymbol{\zeta}} = 0$$

q :
$$\frac{d\boldsymbol{\phi}'}{dt} - RT_*\dot{\boldsymbol{\zeta}} - gw = 0$$

$\boldsymbol{\mu}$:
$$\boldsymbol{\mu} - \frac{\partial q}{\partial(\boldsymbol{\zeta} + Bs)} = 0$$

$\boldsymbol{\phi}$:
$$\frac{T}{T_*} - \frac{\partial(\boldsymbol{\zeta} - \boldsymbol{\phi}'/RT_*)}{\partial(\boldsymbol{\zeta} + Bs)} = 0$$

8 variables, 8 equations (6 prognostic)

OUR EQUATIONS,

ARE’NT THEY BEAUTIFUL?

beginning/end

hydrostatic/non-hydrostatic

linear/non-linear

analytic/discrete

❁

Plutôt jolies, non!

Merci